In memory of M.S. Marinov

CLASSICAL INTEGRABLE SYSTEMS: SELECTED TOPICS†

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In this paper two topics are considered:

- Poisson structures and Hamiltonian systems.
- Explicit integration of equations of motion for the Calogero–Sutherland and Toda systems.

It must be said that a part of the material is well known to specialists in this field for a long time (see for example the book[57]). Even so, it may not always be easily accessible to nonspecialists. Within the limited space it was not possible to give a completely systematic exposition of these topics. Rather I tried to illustrate the main ideas on simple examples and in an as elementary and explicit way as possible. The examples considered here are therefore restricted to the systems related to the algebras $A_n \sim su_{n+1}$. For the case of arbitrary simple Lie algebra, we have only mentioned the results at best. Due to insufficient space many important results are omitted here. Nevertheless I hope that this paper may be useful and may serve as a guide, so that the interested reader may proceed to read original articles.

Contents

† This paper based partially on Lausanne lectures.[59]

1 Poisson Structures and Hamiltonian Systems

1.1 Basic definitions

In order to motivate the basic concepts of Hamiltonian systems, let us begin with the simplest example.

Suppose a particle with mass m moves in a field $U(q)$, where $q = (q^1, \ldots, q^n)$ is a vector of n-dimensional space. The motion of the particle is described by Newton's equations

$$m\ddot{q}^j = -\frac{\partial U}{\partial q^j}, \tag{1}$$

where "dot" means the differentiation in time.

Let us introduce as usual the momentum vector

$$p = (p_1, \ldots, p_n), \quad p_j = m\dot{q}^j, \tag{2}$$

and the Hamilton's function or Hamiltonian

$$H = \frac{1}{2m}\, p^2 + U(q), \quad p^2 = p_j p_j \equiv \sum_{j=1}^{n} p_j p_j. \tag{3}$$

The Newton equations may then be rewritten in the form of the Hamilton equations

$$\dot{q}^j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q^j}, \tag{4}$$

and, in general, the dependence of H on p and q may be arbitrary. Equations (4) may be rewritten as one equation. To do this, let us unify the vectors p and q into one $2n$-dimensional vector $x = (p, q)$, the quantities $\left(\frac{\partial H}{\partial p_j}, \frac{\partial H}{\partial q^k}\right)$ into the $2n$-dimensional vector ∇H, and let us introduce the matrix J of order $2n$:

$$\mathrm{J} = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix},$$

where I is the unit matrix of order n. Then the Hamilton equations (4) may be written as [a]

$$\dot{x} = J \cdot \nabla\, H(x) \quad (\text{or } J \cdot \dot{x} = -\nabla H), \tag{5}$$

i.e. as the equation in the phase space $M = \{x\}$, $x = (x^1, \ldots, x^{2n})$, which in our case is the $2n$-dimensional Euclidean space.

[a] Such a form of the Hamilton equations was used first, probably, by Lagrange[32,33] for the description of the variation of elements of planets perturbed by the action of other planets.

Using the matrix J we may define the Poisson bracket [b] in the space $\mathcal{F}(M)$ of the smooth functions on M, for which we use the name *standard* or *canonical Poisson bracket*:

$$\{f(x),\, g(x)\} = (\nabla f,\ J\,\nabla g) = J^{jk}\,\partial_j f\,\partial_k g \equiv \sum_{j=1}^{n}\left(\frac{\partial f}{\partial p_j}\,\frac{\partial g}{\partial q^j} - \frac{\partial f}{\partial q^j}\,\frac{\partial g}{\partial p_j}\right). \quad (6)$$

It is easy to check that the standard Poisson bracket satisfies the following three conditions:

(i) antisymmetry

$$\{f,g\} = -\{g,f\}; \quad (7)$$

(ii) the Leibniz rule

$$\{f,gh\} = g\,\{f,h\} + h\,\{f,g\}; \quad (8)$$

(iii) the Jacobi identity

$$\{\,f\,\{g,h\}\,\} + \{\,g\,\{h,f\}\,\} + \{h\,\{f,g\}\,\} = 0. \quad (9)$$

Using this Poisson bracket we may rewrite the Hamilton equations in the form:

$$\dot{x} = \{H,x\}. \quad (10)$$

It follows from the Jacobi identity (9) that if the functions $H_1(x)$ and $H_2(x)$ are both integrals of motion, i.e. $\dot{H}_1(x) = 0$ and $\dot{H}_2(x) = 0$ due to the equation of motion (10), then the quantity

$$H_3(x) = \{\,H_1(x),\, H_2(x)\,\} \quad (11)$$

will be also an integral of motion.

Now we are ready to give the general (or abstract) definition of the Hamiltonian system.

Let M be a manifold and $\mathcal{F}\,(M)$ be the space of smooth functions on M.

Definition 1. *The Poisson bracket on M is a bilinear operation on $\mathcal{F}\,(M)$ satisfying the conditions (7), (8) and (9).*

The conditions (7) and (9) are none other than the axioms of the Lie algebra. So the space $\mathcal{F}\,(M)$ equipped with the Poisson bracket $\{\,\cdot,\cdot\,\}$ becomes the Lie algebra (in the general case, an infinite-dimensional one).

[b] Such brackets, for which we use the name "standard", were introduced by Poisson.[60]

As for the condition (8), it follows from the fact that the Poisson bracket is defined by the Poisson bracket for the local coordinates

$$\{ x^j, x^k \} = \omega^{jk}(x), \tag{12}$$

i.e. by the antisymmetric tensor $\omega^{jk}(x)$.

