ASPECTS OF TACHYON CONDENSATION IN STRING THEORY

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We suggest a correspondence between brane-antibrane systems and stable triples (E_1, E_2, T), where E_1, E_2 are holomorphic vector bundles and the tachyon T is a map between them. Under the assumption of holomorphicity, the brane-antibrane field equations reduce to a set of vortex equations, which are equivalent to the mathematical notion of stability of the triple. We discuss some examples and show that the theory of stable triples suggests a new notion of BPS bound states and stability. We then propose a form of the effective action of the tachyon and gauge fields for brane-antibrane systems, written in terms of the supercurvature. Kink and vortex solutions with constant infinite gauge field strength reproduce the exact tensions of the lower-dimensional D-branes.

Contents

1 Introduction

Systems of non-BPS brane configurations have been extensively studied recently (for a review see Ref. 1). A basic non-BPS system is the coincident brane-antibrane configuration, which is not stable. It has a tachyon on the world-volume of the branes that arises from the open string stretched between the branes and the antibranes, and it is charged under the world-volume gauge groups. The decay of the system can be seen by the tachyon rolling down to the minimum of its potential. Upon tachyon condensation one can end up with lower dimensional BPS branes, if the original brane-antibrane system contained the corresponding charges.

Most of the analysis of branes-antibranes systems is carried out in flat space. It is clearly of interest to extend the available methods to consider curved closed string backgrounds. In the following we will discuss some aspects of such systems in curved spaces. First, we will propose a correspondence between brane-antibrane systems and stable triples (E_1, E_2, T), where E_1, E_2 are holomorphic vector bundles and the tachyon, T, is a map between them. Under the assumption of holomorphicity, the brane-antibrane field equations reduce to a set of vortex equations. The latter are equivalent to the topological notion of stability of the triple (E_1, E_2, T). This is quite analogous to the case of a single vector bundle where solutions of Hermitian Yang-Mills equations correspond to stable holomorphic bundles. We discuss some examples and show that the theory of stable triples suggests a new notion of BPS bound states upon tachyon condensation.

Second, we will consider the higher order terms in the tachyon and gauge fields effective action. We will use the notion of superconnections, which when considering the branes-antibranes system appears naturally via the Chan-Paton factors. We will make the assumption that the effective action of tachyon and gauge fields for the $Dp-\bar{D}p$-branes system can be written in a Quillen-like framework in terms of the supercurvature. The tachyon potential that arises in this framework is exponential in the tachyon field. We will suggest a form of the effective action and use it to study the process of tachyon condensation. Kink solutions that we will find, with infinite constant value of the gauge field strength, reproduce the exact tensions of the lower-dimensional D-branes at the minimum of the tachyon potential.

This article is organized as follows. In Section 2 we will introduce the triples, the vortex equations and propose the correspondence. We will then consider some examples. In Section 3 we will relate our description of BPS D-branes as stable triples to existing descriptions of BPS states. In particular, we will show that the theory of stable triples suggests a new notion of BPS bound states and stability. In Section 4 we will introduce the notion of superconnections and supercurvatures, propose an effective action of the tachyon and gauge fields for branes-antibranes systems, study the kink solutions and derive the exact tensions of the lower-dimensional Dp-branes.

This article is mainly based on Refs. 2 and 3. I omitted many relevant works, which can be found in the reference lists of these papers.

2 Branes-antibranes systems

In this section we will propose a correspondence between systems of coinciding branes and antibranes wrapping a manifold X and stable triples (E_1, E_2, T). E_1 and E_2 are holomorphic vector bundles on X. Physically, they correspond to the branes and antibranes respectively. T is a homomorphism between the vector bundles $T : E_2 \to E_1$. It is the tachyon field that arises from the open string stretched between the branes and the antibranes. With a holomorphic ansatz, we will recast the field equations of the brane-antibrane system as a set of vortex equations. The solutions of the vortex equations represent BPS configurations. Such solutions correspond to a mathematical construction of stable triples on X.[4] With this correspondence the analysis of tachyon condensation leading to BPS branes will be replaced by a stability analysis.

2.1 Tachyons and triples

Let us briefly review the construction of D-branes from coincident brane-antibrane configurations. We will consider Type IIA string theory compactified on Calabi–Yau manifold X of complex dimension d.

A configuration of n branes wrapping the Calabi–Yau manifold is described by a $U(n)$ vector bundle E on X. This carries charges for various RR fields

$$Q = \mathrm{ch(E)}\sqrt{\widehat{\mathrm{A}}(\mathrm{X})} = \mathrm{ch(E)}\sqrt{\mathrm{Td(X)}}\ , \tag{1}$$

which is an element of the cohomology $H^*(X, Z)$ known as the Mukai vector. For the equality in (1) we used the fact that $Td(X)$ on a Calabi–Yau manifold is equal to the A-roof genus $\widehat{A}(X)$. In this expression ch(E) is the Chern character of the vector bundle E

$$\mathrm{ch(E)} = \mathrm{Tr}\exp\left[\frac{\mathrm{F}}{2\pi}\right]\ , \tag{2}$$

where F is the field strength of the gauge field on the brane. It has an expansion in terms of the Chern classes

$$\mathrm{ch(E)} = \mathrm{n} + \mathrm{c_1(E)} + \frac{1}{2}(\mathrm{c_1^2(E)} - 2\mathrm{c_2(E)}) + \dots\ . \tag{3}$$

The A-roof genus $\widehat{A}(X)$ and has an expansion in terms of the Pontrjagin classes

$$\widehat{A}(X) = 1 - \frac{p_1(X)}{24} + \dots\ . \tag{4}$$

Consider a configuration of n_1 branes wrapped on X, together with n_2 antibranes. In general, the configuration is described by specifying a $U(n_1)$ vector bundle E_1 on X for the branes together with a second $U(n_2)$ bundle E_2 for the antibranes. The net D-brane charge is then the difference of the Mukai vectors for the two bundles

$$Q = Q_1 - Q_2 = (\mathrm{ch(E_1)} - \mathrm{ch(E_2)})\sqrt{\widehat{A}(X)}\ . \tag{5}$$

In general, we would expect the antibranes to annihilate against the branes. However, if the bundles are different then there is a net D-brane charge and the branes cannot completely annihilate and still conserve charge.