In fact, we have

$$\{ f(x),\, g(x) \} = \omega^{jk}(x)\, \partial_j\, f(x)\, \partial_k\, g(x). \tag{13}$$

Let us note that for this Poisson bracket the Leibniz rule (8) is satisfied automatically while the two conditions (7) and (9) are equivalent to the conditions

$$\omega^{jk}(x) = -\omega^{kj}(x), \tag{14}$$

and

$$\omega^{jk}\, \partial_k\, \omega^{lm} + \omega^{lk}\, \partial_k\, \omega^{mj} + \omega^{mk}\, \partial_k\, \omega^{jl} = 0. \tag{15}$$

We say that we have the Poisson structure on the manifold M, if the Poisson bracket on $\mathcal{F}(M)$ satisfies the conditions (7), (8) and (9). Such Poisson structure may be defined by the bi-vector

$$\Lambda = \omega^{jk}\, \partial_j \wedge \partial_k, \tag{16}$$

where $\partial_j \wedge \partial_k$ means the wedge product of the vector fields ∂_j and ∂_k.

Now we can define the notation of the Hamiltonian system.

Definition 2. *The dynamical system*

$$\dot{x} = X \tag{17}$$

is called a Hamiltonian system, if the vector field X is a Hamiltonian vector field, i.e. if

$$X = X_H = \omega^{jk}(x)\, (\partial_j H)\, \partial_k, \tag{18}$$

or in another way,

$$\dot{x} = \{ H, x \}. \tag{19}$$

So a general Hamiltonian system is the triple

$$M,\ \{\, ,\, \},\ H(x),$$

where M is a manifold, $\{\, ,\, \}$ is the Poisson structure on M and $H(x)$ is a real function on M.

Let us emphasize that the tensor $\omega^{jk}(x)$ appearing in (16) need not be non-degenerate. In particular, the dimension of M may be odd. The simplest case of such a situation is the three-dimensional case.

1.2 Examples of Poisson structures

Following the paper[21] (see also Ref. 57) let us discuss typical examples of the Poisson bracket.

- **Example 1.** The standard or canonical Poisson bracket on a $2n$-dimensional vector space may be written in canonical coordinates $(\mathbf{q}, \mathbf{p})$ in the form
$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right). \tag{1}$$

- **Example 2.** If a tensor $\omega^{jk}(x)$ does not depend on x, by a linear change of the variables this case can be reduced to the bracket as above, but with $(\mathbf{q}, \mathbf{p})$ being only part of the coordinates. The canonical Poisson bracket will be produced if these are all the coordinates, which is possible only in an even dimension.

- **Example 3.** The next important case is that of the linear dependence of $\omega^{jk}(x)$ on x (here let us assume for simplicity that M is a linear space):
$$\omega^{jk}(x) = C_l^{jk} x^l. \tag{2}$$
Since
$$\{x^j, x^k\} = C_l^{jk} x^l, \tag{3}$$
C_l^{jk} are structure constants of some Lie algebra. Conversely, having the Lie algebra $(\mathcal{G}, [\cdot, \cdot])$ with a basis e^l, we can regard this basis as the coordinate x^l (in fact linear) system for the dual space $\mathcal{G}^*$, and consider the unique Poisson bracket $\{\cdot, \cdot\}$ on $\mathcal{G}^*$ for which $\{x^i, x^j\} = [e^i, e^j]$. The Jacobi identity for the bracket in $\mathcal{G}$ implies that this bracket on $\mathcal{F}(\mathcal{G}^*)$ given by
$$\{f(x), g(x)\} = C_l^{jk} x^l \partial_j f \, \partial_k g \tag{4}$$
is really the Poisson bracket. This is usually called the *Kirillov-Kostant-Souriau bracket* (or simply KKS-bracket)[c] and can also be written formally as
$$\{f(x), g(x)\} = \left(x, \left[\frac{\partial f}{\partial x}, \frac{\partial g}{\partial x} \right] \right). \tag{5}$$
In this case the Poisson bracket of two polynomial functions on $\mathcal{G}^*$ is again a polynomial function, so that the space $\mathcal{P}(\mathcal{G}^*)$ of all polynomials on $\mathcal{G}^*$ is the Lie subalgebra.

[c] Note, however, that this bracket was known to S. Lie.[35]

- **Example 4.** Having associated our bracket with a tensor field, it is clear that we can perform a coordinate transformation and generate "new" brackets, if the transformation is not canonical.

  Let M be the dual space of the Lie algebra of the group $SO(3)$, $M = \{x : x = (x_1, x_2, x_3)\}$. In this case, the KKS-bracket is given by
$$\{x_j, x_k\} = \varepsilon_{jkl} x_l, \quad j, k, l = 1, 2, 3, \tag{6}$$
  where ε_{jkl} is a totally skew-symmetric tensor, $\varepsilon_{123} = 1$. The Poisson structure is degenerate, and on the orbits $\mathcal{O}_r = \{x : |x|^2 = x_1^2 + x_2^2 + x_3^2 = r^2\}$ of the co-adjoint representation it can be expressed in spherical coordinates as
$$\{f(\theta,\varphi),\, g(\theta,\varphi)\} = \frac{1}{r\sin\theta}\left(\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\varphi} - \frac{\partial f}{\partial\varphi}\frac{\partial g}{\partial\theta}\right). \tag{7}$$
  Here $x_1 = r\sin\theta\cos\varphi,\ x_2 = r\sin\theta\sin\varphi,\ x_3 = r\cos\theta$. The dynamics in M is given by the equation
$$\dot{x} = \left[x, \frac{\partial H}{\partial x}\right], \tag{8}$$
  where $[\,,\,]$ is the vector product. We notice that for a quadratic Hamiltonian
$$H = \frac{1}{2}\sum a_j x_j^2 \tag{9}$$
  these equations turn into Euler's equations describing the motion of a rigid body about a fixed point.