Since identical bundles can annihilate, adding the same bundle to E_1 and E_2 gives the same physical configuration. That is we should identify $(E_1 \oplus V, E_2 \oplus V)$ with (E_1, E_2), which is the equivalence class identification made in K-theory. In K-theory one identifies the Chern class of the pair (E_1, E_2) with the difference of Chern classes $\mathrm{ch(E_1)} - \mathrm{ch(E_2)}$, thus providing the map from K-theory to the Mukai charge (5). In fact, a D-brane charge is more accurately measured by the K-theory class rather than the cohomological Mukai charge. In particular, K-theory includes more information than the Chern classes themselves. For instance, Chern classes miss torsion.

Physically, the annihilation happens because there is a tachyonic mode T in the open string connecting the branes and antibranes. The tachyon potential has a minimum away from zero. However, if the bundles E_1 and E_2 are different, there is a topological obstruction to the tachyon being at the minimum everywhere on X. The tachyon T transforms in the fundamental representation of each bundle $(\mathbf{n_2}, \bar{\mathbf{n}}_\mathbf{1})$, and it must respect the twisting of each bundle. In general, even if $n_1 = n_2$ if the bundles are different, it cannot do so and remain everywhere at the minimum of the tachyon potential. Instead it must be zero on some sub-manifold C of X. There is a vortex solution representing a lower-dimensional brane localized on C. In particular, all the lower dimensional branes can be built out of D9-branes in this way.[5]

To specify a general D-brane configuration we need to specify the bundles E_1 and E_2 together with the condensed tachyon T. Since T is in the bi-fundamental representation, it represents a map between the bundles. Thus the full information is the triple (E_1, E_2, T) giving the complex

$$E_2 \xrightarrow{T} E_1 \ . \tag{6}$$

As we have noted, the D-brane charges are characterized by the K-theory class of the pair (E_1, E_2). However for a given D-brane charge there is generically a moduli space of different D-brane states. It is natural to ask what characterizes these distinct D-brane states. In general, this should be some equivalence class of triples (E_1, E_2, T), giving a finer classification than simply the K-theory class. In the following, we will consider BPS configurations. We will see that this implies that the bundles and maps are holomorphic. A possible equivalence class is that for holomorphic bundles E_1 and E_2, we should identify triples in the same derived category, which essentially means considering complexes of bundles of the form (6), modulo exact sequences.

2.2 The vortex equations

Consider the low-energy effective action of the world-volume theory of a configuration of coinciding n_1 branes and n_2 antibranes wrapping a manifold X

$$S = \int_X \left[\frac{1}{4}\mathrm{Tr}_1\mathrm{F}_1^2 + \frac{1}{4}\mathrm{Tr}_2\mathrm{F}_2^2 + (\mathrm{DT})^{\mathrm{a}}_{\bar{\mathrm{b}}}(\mathrm{DT}^*)^{\bar{\mathrm{b}}}_{\mathrm{a}} + \lambda\left(\mathrm{T}^{\mathrm{a}}_{\bar{\mathrm{b}}}\mathrm{T}^{*\bar{\mathrm{b}}}_{\mathrm{b}} - \alpha^2\delta^{\mathrm{a}}_{\mathrm{b}}\right)^2\right] \ . \tag{7}$$

Typically, we will take $n_1 = n_2$, so only lower-dimensional branes remain after condensation. There are higher order corrections to (7), and in general one also expects the kinetic terms of the tachyon and the gauge fields to depend on the tachyon background. Such corrections modify the field equations and the precise description of the tachyon rolling to the minimum of its potential. We expect, however, that it should not matter for the topological construction of the lower-dimensional branes upon the condensation of the tachyon. Here we think about the lower-dimensional branes as the BPS branes. Quantitative properties of the lower-dimensional branes such as the size of vortex solutions will be modified, upon the inclusion of the corrections.

We further assume the same gauge coupling for the two gauge groups and rescaled the gauge and tachyon fields in (7). In (7), a is the index of the fundamental representation of E_1 and $\bar{a}$ the anti-fundamental of E_2. The parameter α^2 in the tachyon potential is related to the value of the tachyon field at the minimum of the potential

$$\alpha^2 = \frac{1}{n_1}\mathrm{Tr}(\mathrm{TT}^*)|_{\mathrm{minimum}} \ . \tag{8}$$

Since T has charge $\pm e$ under the gauge groups its covariant derivative is

$$D_M T^a_{\bar{b}} = \partial_M T^a_{\bar{b}} + ie(A^1_M)^a_b T^b_{\bar{b}} - ieT^a_{\bar{a}}(A^2_M)^{\bar{a}}_{\bar{b}} \ . \tag{9}$$

From now on we will suppress the indices and write, for instance $DT = dT + ieA_1 T - ieTA_2$.

The equations of motion read

$$\begin{aligned} D_1^M F^1_{MN} &= ie\left[T\left(D_N T^*\right) - \left(D_N T\right)T^*\right] \ , \\ D_2^M F^2_{MN} &= ie\left[T^*\left(D_N T\right) - \left(D_N T^*\right)T\right] \ , \\ D^2 T &= 2\lambda\left(TT^*T - \alpha^2 T\right) \ , \end{aligned} \tag{10}$$

where $D_1 = d + ieA_1$ and $D_2 = d + ieA_2$.