- **Example 5.** Consider a Poisson structure $\Lambda = \omega^{jk}\partial_j \wedge \partial_k$ with constant coefficients. Using the coordinate transformation $\xi_i = e^{x_i}$ we get the quadratic bracket $\{\xi_j, \xi_k\} = \omega^{jk}\xi_j\xi_k$ corresponding to the Poisson structure
$$\Lambda = \omega^{jk}\xi_j\xi_k \frac{\partial}{\partial\xi_j} \wedge \frac{\partial}{\partial\xi_k}. \tag{10}$$

1.3 The three-dimensional case

In this section, following Ref. 21, we give the description of the general Poisson structure for the case of a three-dimensional manifold.

Let us consider for simplicity the case of three-dimensional Euclidean space $M = \{\mathbf{R}\}^{\mathbf{3}}$ with standard coordinates (x^1, x^2, x^3). Then the Poisson structure is defined by three Poisson brackets
$$\{x^2, x^3\} = A_1(\mathbf{x}),\ \{x^3, x^1\} = A_2(\mathbf{x}),\ \{x^1, x^2\} = A_3(\mathbf{x}) \tag{1}$$

or by the three-dimensional vector

$$\mathbf{A} = (A_1(\mathbf{x}),\ A_2(\mathbf{x}),\ A_3(\mathbf{x})). \tag{2}$$

We have here just one Jacobi identity, which has the form

$$\mathbf{A}\,\mathrm{rot}\,\mathbf{A} = 0. \tag{3}$$

It is easy to check that the vector-field $\mathbf{A}(\mathbf{x}) = u(\mathbf{x})\,\nabla\varphi(\mathbf{x})$ is the solution of this equation.

Let us show that this solution is in fact the general local solution of this equation. Let us introduce the one-form

$$\alpha = A_j(\mathbf{x})\,\mathrm{d}x^j. \tag{4}$$

Then equation (3) takes the form

$$\alpha \wedge \mathrm{d}\alpha = 0, \tag{5}$$

where $\wedge$ is the exterior product. However, it is known that in this case the one-form α admits an integrating factor. So we can write $\alpha = u\,\mathrm{d}\varphi$ for some smooth functions u, φ defined at least on an open and dense subset of the space $\mathbf{R}^3$. Thus we get the following theorem.

Theorem 1. *Every Poisson structure* $\mathbf{R}$ *on* $\mathbf{R}^3$ *corresponds to a one-form* $\alpha = A_j\,dx^j$ *which can be written as* $\alpha = u\,d\varphi$ *for smooth functions* u, φ *defined on an open and dense subset of* $\mathbf{R}^3$*. The most general Poisson bracket in* $\mathbf{R}^3$ *can be therefore written in the form*

$$\{x_i, x_j\} = \varepsilon_{ijk} u(\mathbf{x})\,\frac{\partial\varphi(\mathbf{x})}{\partial x_k}. \tag{6}$$

Remark. It is worth noting that we really have to admit not everywhere defined (singular) functions. For example, the bracket in $\mathbf{R}^3$ defined by $\{x_1, x_3\} = x_1$, $\{x_2, x_3\} = x_2$, $\{x_1, x_2\} = 0$, corresponds to the one-form $\alpha = x_2\mathrm{d}x_1 - x_1\mathrm{d}x_2$. Here the integrating factor is $(x_1^2 + x_2^2)^{-1}$, so we can write

$$\alpha = (x_1^2 + x_2^2)\,\mathrm{d}\left(\mathrm{arctg}\,\frac{x_1}{x_2}\right). \tag{7}$$

We list now some relevant examples.

I. Assume that $u = 1$.

1. The function φ is linear in x_1, x_2, x_3. Then after orthogonal transformation we may consider $\varphi = ax_3$ and we obtain

$$\{x_1, x_2\} = a, \quad \{x_3, x_1\} = \{x_3, x_2\} = 0. \tag{8}$$

We may take $x_3 = c$ and we have the standard Poisson bracket.

2. The function φ is quadratic in x_1, x_2, x_3. Then by means of the orthogonal transformation we obtain

$$\varphi = \frac{1}{2}(a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2). \tag{9}$$

We have

$$\{x_1, x_2\} = a_3 x_3, \quad \{x_2, x_3\} = a_1 x_1, \quad \{x_3, x_1\} = a_2 x_2. \tag{10}$$

Let us consider some special cases:

(i)

$$a_1 = a_2 = a_3 = 1; \tag{11}$$

we get the KKS-bracket corresponding to the Lie algebra $so(3)$.

(ii)

$$a_1 = a_2 = -a_3 = 1; \tag{12}$$

we get the KKS-bracket corresponding to the Lie algebra $so(2,1)$.

(iii)

$$a_1 = a_2 = 1; \quad a_3 = 0; \tag{13}$$

$$\{x_1, x_2\} = 0, \quad \{x_2, x_3\} = x_1, \quad \{x_3, x_1\} = x_2; \tag{14}$$

we get the KKS-bracket corresponding to the Lie algebra $e(2)$.

(iv)

$$a_1 = -a_2 = 1; \quad a_3 = 0, \tag{15}$$

we get the KKS-bracket corresponding to the Lie algebra $e(1,1)$

$$\{x_1, x_2\} = 0, \quad \{x_2, x_3\} = x_1, \quad \{x_3, x_1\} = -x_2. \tag{16}$$

(v)

$$a_1 = a_2 = 0, \tag{17}$$

we get the KKS-bracket corresponding to the Heisenberg-Weyl algebra $\mathcal{W}_1$

$$\{x_1, x_2\} = x_3, \quad \{x_2, x_3\} = \{x_3, x_1\} = 0. \tag{18}$$

2′. When $\varphi = x_1x_2 + x_2x_3 + x_1x_3$, the quadratic form associated with φ can be reduced to the canonical form $\varphi = -2y_1^2 + y_2^2 + y_3^2$ and we get again $so(2,1)$.

2″. In the examples (i)-(v) we take $u(x) = 1$ and obtain the linear Poisson brackets. There are also four linear Poisson brackets for the case $u(x) \neq \text{const}$. This gives the complete list of three-dimensional Lie algebras, obtained in the previous century by L. Bianchi.

3. The function φ is cubic. Here we have 10 monomials, i.e. in the general case, we have a 10-parameter family of brackets. We give only two examples.

(i) $\varphi = x_1x_2x_3$. We get the bracket

$$\{x_1, x_2\} = x_1x_2, \quad \{x_2, x_3\} = x_2x_3, \quad \{x_3, x_1\} = x_3x_1. \tag{19}$$

(ii) $\varphi = \frac{1}{3}(x_1^3 + x_2^3 + x_3^3)$. We get the bracket

$$\{x_1, x_2\} = x_3^2, \quad \{x_2, x_3\} = x_1^2, \quad \{x_3, x_1\} = x_2^2. \tag{20}$$

By using a polynomial function of degree k we get a polynomial bracket of degree $k-1$.

II. **1.** Put now $\alpha = u\, d\varphi$, where

$$\begin{array}{rcl} \varphi &=& \psi(x_1) + \psi(x_2) + \psi(x_3), \\ u &=& \chi(x_1)\chi(x_2)\chi(x_3), \\ \chi(x) &=& (\psi'(x))^{-1}. \end{array} \tag{21}$$

We get

$$\{x_1, x_2\} = \chi(x_1)\chi(x_2),\ \{x_2, x_3\} = \chi(x_2)\chi(x_3), \tag{22}$$

$$\{x_3, x_1\} = \chi(x_3)\chi(x_1). \tag{23}$$

For example, if $\psi(x) = (1-m)^{-1}x^{1-m}$, we find

$$u\, d\varphi = (x_2x_3)^m \mathrm{d}x_1 + (x_3x_1)^m \mathrm{d}x_2 + (x_1x_2)^m \mathrm{d}x_3. \tag{24}$$

2. Here

$$\varphi = \psi(x_1)\,\psi(x_2)\,\psi(x_3)\left(\frac{1}{\psi(x_1)} + \frac{1}{\psi(x_2)} + \frac{1}{\psi(x_3)}\right) \tag{25}$$

and u is as above.

Now

$$u\,d\varphi = \left(\frac{\psi(x_2)}{\psi'(x_2)}\,\frac{1}{\psi'(x_3)} + \frac{\psi(x_3)}{\psi'(x_3)}\,\frac{1}{\psi'(x_2)}\right)\mathrm{d}x_1 + \text{cyclic.} \tag{26}$$

For instance, with $\psi(x) = (1-m)^{-1}x^{1-m}$, we get

$$u\,d\varphi = c\,(x_2x_3^m + x_3x_2^m)\,\mathrm{d}x_1 + \text{cyclic.} \tag{27}$$

3. For

$$u\,d\varphi = f_1(x_1,x_2)\,\mathrm{d}x_1 + f_2\,(x_1,x_2)\,\mathrm{d}x_2 \tag{28}$$

the Poisson bracket is

$$\{x_1,x_2\} = 0, \quad \{x_2,x_3\} = f_1, \quad \{x_3,x_1\} = f_2. \tag{29}$$

4. $u\,d\varphi = F(x_1,x_2,x_3)\,\mathrm{d}x_3$ gives

$$\{x_1,x_2\} = F, \quad \{x_2,x_3\} = 0, \quad \{x_3,x_1\} = 0. \tag{30}$$

1.4 Examples of three-dimensional Hamiltonian systems

In this section, we give some examples of three-dimensional Hamiltonian systems related to nonstandard Poisson structures. For more details on these systems see Ref. 22 and, especially, very informative and clearly written book of Goriely.[18]

Euler top

The Poisson structure for this system is related to the Lie algebra $so(3)$

$$\{x_1,x_2\} = x_3, \quad \{x_2,x_3\} = x_1, \quad \{x_3,x_1\} = x_2, \tag{1}$$

and the Hamiltonian is quadratic in x_j:

$$H = \frac{1}{2}\,(a_1x_1^2 + a_2x_2^2 + a_3x_3^2). \tag{2}$$

Then Hamilton's equations have the form:

$$\dot{x}_1 = (a_3-a_2)\,x_2x_3, \quad \dot{x}_2 = (a_1-a_3)x_3x_1, \quad \dot{x}_3 = (a_2-a_1)\,x_1x_2 \tag{3}$$

and coincide with Euler's famous equations describing the rotation of a rigid body around a fixed point.