We denote the Kähler metric on X by $g_{m\bar{n}}$, where m is a holomorphic index and $\bar{n}$ an anti-holomorphic index. There is then a set of equations which imply the equations of motion.[2] They are, first, that all the fields are holomorphic, namely

$$\begin{aligned} F^1_{mn} &= F^1_{\bar{m}\bar{n}} = 0 \ , \\ F^2_{mn} &= F^2_{\bar{m}\bar{n}} = 0 \ , \\ D_{\bar{m}} T &= 0 \ . \end{aligned} \tag{11}$$

Then in addition we have a Hermitian condition

$$\begin{aligned} ig^{m\bar{n}} F^1_{m\bar{n}} + eTT^* &= 2\pi\tau_1 I_1 \ , \\ ig^{m\bar{n}} F^2_{m\bar{n}} - eT^*T &= 2\pi\tau_2 I_2 \ , \end{aligned} \tag{12}$$

where I_1, I_2 are the identity matrices for the E_1 and E_2 bundles respectively. Together we shall call equations (11) and (12) the *vortex equations*. The important point, as we discuss in the next subsection, is that solutions of the vortex equations (11) and (12) are in one-to-one correspondence with the topological notion of stability of the triple (E_1, E_2, T).[4]

This requires that the parameters in the action are related so that

$$\lambda = e^2 \ . \tag{13}$$

We also get a relation between τ_1, τ_2 and α

$$e\alpha^2 = \pi\left(\tau_1 - \tau_2\right) \ . \tag{14}$$

Note that the relation between λ and e, together with the assumption that the height of the tachyon potential is the tension of the brane system $\mathcal{T}_p$, imply that the tachyon charge e is related to the value of the tachyon field at the minimum of its potential by

$$e^2 \sim \mathcal{T}_p \left(\frac{1}{n_1} \, \mathrm{Tr}(\mathrm{TT}^*)|_{minimum} \right)^{-2} \; . \tag{15}$$

If we add that vortex equations, take a trace and integrate over X, we find

$$\tau_1 n_1 + \tau_2 n_2 = \mathrm{degE}_1 + \mathrm{degE}_2 \; , \tag{16}$$

where the degree of a vector bundle degE is defined as

$$\mathrm{degE} = \frac{1}{\mathrm{V(d-1)!}} \int \mathrm{c}_1(\mathrm{E}) \wedge \mathrm{J}^{\mathrm{d}-1} \; , \tag{17}$$

with J the Kähler form on X and $c_1(E)$ is the first Chern class. Thus we see that τ_1 and τ_2 are completely determined by the parameter α and the bundles E_1 and E_2. In particular

$$\begin{aligned} 2\pi\tau_1 &= 2\pi \frac{\mathrm{degE}_1 + \mathrm{degE}_2}{n_1 + n_2} + \frac{2n_2}{n_1 + n_2} \alpha^2 \; , \\ 2\pi\tau_2 &= 2\pi \frac{\mathrm{degE}_1 + \mathrm{degE}_2}{n_1 + n_2} - \frac{2n_1}{n_1 + n_2} \alpha^2 \; . \end{aligned} \tag{18}$$

We expect that the solutions of the vortex equations are supersymmetric BPS states. One way to establish this is to analyze the supersymmetry directly. Another way, which is to show that these solution satisfy the Bogomol'nyi bound.[2]

With the relations (13) and (18) we have an interesting correspondence. Brane configurations where all fields are holomorphic and that arise via the process of tachyon condensation are described by solutions to the vortex equations (11) and (12). There is one dimensionful parameter, α^2, in the equations which is related to the value of the tachyon at the minimum of the potential and so scales as the string scale. We should emphasize that the holomorphicity conditions (11) limit our discussion to tachyon condensation that leads to BPS branes. In order to study stable non-BPS branes we would have to relax these conditions.

2.3 Stable triples

A particularly useful property of the vortex equations is that their solutions are in one-to-one correspondence with a topological notion of stability of the triple

(E_1, E_2, T). Thus we can use stability to analyze the existence of solutions on a general X, rather than looking for solutions explicitly.

The analogy here is to the Hermitian Yang–Mills equations (HYM) describing the supersymmetric compactification of a single gauge bundle E on a Calabi–Yau manifold. They read

$$F_{mn} = F_{\bar{m}\bar{n}} = 0 \ , \qquad ig^{m\bar{n}} F_{m\bar{n}} = 2\pi\tau \ , \tag{19}$$

which are a simple subset of the vortex equations. By the Donaldson–Uhlenbeck–Yau theorem, solutions of the HYM equations are in one-to-one correspondence with holomorphic vector bundles of a particular type: those that are poly-stable. This is defined as follows. Let the slope of a bundle be given by

$$\mu(E) = \frac{\mathrm{degE}}{\mathrm{rankE}} \ . \tag{20}$$

A bundle E is stable if for any non-trivial sub-bundle $E' \subset E$, one has $\mu(E') < \mu(E)$. Poly-stability means that E is the direct sum of stable bundles each with the same slope. By this theorem, an analytic problem — solutions of the HYM equations — is a equivalent to a topological problem — listing the stable bundles — and the latter problem is generally much easier to solve. We should note that the stability problem is not quite topological: it also depends on the choice of Kähler form J via the definition of degE.

It turns out there that is an analogous notion of stability for a triple (E_1, E_2, T), such that there is a solution to the vortex equations if and only if the triple is stable.[4] One first needs to define what is meant by a sub-triple. We take the definition that (E_1', E_2', T') is a sub-triple if

(1) E_i' is a coherent sub-sheaf of E_i with $i = 1, 2$,

(2) T' is the restriction of T.