Lotka-Volterra system

This system was considered first by Lotka[38] and Volterra.[65] According to Ref. 20 we have

$$\begin{aligned}\dot{x} &= x\,(cy + z + \lambda), &(4)\\ \dot{y} &= y\,(x + az + \mu), &(5)\\ \dot{z} &= z\,(bx + y + \nu). &(6)\end{aligned}$$

Two cases are known with two functionally independent integrals of motion:

i.

$$abc = -1, \quad \nu = \mu b - \lambda ab \tag{7}$$

and

ii.

$$a = b = c = 1, \quad \lambda = \mu = \nu = 0. \tag{8}$$

In the first case, the conserved quantities have the form:

$$\begin{aligned}I_1 &= ab\ln x - b\ln y + \ln z, &(9)\\ I_2 &= abx + y - az + \nu\ln y - \mu\ln z &(10)\end{aligned}$$

and this system is Hamiltonian with respect to the Poisson structure

$$\{x, y\} = cxy, \quad \{y, z\} = -yz, \quad \{z, x\} = -bcxz. \tag{11}$$

The second case is the limiting case of the Halphen system:

$$\begin{aligned}\dot{x} &= x(y + z),\\ \dot{y} &= y(z + x),\\ \dot{z} &= z(x + y),\end{aligned} \tag{12}$$

and conserved quantities have the form:

$$I_1 = (xyz)^{1/2}\left(\frac{1}{x} - \frac{1}{y}\right), \quad I_2 = (xyz)^{1/2}\left(\frac{1}{y} - \frac{1}{z}\right). \tag{13}$$

This system is Hamiltonian with respect to the Poisson structure

$$\{x, y\} = xy\,(x - y), \quad \{y, z\} = yz\,(y - z), \quad \{z, x\} = zx\,(z - x) \tag{14}$$

and the Hamiltonian has the form

$$H = \ln\left\{\left[\frac{z}{xy}\,(x - y)^2\right]^{\alpha}\left[\frac{x}{yz}\,(y - z)^2\right]^{1-\alpha}\right\}, \tag{15}$$

where α is an arbitrary constant.

Halphen system

This system has the form:

$$\begin{aligned} \dot{x} &= yz - x\,(y+z), \\ \dot{y} &= zx - y\,(z+x), \\ \dot{z} &= xy - z\,(x+y). \end{aligned} \tag{16}$$

It is equivalent to the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(x+y) = xy, \quad \frac{\mathrm{d}}{\mathrm{d}t}\,(y+z) = yz, \quad \frac{\mathrm{d}}{\mathrm{d}t}\,(z+x) = zx, \tag{17}$$

which was considered first by Halphen.[24] This system has been investigated extensively recently because of its relation to the two-monopole system.[2] After the change of variables

$$x = bc, \quad y = ac, \quad z = ab \tag{18}$$

we obtain the system

$$\begin{aligned} \frac{2}{a}\,\frac{\mathrm{d}a}{\mathrm{d}t} &= a^2 - (b-c)^2, \\ \frac{2}{b}\,\frac{\mathrm{d}b}{\mathrm{d}t} &= b^2 - (c-a)^2, \\ \frac{2}{c}\,\frac{\mathrm{d}c}{\mathrm{d}t} &= c^2 - (a-b)^2. \end{aligned} \tag{19}$$

The solution of equations (16) gives all $SO(3)$-invariant anti-self-dual Einstein metrics. These equations are derived from the metric of relative monopole space, i.e. the space of motion of two monopoles with respect to their center of mass, which is a four-dimensional irreducible hyperkähler manifold.[2] In this case it may be shown[22] that the system cannot be globally completely integrable.

Lorenz model

This model was introduced in Ref. 37 and it is described by the equations

$$\begin{aligned} \dot{x} &= \sigma\,(y-x), \\ \dot{y} &= (\rho x - xz - y), \\ \dot{z} &= (-\beta z + xy), \end{aligned} \tag{20}$$

where σ and ρ are the Prandtl and Rayleigh numbers, respectively.

This system has a chaotic behavior for most values of parameters σ, ρ and β, however, it becomes Hamiltonian for two limiting cases. The first of them is

$$\rho = 0, \quad \sigma = \frac{1}{2}, \quad \beta = 1. \tag{21}$$

In this case, the quantity

$$I = \frac{y^2 + z^2}{(x^2 - z)^2} \tag{22}$$

is the integral of motion and this system is the Hamiltonian with respect to the Poisson structure

$$\{y, z\} = -2(y^2 + z^2)\, x, \quad \{z, x\} = (x^2 - z)\, y, \quad \{x, y\} = x^2 z + y^2. \tag{23}$$

Kermack-McKendrick models for epidemics

These models describe the spread of epidemics for a closed population and have the form[27,28,29]

$$\dot{S} = -rSI, \quad \dot{I} = rSI - aI, \quad \dot{R} = aI, \tag{24}$$

where S denotes the number of susceptibles, I the number of infected and R the number of removals. The constants a, r determine the infection and removal rates of infectives, respectively.

The equation of motion here admits two integrals of motion

$$\begin{aligned} I_1 &= S + I + R, \\ I_2 &= R + \frac{a}{r} \log S \end{aligned}$$

which form the Hamiltonian system relative to the Poisson structure

$$\{S, I\} = SI, \quad \{I, R\} = SI, \quad \{R, S\} = SI. \tag{25}$$

1.5 The four-dimensional case

Let us consider the simplest case of four-dimensional Euclidean space $\mathbf{R}^4$. In this case, the Poisson structure is determined by the antisymmetric tensor $\omega^{\mu\nu}$ or by the vector Λ:

$$\omega^{\mu\nu} = \{x^\mu, x^\nu\}, \ \Lambda = \omega^{\mu\nu} \partial_\mu \wedge \partial_\nu, \ \partial_\mu = \frac{\partial}{\partial x^\mu}, \ \mu, \nu = 1, \ldots, 4. \tag{1}$$

Following Ref. 21 we should consider two different cases.

I. Tensor $\omega^{\mu\nu}$ is non-degenerate (i.e. $\det(\omega^{\mu\nu}) \neq 0$) on some open dense submanifold of $\mathbf{R}^4$.

On this submanifold we may consider the inverse tensor $F_{\mu\nu}$ and the two-form

$$F = F_{\mu\nu}\, dx^\mu \wedge dx^\nu. \tag{2}$$

We may also write this form analogously as in electrodynamics

$$F = (E_j\, dx^j) \wedge dx^4 - B_{jk}\, dx^j \wedge dx^k, \qquad j, k = 1, \ldots, 3. \tag{3}$$

If we introduce $H^k = \varepsilon^{ijk} B_{jk}$ we may also rewrite F in terms of the vector $\mathbf{H}$. We have

$$F \wedge F = 2\mathbf{EH}\, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4. \tag{4}$$

Therefore our bivector field will be degenerate at the points where $\mathbf{EH} = 0$. It is easy to show that in the case under consideration, the Jacobi identity is equivalent to the equation

$$dF = 0, \tag{5}$$

or, in other words, the vectors $\mathbf{E}$ and $\mathbf{H}$ should satisfy the Maxwell equations.

Thus we may express the tensor $F_{\mu\nu}$ in terms of the vector potential A_μ:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{6}$$

which is determined up to the "gauge" transformation

$$A_\mu \to A_\mu + \partial_\mu f. \tag{7}$$

Therefore in this case, the Poisson structure is determined by three real functions of four variables.

II. The tensor $\omega^{\mu\nu}$ is completely degenerate:

$$\det(\omega^{\mu\nu}) \equiv 0. \tag{8}$$

The general Poisson structure for this case is given by the theorem of the paper.[21]

Theorem 2. *If the Poisson structure on* $\mathbf{R}^4$ *is degenerate, i.e.* $\det(\omega^{\mu\nu}) \equiv 0$*, it is given by the formula*

$$\omega^{\mu\nu} = \{x^\mu, x^\nu\} = u(x)\, \varepsilon^{\mu\nu\lambda\sigma} \partial_\lambda C_1(x)\, \partial_\sigma C_2(x). \tag{9}$$

where the functions $u(x)$, $C_1(x)$ and $C_2(x)$ are defined on an open and dense submanifold in $\mathbf{R}^4$.

Note that in general, the exact forms $\mathrm{d}C_1$ and $\mathrm{d}C_2$ can be replaced by closed one-forms.

We close these general considerations by saying that whenever F reduces to a pure "electric", or "magnetic", or "radiation" field, we will have the degenerate Poisson bracket.
Let us give some examples.

- **Example 6**. Due to Theorem 2, the form

$$F = \mathrm{d}(a_\mu x_\mu^2) \wedge \mathrm{d}(b_\nu x_\nu^2) \tag{10}$$

corresponds to the Poisson structure Λ on $\mathbf{R}^4$. Since

$$F = 2(a_\mu b_\nu - b_\mu a_\nu)\, x_\mu x_\nu \, \mathrm{d}x_\mu \wedge \mathrm{d}x_\nu, \tag{11}$$

so

$$\Lambda = \varepsilon_{\mu\nu\lambda\sigma}\,(a_\mu b_\nu - b_\mu a_\nu)\, x_\mu x_\nu \,\partial_\lambda \wedge \partial_\sigma, \tag{12}$$

and we get the the covariant form of Poisson bracket

$$\{x_\mu, x_\nu\} = \varepsilon_{\mu\nu\lambda\sigma}(a_\lambda b_\sigma - b_\lambda a_\sigma)\, x_\lambda x_\sigma. \tag{13}$$

In particular case $b_4 = 0,\ b_1 = b_2 = b_3 = 1,\ a_4 = 1$, we get the bracket considered by Sklyanin[61]

$$\{x_k, x_l\} = \varepsilon_{jkl}\, x_4 x_j, \quad \{x_j, x_4\} = \varepsilon_{jkl}\,(a_k - a_l)\, x_k x_l, \tag{14}$$

where $j, k, l = 1, 2, 3$.

- **Example 7**. Put

$$F = (E_i dx^i) \wedge dx^0 - \varepsilon_{ijk}\, H_i \, dx^4 \wedge dx^k. \tag{15}$$

Then

$$\Lambda = \partial_4 \wedge (H_i \partial_i) + \varepsilon_{ijk}\, E_k \,\partial_i \wedge \partial_j, \tag{16}$$

i.e.

$$\{x^4, x^i\} = H_i,\ \{x^i, x^j\} = \varepsilon_{ijk} E_k. \tag{17}$$

Let us look for quadratic brackets. From $F = \mathrm{d}A$ and the quadratic requirement, $A = A_i \mathrm{d}x^i$ must have A_i of degree three. Therefore

$$\mathrm{d}(x_4(x_i^2 \mathrm{d}x_i) + (x_1^3 + x_2^3 + x_3^3)\mathrm{d}(x_1 + x_2 + x_3)) = F \tag{18}$$

will provide us with an instance of a quadratic bracket in four dimensions. It is possible to give many more examples. For instance, by taking $C_1 = x_1x_4 - x_2x_3$, $C_2 = x_1^2 + ... + x_4^2$ we again get the quadratic Poisson bracket. It should be noticed that if we consider the Ampere two-form

$$G = (H_i \mathrm{d}x^i) \wedge \mathrm{d}x^4 + D_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j$$

with $G \wedge G \neq 0$, the associated bivector field will satisfy the Jacobi identity except at the world lines of the sources. For the electromagnetic field of a moving charge the Jacobi identity fails only along the world line of the charge. This situation generalizes into four dimensions the situation for magnetic monopoles in $\mathbf{R}^3$.