Next one needs the analog of the μ-slope $\mu(E)$. With σ a real number, one defines the σ-slope of a triple (E_1, E_2, T) by

$$\mu_\sigma(T) = \frac{\mathrm{deg(E_1 \oplus E_2)} + \sigma \mathrm{n_2}}{n_1 + n_2} \ . \tag{21}$$

A triple is then called σ-stable if for all nontrivial sub-triples (E_1', E_2', T') we have

$$\mu_\sigma(T') < \mu_\sigma(T) \ . \tag{22}$$

The relation between solutions to the vortex equations (11) and (12), and the σ-stability of the triple is for $\sigma = \tau_1 - \tau_2$. As seen, for instance, from (18), for the brane-antibrane system it means

$$\sigma \equiv \frac{\alpha^2}{\pi} \ . \tag{23}$$

The BPS branes that arise upon the condensation of the tachyon will be σ-stable with σ given by the relation (23). As will be very relevant later, since α goes like the string scale M_s, the large volume limit corresponds to large σ. This is the regime where we can trust the vortex equations to provide an adequate description. Note, in particular in the large σ limit, the stability condition (22) reads

$$n_2 n_1' - n_2' n_1 > 0 \ , \tag{24}$$

where $n_i' = \mathrm{rank}\,\mathrm{E_i'}$, for $i = 1, 2$, which is similar to a stability condition on a quiver in the orbifold limit.

2.4 BPS branes via tachyon condensation

As a simple illustration of solutions to the vortex equations and some of the discussion to follow, let us consider how we can realize a D0-brane on C via the condensation of a D2-brane and an anti-D2-brane.

The D2-branes will be realized as $U(1)$ bundles. For finite energy, we require that the connection is pure gauge at infinity (thus we are effectively considering bundles on $S^2 = P^1$). To ensure that we have a zero brane we need the difference of the bundle charges (5) to be one:

$$c_1(E_1) - c_1(E_2) = 1 \ . \tag{25}$$

For simplicity we can take E_2 to be trivial, while E_1 has $c_1(E_1) = 1$. This means that E_1 has a non-trivial holonomy at infinity.

Now consider a solution of the vortex equations (11). First T must be holomorphic. This implies that

$$\bar{\partial} T + i A^1_{\bar{z}} T - i A^2_{\bar{z}} T = 0 \ . \tag{26}$$

Equation (26) can be solved and gives

$$\begin{aligned} A^1_z - A^2_z &= i\partial \ln \mathrm{T} \\ &= \partial\chi + i\partial \ln \mathrm{f} \ , \end{aligned} \tag{27}$$

where we have written the tachyon as $T = f e^{i\chi}$. Note that locally gauge transformations can always remove χ. Since fields must be pure gauge at infinity, we have $\ln \mathrm{f} \to \mathrm{const}$ at infinity. The fact that E_2 is trivial and E_1

has $c_1(E_1) = 1$ means that A_2 can be gauged to zero at infinity while A_1 can be gauged to the form

$$A_1|_\infty = \partial\theta \ , \tag{28}$$

where $z = re^{i\theta}$, which is locally trivial but has global holonomy.

From this we see that we can gauge T such that

$$\chi|_\infty = \theta \ . \tag{29}$$

There must be some point $z = z_0$ in C where this non-trivial holonomy in T untwists, at which point $T = 0$. We can choose this to be the origin $z = 0$. In general this means we can globally gauge T to the form

$$T = f(r)e^{i\theta} \tag{30}$$

such that $f(0) = 0$.

The form of f can be determined by subtracting the vortex equations (12). We have

$$\begin{aligned} ig^{z\bar{z}}F^1_{z\bar{z}} - ig^{z\bar{z}}F^2_{z\bar{z}} &= i\partial\left(A^1_{\bar{z}} - A^2_{\bar{z}}\right) - i\bar{\partial}\left(A^1_z - A^2_z\right) \\ &= \partial\bar{\partial}\ln \mathrm{f}^2 \ , \end{aligned} \tag{31}$$

so the difference of the vortex equations (12) reads

$$\partial\bar{\partial}\ln \mathrm{f}^2 + 2\mathrm{f}^2 = 2\pi(\tau_1 - \tau_2) = 2\alpha^2 \ . \tag{32}$$

Finding the solution of this equation such that $f(0) = 0$ and $f(\infty) = \alpha$ then completely determines the vortex solution up to gauge transformations. In particular, the individual vortex equations give

$$\begin{aligned} F^1_{z\bar{z}} &= i\left(f^2 - \alpha^2\right) - 2\pi i \ , \\ F^2_{z\bar{z}} &= -i\left(f^2 - \alpha^2\right) \end{aligned} \tag{33}$$

for the two field strengths.

We see that at infinity $|T| = \alpha$ and the tachyon is at the minimum of its potential. However, as one moves around the S^1 at infinity the phase of the tachyon rotates giving a vortex. The vortex untwists at the origin where $T = 0$. This is the position of the D0-brane. In general there is a modulus to move the D0-brane to at any point in the complex plane.

We can do the same analysis without referring directly to the vortex equations. On P^1 we have $E_1 = \mathcal{O}(\infty)$ and $E_2 = \mathcal{O}$ the trivial bundle. Thus we have the triple

$$\mathcal{O} \xrightarrow{\mathcal{T}} \mathcal{O}(\infty) \ . \tag{34}$$

We can think of this as a map between holomorphic functions on P^1 (that is constant functions) to meromorphic functions with, at most, a single pole (at the zero-brane). However, such maps lie in an exact sequence

$$\mathbf{0} \longrightarrow \mathcal{O} \xrightarrow{\mathcal{T}} \mathcal{O}(\infty) \longrightarrow \mathcal{O}_{\mathcal{P}} \longrightarrow \mathbf{0} \ . \tag{35}$$

That is the kernel of T is zero. The cokernel, meanwhile, is simply the sheaf of functions localized at a point $\mathcal{O}_{\mathcal{P}}$. This is precisely the set of points where T vanishes. But since $\mathcal{O}_{\mathcal{P}}$ is localized on a point p it is precisely the description of a D0-brane on p. Thus after condensation, E_1 and E_2 are effectively replaced by their cokernel, representing a D0-brane. Depending on the particular choice of the map T, the D0-brane lies at different points p in P^1.

It is straightforward to show that the triple (34) is σ-stable in the sense of (21) and (22), since any sub-triple has E_2' zero.