- **Example 8**. Let us take

$$u(x) = x_3^2 + x_4^2,\ C_1 = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2),\ C_2 = \operatorname{arctg}\frac{x_3}{x_4}.$$

The corresponding Poisson bracket in $\mathbf{R}^4$ is quadratic and has the form

$$\{x_2, x_1\} = x_3^2 + x_4^2; \qquad \{x_1, x_4\} = x_2x_4; \qquad \{x_1, x_3\} = x_2x_3; \tag{19}$$

$$\{x_2, x_4\} = -x_1x_4; \qquad \{x_2, x_3\} = -x_1x_3; \qquad \{x_3, x_4\} = 0. \tag{20}$$

This bracket is invariant with respect to the left and right action of the group of matrices

$$\begin{pmatrix} e^{\mathrm{i}t} & 0 \\ 0 & e^{-\mathrm{i}t} \end{pmatrix}$$

and is related to the so-called Lie-Poisson structure on the group $G = \mathbf{R}_+ \otimes SU(2)$ (see Ref. 21) – group of 2×2 matrices of the type

$$\begin{pmatrix} x_1 + \mathrm{i}x_2 & -x_3 + \mathrm{i}x_4 \\ x_3 + \mathrm{i}x_4 & x_1 - \mathrm{i}x_2 \end{pmatrix}$$

- **Example 9**. As above, if we take $u(x) = (x_2^2 + x_3^2)$, $C_1 = x_1x_4 - x_2x_3$ and $C_2 = \operatorname{arctg}(x_2/x_3)$, then we get the Poisson brackets:

$$\{x_2, x_1\} = x_2x_1; \quad \{x_1, x_4\} = -2x_2x_3; \quad \{x_1, x_3\} = -x_1x_3; \tag{21}$$

$$\{x_2, x_4\} = -x_2 x_4; \quad \{x_4, x_3\} = x_4 x_3; \quad \{x_3, x_2\} = 0. \tag{22}$$

These brackets are invariant with respect to the left and right action of the group of matrices

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

and are related to the Lie-Poisson structure on the group $G = GL\,(2, \mathbf{R})$.

At the end of this section we mention the paper[11] where the Poisson structure approach was used for classification of low-dimensional Lie algebras.

1.6 Generalized Poisson structures

In this section we just mention without going into any detail two generalizations of Poisson structures.

1. At 1973 Nambu[43] proposed the generalization of the standard classical Hamiltonian mechanics based on a threedimensional 'phase space' spanned by a canonical triplet of dynamical variables and on two 'Hamiltonians' h_1 and h_2. For n-dimensional space the dynamics of the system is determined by $n-1$ Hamiltonians. The typical equations of motion are of the form

$$\dot{f} = \frac{D(f, h_1, ..., h_{n-1})}{D(x_1, x_2, ..., x_n)}$$

where the r.h.s. is the Jacobian. For more details on Nambu mechanics see Refs. 63 and 1. In the last paper the important theorem was proven that the general Nambu case may be reduced to several typical cases.

2. Another generalization of standard Poisson brackets was given in papers by Azcárraga, Perelomov and Pérez Bueno.[5,6] Let us consider here the special case of this bracket when the space $\mathbb{R}^n$ is endowed by the structure n-dimensional compact simple Lie algebra $\mathcal{G}$ with the orthonormal basis e_i relative the Killing-Cartan form. So we have

$$[e_i, e_j] = c_{ijk} e_k$$

where the tensor c_{ijk} is totally antisymmetric one.

The generalized Hamilton's equations here have the form.

$$\dot{x}_i = c_{ij_1 ... j_{k-1}} \partial_{j_1} h_1 ... \partial_{j_{k-1}} h_{k-1}$$

where $c_{ij_1 ... j_{k-1}}$ is $\mathcal{G}$-invariant totally antisymmetric tensor (which may be considered as an element of k-th cohomology space $\mathcal{H}^k$).

Note that these brackets are generalized to the case of Lie superalgebras (see Ref. 7 where interesting reader may find the details).

2 Explicit Integration of Equations of Motion

2.1 Projection method

Let us recall first of all that the Hamiltonian system with n degrees of freedom will be completely integrable if there are n integrals of motion in involution

$$\{I_j, I_k\} = 0 \quad , \quad j,k = 1,\ldots,n \quad , \quad I_1 = H, \tag{1}$$

which are functionally independent.

In this case, the Bour-Liouville-Arnold theorem[10,36,4] states that the motion may be reduced to the motion on a torus (or on Euclidean space). Unfortunately, this theorem does not give the recipe for explicit integration of the equation of motion.

In the standard approach–the so-called Hamiltonian reduction (Jacobi (around 1830), Lie[35] and Marsden–Weinstein[39]) one try to use the integrals of motion to decrease the number of degrees of freedom. But the dynamical behavior on the reduced manifold becomes more complicated and except of some simple cases this method does not help to integrate the equations of motion.

The main idea of the projection method[46,47,58] is to inverse the process of the Hamiltonian reduction. Suppose we have a dynamical system on the manifold $M = \{x\}$ which generate the flow

$$g_t : M \to M, \quad x_t = g_t x_0.$$

Let us try to find another system with a larger number of degrees of freedom $N > n$ on the another manifold $\tilde{M} = \{y\}$

$$\tilde{g}_t : \tilde{M} \to \tilde{M}, \quad y_t = \tilde{g}_t y_0$$

so that the following conditions are satisfied:

(i) The dynamical behavior of the new system is sufficiently simple so that the equation of motion of this system may be integrated explicitly.