We can naturally generalize the previous example to construct, via the process of tachyon condensation, supersymmetric $(2d-2)$-branes on a d complex dimensional Calabi–Yau manifold X. Such branes are described by sheaves localized on a holomorphic hypersurface C in X. As above let us assume that E_2 is the trivial $U(1)$ bundle $\mathcal{O}_{\mathcal{X}}$. We then require $c_1(E_1) = [C]$, the class of C. This can be achieved by taking E_1 to be the bundle $\mathcal{O}_{\mathcal{X}}(\mathcal{C})$. Note that in general this bundle also induces lower-dimensional brane charges. Then, for any map T we have the exact sequence

$$\mathbf{0} \longrightarrow \mathcal{O}_{\mathcal{X}} \xrightarrow{\mathcal{T}} \mathcal{O}_{\mathcal{X}}(\mathcal{C}) \longrightarrow \mathcal{O}_{\mathcal{C}}(\mathcal{C}) \longrightarrow \mathbf{0} \ , \tag{36}$$

where $\mathcal{O}_{\mathcal{C}}(\mathcal{C})$ is a sheaf localized on C. As in the previous example, this means we can replace the triple $E_2 \xrightarrow{T} E_1$ with the sheaf $\mathcal{O}_{\mathcal{C}}(\mathcal{C})$. Since this represents a bundle localized on C, it describes a supersymmetric $(2d-2)$-brane configuration as required.

Again, it is easy to see that the triple is σ-stable since any sub-triple has E_2' zero.

3 BPS bound states as stable triples

In this section we will discuss and illustrate the description of BPS branes as stable triples in relation with other existing descriptions of BPS branes.

3.1 Stable sheaves

Let us now relate our description of BPS D-branes as stable triples to existing descriptions of BPS states. Locally, the bosonic D-brane degrees of freedom are a bundle E with gauge fields A_m on the brane together with scalars Φ living in the normal bundle to the brane world-volume C and the adjoint of E. If the brane is an embedding $C \subset X$, then the normal bundle is generically non-trivial, and the scalars are its sections. The conditions that the background on the brane preserves supersymmetry then lead to a set of first-order holomorphic differential equations on A_m and Φ. For instance, for a brane wrapping a holomorphic two-cycle in K3, one has the Hitchin equations

$$\begin{aligned} F_{z\bar{z}} &= [\Phi_z, \Phi_{\bar{z}}] \ , \\ \bar{D}_{\bar{z}}\Phi_z &= D_z\bar{\Phi}_{\bar{z}} \ . \end{aligned} \tag{37}$$

These conditions can be reinterpreted geometrically as a generalized stability condition on the pair (E, Φ).

A second description of a D-brane is as a sheaf $\mathcal{S}$ on X. More precisely, BPS branes correspond to "coherent semistable" sheaves on X. The coherent condition means that $\mathcal{S}$ fits into an exact sequence

$$E_2 \longrightarrow E_1 \longrightarrow \mathcal{S} \longrightarrow \mathbf{0} \ , \tag{38}$$

where E_1 and E_2 are vector bundles (or more precisely, the sheaves of sections of vector bundles). The semi-stability condition is the generalization to sheaves of the geometrical condition of stability for vector bundles. However, in contrast to the case for vector bundles, there is no differential equation on the sheaf corresponding to the condition of stability. From this point of view, the requirement that the sheaves are stable is a conjecture.

The sheaf description is related to the description in terms of fields on the embedded brane C as follows. One requires that on C the sheaf $\mathcal{S}$ reduces to the vector bundle E. Furthermore, away from C the sheaf must be zero. Mathematically this means that the support of $\mathcal{S}$ is C, and the restriction of $\mathcal{S}$ to C is E,

$$\mathrm{supp}(\mathcal{S}) = \mathcal{C} \ , \qquad \mathcal{S}|_{\mathcal{C}} = \mathcal{E} \ . \tag{39}$$

In fact, these conditions do not completely determine $\mathcal{S}$. One can show that the additional information required is precisely the twisting of the scalar fields Φ. Thus, locally, $\mathcal{S}$ is equivalent to the pair (E, Φ). In the case of C being a curve in K3 one can then see a close relation between stable sheaves $\mathcal{S}$ and solutions of the local Hitchin equations (37). However, in general, there is no explicit justification of the requirement that $\mathcal{S}$ be stable.

3.2 Stable triples

We are proposing a description of BPS brane states as stable triples. How does this relate to the sheaf and local-field descriptions? Consider the large volume limit where we can neglect stringy corrections. In this limit we can derive an important result:

In the large σ limit, any stable triple will necessarily be without a kernel, that is, the map T will be injective (one-to-one).

To see this, suppose there is a kernel, $\ker(T) \subset E_2$. By definition, there is then a non-trivial sub-triple $T' : \ker(T) \to 0$. Since E_2 is torsion-free, any sub-sheaf of E_2 must be torsion-free. This implies that if $\ker(T)$ is non-trivial, it must be supported on the whole of X. In particular, we must have $\mathrm{rank}(\ker(\mathrm{T})) > 0$. Recall that for large σ the stability condition reduced to a condition on the ranks (24). However, for the sub-triple $T' : \ker(T) \to 0$ we have

$$n_2 n_1' - n_1 n_2' = -n_1 n_2' < 0 \ , \tag{40}$$

since $n_1' = 0$, $n_2' > 0$ and so the stability condition (24) is violated. Thus any stable triple has $\ker(T) = 0$. In general, it will have a cokernel however. The fact that $\ker(T) = 0$ implies that there is always an exact sequence

$$\mathbf{0} \longrightarrow \mathbf{E_2} \xrightarrow{\mathbf{T}} \mathbf{E_1} \longrightarrow \mathrm{coker(T)} \longrightarrow \mathbf{0} \ . \tag{41}$$

Comparing with (38), we see that, in the large σ limit, the coherent sheaf $\mathcal{S}$ is simply the cokernel coker(T) of the tachyon map. In particular, it will be supported on some holomorphic subspace of X (or X itself). For example on K3, if E_1 and E_2 of the same rank, then the brane charge $c_1(E_1) - c_1(E_2) = c_1(\mathcal{S})$ must be effective, *i.e.* the BPS branes are realized as a sheaf coker(T) localized on a holomorphic curve C. In particular, $c_1(E_1) - c_1(E_2) = n[C]$, where n is the rank of coker(T) on C.

This appears to justify the conjecture that D-branes are described by stable coherent sheaves. However, it turns out that the notion of stability for an injective triple (with T injective), is different from the notion of stability of the cokernel $\mathcal{S} = \mathrm{coker(T)}$, considered either as a torsion sheaf on X or as a vector bundle on its support. The reason being that the σ-slope (21) in the σ-stability of the triple (22) involves only the ranks and the first Chern classes of E_1 and E_2. On the other hand, the μ-slope (20) in the μ-stability of the cokernel $\mathcal{S}$ involves its first Chern class. The latter is related to the second Chern classes of E_1 and E_2, which do not enter the σ-slope stability.

One might expect that the vortex equations receive corrections involving higher-order Chern classes, which could correct this discrepancy. However, in fact, there is a basic difference between the stability of the triple and other notions of D-branes stability. In general, one considers the charges of a brane-antibrane system as elements in K-theory, and searches for geometric objects that correspond to these charges. In this paper, we suggest a representation of the brane-antibrane charges by holomorphic triples, which satisfy the vortex equations. For an injective triple, the K-theory charges are then related to the Chern classes of coker(T), the difference of $\mathrm{ch}(\mathrm{E}_1)$ and $\mathrm{ch}(\mathrm{E}_2)$. However, this proposal implies that the stability condition does not depend only on the K-theory classes of a complex and its sub-complexes, but rather on the individual terms. The σ-slope is expressed in terms of the sums of the ranks and degrees of the individual terms in the complex, rather than their alternating differences.

We can also consider more general webs of vector bundles. Basically we are working with the representation of some quiver in the category of vector bundles on X. We fix the 'shape' of the web of bundles and maps and then look for stability of all configurations of bundles and maps for that shape. Thus, for instance, for vector bundles we take the quiver of type A_1, for stable triples we take the quiver A_2 and for a sequence of n vector bundles we will take the quiver A_n. For all such objects one can define stability, but there are many notions of slope and stability (depending on discrete and continuous parameters). All of these stability notions specialize to the ordinary slope stability of vector bundles when working with A_1. However already for A_2 there are many different stabilities. For example there is one continuous family of stabilities (depending on σ or τ) that we use.

Finally, note that since there are corrections to the effective action (7), we expect the vortex equations to be deformed. This deformation is likely to influence the stability notion when the corrections are not negligible, and in particular in the finite σ regime.

4 Superconnections and branes-antibranes

In this section we will introduce the notion of superconnections.[6] We will make the assumption that the effective action of tachyon and gauge fields for branes-antibranes system can be written in a Quillen-like framework in terms of the supercurvature, and propose the form of the effective action.

4.1 Superconnections

The superconnections which will be relevant for us appear in the work of Quillen[6] on the Chern character of a K-class. Let us briefly review some of its elements.

One considers a pair of complex vector bundles E_1, E_2 over a manifold M and a homomorphism $T : E_2 \to E_1$. In the branes-antibranes system the vector bundles E_1 and E_2 correspond to the branes and antibranes respectively, and the map T corresponds to the tachyon arising from the open string stretched between them. One can regard $E = E_1 \oplus E_2$ as a superbundle, that is a bundle that carries a Z_2-graded structure. The fiber $V = V_1 \oplus V_2$ is a vector space with Z_2 grading. Denote the involution that gives the grading by ε: $\varepsilon(v) = (-1)^{deg(v)}v$. The algebra of endomorphisms of V, $End(V)$, is a superalgebra with even and odd elements. The even endomorphisms commute with ε, while the odd ones anticommute with it. The supertrace is defined by

$$\mathrm{Tr}_s(X) \equiv Tr(\varepsilon X), \qquad X \in End(V)\ . \tag{42}$$

It vanishes for odd endomorphisms, and gives the difference of the traces on V_1 and V_2 for the even ones.

When considering differential forms on M there is a natural Z-grading corresponding to the degree of the forms. Thus, differential forms on M with values in E have a $Z \times Z_2$ grading. What will be relevant is the total Z_2 grading.

Let D be an odd degree connection on E preserving the Z_2 grading

$$D = \begin{pmatrix} d + A^1 & 0 \\ & \\ 0 & d + A^2 \end{pmatrix}\ . \tag{43}$$

Denote by $\mathcal{T}$ the odd degree endomorphism of E

$$\mathcal{T} = \begin{pmatrix} 0 & iT \\ & \\ i\bar{T} & 0 \end{pmatrix}\ . \tag{44}$$

The superconnection $\mathcal{A} = D + \mathcal{T}$ on E is an operator of odd degree acting on differential form on M with values in E

$$\mathcal{A} = \begin{pmatrix} d + A^1 & iT \\ & \\ i\bar{T} & d + A^2 \end{pmatrix}\ . \tag{45}$$

When considering the branes-antibranes system the superconnection (45) appears naturally via the Chan-Paton factors, where the gauge fields of the branes A^1_μ and antibranes A^2_μ are the diagonal elements and the off-diagonal elements are the tachyon T and its conjugate $\bar{T}$. Note, that while the diagonal elements are 1-forms the off-diagonal elements are 0-forms. However, the total grading of all the matrix elements is one.

The supercurvature $\mathcal{F} = \mathcal{A}^2$ is given by

$$\mathcal{F} = \begin{pmatrix} F^1 - T\bar{T} & i\mathcal{D}T \\ & \\ i\overline{\mathcal{D}T} & F^2 - T\bar{T} \end{pmatrix}\ , \tag{46}$$

where the covariant derivatives are defined by

$$\mathcal{D}T \equiv dx^\mu D_\mu T = dx^\mu(\partial_\mu T + A^1_\mu T - T A^2_\mu) \ ,$$
$$\overline{\mathcal{D}T} \equiv dx^\mu \overline{D_\mu T} = dx^\mu(\partial_\mu \bar{T} + A^2_\mu \bar{T} - \bar{T} A^1_\mu) \ . \tag{47}$$

$F^i,\ i=1,2$ are the gauge fields strength associated with the gauge potentials $A^i,\ i=1,2$. Note, that we used the fact that in this framework T and $\bar{T}$ anti commute with dx^μ. The Chern character $ch(E_1) - ch(E_2)$ is represented by $Tr_s\ e^{\mathcal{F}}$.[6]

4.2 The effective action

One can rewrite the supercurvature (47) using the Clifford algebra. We replace $dx^{\mu_1}...dx^{\mu_n} \rightarrow (1/n!)\,\gamma^{\mu_1}...\gamma^{\mu_n}$, where γ^μ satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

The supercurvature reads now

$$\mathcal{F} = \begin{pmatrix} \frac{1}{2}\gamma^{\mu\nu}F^1_{\mu\nu} - (T\bar{T} - m\bar{m}) & i\gamma^\mu D_\mu T \\ i\gamma^\mu \overline{D_\mu T} & \frac{1}{2}\gamma^{\mu\nu}F^2_{\mu\nu} - (T\bar{T} - m\bar{m}) \end{pmatrix} \ , \tag{48}$$

where $\gamma^{\mu\nu} = [\gamma^\mu, \gamma^\nu]/2$, namely $dx^\mu \wedge dx^\nu \rightarrow \gamma^{\mu\nu}$. Note that in (48) we used the freedom to add a constant part, represented by $m\bar{m}$.

We denote by tr the trace taken over the Clifford algebra elements, e.g. $\mathrm{tr}(\gamma^\mu\gamma^\nu) = 2^{[(p+2)/2]}g^{\mu\nu}$ in a $(p+1)$-dimensional space. We denote by Tr the one taken over the matrix structure of $\mathcal{F}$, and Tr_s is as in (42).

Since the superconnection appears naturally in the description of the branes-antibranes system it is natural to ask whether we can write the effective action in terms of the supercurvature. The first hint is the $Dp - D\bar{p}$ effective action up to second order

$$S_2 = T_p \int d^{p+1}x \left(\frac{1}{4}\, F^{1\mu\nu}F^1_{\mu\nu} + \frac{1}{4}F^{2\mu\nu}F^2_{\mu\nu} - D^\mu T \overline{D_\mu T} - (T\bar{T} - m\bar{m})^2 \right), \tag{49}$$

where by T_p we denote the tension of a BPS Dp-brane. This action can be written as

$$S_2 = -\frac{T_p}{2^{[(p+2)/2]}} \int d^{p+1}x\ \mathrm{Tr}(\mathrm{tr}\ \mathcal{F}^2) \ . \tag{50}$$

One may suspect then that the higher order terms in the effective action, in the slowly varying fields approximation, where we neglect terms like $\partial^k F$ and $\partial^l T,\ l > 1$, could be of the form $\mathcal{F}^n,\ n > 2$. We will work in the slowly varying fields approximation in the following. We will see that this approximation is sufficient for the analysis of some exact properties of the tachyon condensation.

The second hint comes from the form of the Wess-Zumino (WZ) term of the branes-antibranes system. In the language of differential forms it reads

$$S_{WZ} = \mu_p e^{-m\bar{m}} \int \mathcal{C} \wedge \mathrm{Tr_s}(\, e^{\mathcal{F}}) \ , \tag{51}$$

where

$$\mathcal{C} = \sum \frac{1}{n!} \gamma^{\mu_1, \cdots \mu_n} C_{\mu_1, \cdots \mu_n} \ , \tag{52}$$

and $C_{\mu_1, \cdots \mu_n}$ is an n-form corresponding to the RR n-form field.

This WZ action is expected in view of the discussion in Sec. 4.1 and the fact that D-branes charge is measured by the K-theory class.

The supercurvature $\mathcal{F}$ can be decomposed as

$$\begin{aligned} \mathcal{F} &= \begin{pmatrix} \frac{1}{2}\gamma^{\mu\nu} F^1_{\mu\nu} & i\gamma^\mu D_\mu T \\ i\gamma^\mu \overline{D_\mu T} & \frac{1}{2}\gamma^{\mu\nu} F^2_{\mu\nu} \end{pmatrix} - (T\bar{T} - m\bar{m}) \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \\ &= \bar{\mathcal{F}} - (T\bar{T} - m\bar{m})\mathbb{1} \ . \end{aligned} \tag{53}$$

Using this form of the curvature the WZ action (51) can be written as

$$S_{WZ} = \mu_p \int d^{p+1}x e^{-T\bar{T}} \mathcal{C} \wedge \mathrm{Tr_s} \left(\sum_{n \le p+1} \frac{\bar{\mathcal{F}}^n}{n!} \right) \ . \tag{54}$$

The WZ action (54) suggests that the tachyon potential is

$$V(T, \bar{T}) \sim e^{-T\bar{T}} \ . \tag{55}$$

This is in accord with the effective field theory, string field theory, and σ-model computations.

We now turn to the non-topological part of the branes-antibranes action, which we will denote by DBI. We expect to get the same tachyon potential (55) in the DBI part. We now make the assumption that we can write it via the supercurvature. Since the superconnection and supercurvature appear as part of the structure of the system via the Chan-Paton factors one may expect this to be the case. However, it is also possible that only the topological part of the branes-antibranes action can be written using the supercurvature. This is related to the question whether the superbundle structure is indeed a structure of the brane-antibrane system or only of its topological part. We will continue with the assumption, bearing in mind that we do not have a proof for it.

The requirement of being able to write the DBI part using the supercurvature, together with the requirement of getting the same tachyon potential (55) in the DBI part, uniquely fixes the DBI action to

$$S_{DBI} = -\tau_0 \int d^{p+1}x \ \mathrm{Tr}\left(\mathrm{tr}\ e^{\mathcal{F}}\right) \ . \tag{56}$$

τ_0 is a normalization constant given by $T_p/2^{[(p+1)/2]} = \tau_0 e^{m\bar{m}}$.

The order $\mathcal{F}^2$ of (56) is precisely (49). Using the form of the curvature (53) we have

$$S_{DBI} = -\frac{T_p}{2^{[(p+2)/2]}} \int d^{(p+1)}x \ e^{-T\bar{T}} \ \mathrm{Tr}\left(\mathrm{tr}\ e^{\bar{\mathcal{F}}}\right) \ . \tag{57}$$

Thus, the proposed effective action of the branes-antibranes system, written in terms of the supercurvature (53,) is $S = S_{DBI} + S_{WZ}$, with S_{DBI} given by (57) and S_{WZ} by (54).

4.3 Tachyon condensation

Consider tachyon condensation on a Dp-$\bar{D}p$ system carrying a D(p-2)-brane charge. The tachyon should form a vortex-like configuration, with the topological charge of the vortex encoding the $D(p-2)$ brane charge.

We take the tachyon configuration $T = \alpha z, \ \bar{T} = \bar{\alpha}\bar{z}$, where $z = x^1 + ix^2$. Inserting into the WZ action (54) we get the coupling of RR p-form to the BPS-brane. It reads

$$\begin{aligned} S^{(2)}_{WZ} &= \mu_p \int d^{p+1}x \ \frac{1}{2\ p!} \ \epsilon^{\mu_0,\ldots\mu_{p-1}\alpha\beta} \\ &\times \ C_{\mu_0\ldots\mu_{p-1}} \left((F^1 - F^2)_{\alpha\beta} + 2D_\alpha T \overline{D_\beta T}\right) e^{-T\bar{T}} \\ &= \mu_p \ (2\pi)(1+\Delta F) \int d^{p-1}x \frac{1}{p!} \epsilon^{\mu_0\cdots\mu_{p-1}} C_{\mu_0\cdots\mu_{p-1}} \ , \end{aligned} \tag{58}$$

where $\Delta F = F^1 - F^2$. Reinstalling $2\pi\alpha'$ one thus finds $\mu_{cond} = 2\pi \ \mu_{p-2}(1+\Delta F)$. Assume that only $F^i_{12}, \ i = 1, 2$ is different from zero. In order to find the exact charge the vortex-like solution should have $F^1_{12} - F^2_{12} = 0$.

In this setup the supercurvature (53) reads

$$\bar{\mathcal{F}} = \begin{pmatrix} \gamma^1\gamma^2 F_{12} & i(\gamma^1 + i\gamma^2)\partial_z T \\ i(\gamma^1 + i\gamma^2)\partial_{\bar{z}}\bar{T} & \gamma^1\gamma^2 F_{12} \end{pmatrix}. \tag{59}$$

Evaluating from this the action (57) we get

$$S = -2T_p \int d^{p+1}x \, e^{-|T|^2} \cosh \sqrt{2\partial_z T \partial_{\bar{z}} \bar{T} - F^2} \ . \tag{60}$$

The field equations read

$$\partial_z \left[\partial_{\bar{z}} \bar{T} e^{-|T|^2} \frac{\sinh(2|\partial T|^2 - F^2)^{1/2}}{(2|\partial T|^2 - F^2)^{1/2}} \right] - \bar{T} e^{-|T|^2} \cosh(\sqrt{2|\partial T|^2 - F^2}) = 0,$$

$$\partial_z \left[e^{-|T|^2} F_{z\bar{z}} \frac{\sinh(2|\partial T|^2 - F^2)^{1/2}}{(2|\partial T|^2 - F^2)^{1/2}} \right] = 0, \tag{61}$$

where we denote $F = F_{12} = -(i/2)\, F_{z\bar{z}}$.

To calculate the vortex tension consider the following kink profile

$$T = \alpha z, \qquad \beta = \sinh(\sqrt{2|\alpha|^2 - F^2}) \ . \tag{62}$$

For this profile the equations of motion read

$$\alpha z e^{-|\alpha|^2 |z|^2} \left[\frac{|\alpha|^2 \beta}{\text{arcsinh}\beta} - \sqrt{1+\beta^2} \right] = 0$$

$$|\alpha|^2 z e^{-|\alpha|^2 |z|^2} \ \frac{|\alpha|^2 \beta \sqrt{|\alpha|^2 - \text{arcsinh}^2 \beta}}{\text{arcsinh}\beta} = 0 \ . \tag{63}$$

This profile can solve the equations of motion for $\alpha = 0$ and $F = 0$. This solution corresponds to the top of the potential where we have the Dp-$\bar{D}p$ system. Condensation of the Dp-$\bar{D}p$ system to a $D(p-2)$ brane, corresponds to non zero fields with the tachyon mostly sitting at the minimum of the potential. This happens for $|\alpha| \to \infty$. F has to be sent to infinity such that the $D(p-2)$ tension saturates the BPS bound.

The correct scaling for the field strength F can be found from calculating the tension of the vortex. Plugging the $\alpha \to \infty$ solution into the action, we get

$$S|_{\text{vortex}} = -2T_p \int d^{p+1}x \, e^{-|\alpha|^2 |z|^2} \sqrt{1+\beta^2} = -2\pi T_p \ \frac{\sqrt{1+\beta^2}}{|\alpha|^2} \int d^{p-1}x \ . \tag{64}$$

Scaling F such that $|\beta| \to |\alpha|^2$ the tension of the vortex is $T_{p-2,cond} = 2\pi \ T_{p-2}$. After reinstalling $2\pi\alpha'$ one finds

$$T_{p-2} = (2\pi)^2 \alpha' \ T_{p-2} \ , \tag{65}$$

which is the correct value of the D(p-2)-brane tension.

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