(ii) The initial system is the projection π of the larger one

$$\pi : \tilde{M} \to M$$

and this projection may be described by explicit formulae.

(iii) There exists a map $\rho : M \to \tilde{M}$ such that $\pi \cdot \rho = \mathrm{id}$ which may be described explicitly.

In this case, we may integrate the equations of motion for the Hamiltonian system on M explicitly. Indeed, we have

$$x_t = \pi \cdot y_t = \pi \cdot \tilde{g}_t \cdot y_0 = \pi \cdot \tilde{g}_t \cdot \rho \cdot x_0, \tag{2}$$

i.e.

$$\boxed{x_t = \pi \cdot \tilde{g}_t \cdot \rho \cdot x_0} \quad . \tag{3}$$

The main difficulty with this method is in finding $\tilde{M}$, π, $\tilde{g}_t$ and ρ. We do not know how to do this in general, but we have a number of interesting and nontrivial examples of application of this method. We consider some of them below.

2.2 Systems of Calogero–Sutherland type

These are one-dimensional systems (systems of n particles on a line) interacting pairwise via potentials $V(q) = g^2\, v(q)$ of the following 5 types [d]

$$\begin{aligned}
&\text{I.} && v(q) = q^{-2}, \\
&\text{II.} && v(q) = a^2 \sinh^{-2} aq, \\
&\text{III.} && v(q) = a^2 \sin^{-2} aq, \\
&\text{IV.} && v(q) = a^2\, \mathcal{P}(aq), \\
&\text{V.} && v(q) = q^{-2} + \omega^2 q^2.
\end{aligned} \tag{1}$$

Here $\mathcal{P}(q)$ is the Weierstrass function, i.e. the meromorphic double-periodic function with periods $2\omega_1$ and $2\omega_2$ in a complex plane and asymptotic behavior near the pole at point q_0

$$\mathcal{P}(q) \sim (q - q_0)^{-2} + c_2\,(q - q_0)^2 + c_3\,(q - q_0)^4 + \dots . \tag{2}$$

The investigation of systems of type I was initiated by F. Calogero[12] and of type III by B. Sutherland[62] (it was continued in Ref. 14). It was later proved by J. Moser[41] that the systems of type I are completely integrable in the Bour-Liouville-Arnold sense.

[d] For more details on these systems see Ref. 49 and books.[57,16]

To prove complete integrability it is convenient to use the Lax representation. We shall first exhibit the Lax pairs for the Calogero-Sutherland systems of types I-IV, which will allow to analyze the integrability of these systems. Next we shall consider systems of type V.

The Lax pairs for systems of types I-IV were found by Moser[41] and by Calogero.[14] They are given by the matrices:

$$L_{jk} = p_j \delta_{jk} + \mathrm{i} g \, (1 - \delta_{jk}) \, x \, (q_j - q_k), \tag{3}$$

$$M_{jk} = g \Big[\delta_{jk} \sum_{l \neq j} z \, (q_j - q_l) - (1 - \delta_{jk}) \, y \, (q_j - q_k) \Big]. \tag{4}$$

Substituting these matrices into the Lax equation

$$\dot{L} = [L, M]$$

and requiring this equation to be equivalent to the original Hamilton equations we find both

$$y(\xi) = x'(\xi) \tag{5}$$

and a functional relation for $x(\xi)$ and $z(\xi)$:

$$x'(\xi) \, x(\eta) - x'(\eta) \, x(\xi) = x(\xi + \eta) \, [z(\xi) - z(\eta)]. \tag{6}$$

The function $v(\xi)$ which occurs in the potential energy is given by the formula:

$$v(\xi) = x(\xi) \, x(-\xi) + \text{const.} \tag{7}$$

The functional equation (6) was solved in several papers.[15,45,52] It turns out that

$$z(\xi) = x''(\xi)/2x(\xi), \tag{8}$$

and all odd solutions $x(\xi)$ have the form:

$$x(\xi) = \begin{cases} \xi^{-1} & & \mathbf{I} \\ a \coth a\xi, & a \sinh^{-1} a\xi & \mathbf{II} \\ a \cot a\xi, & a \sin^{-1} a\xi, & \mathbf{III} \\ a \, \mathrm{cn} \, a\xi / \mathrm{sn} \, a\xi, & a \, \mathrm{dn} \, a\xi / \mathrm{sn} \, a\xi, \;\; a/\mathrm{sn} \, a\xi & \mathbf{IV} \end{cases} \tag{9}$$

Here $\mathrm{cn}\,(\xi)$, $\mathrm{sn}\,(\xi)$ and $\mathrm{dn}\,(\xi)$ are the Jacobi elliptic functions.

It is easy to verify, by using (7), that the potentials of types I-IV in (1) result from these expressions. Substituting ia for a we take the function $x(\xi)$ of type II to the function $x(\xi)$ of type III. In the limit as $a \to 0$, the function

$x(\xi)$ of type II or III goes over into $x(\xi)$ of type I. The same relationship holds for the matrices L.

The Lax representations for more general systems of types I-IV, the so-called systems related to other classical root systems (for details, see Ref. 57) were found by Olshanetsky and Perelomov.[45]

In the general case, both matrices L and M have dimension $2n+1$ and can be written in the form:

$$L = P + X, \quad M = -D + Y, \tag{10}$$

where

$$\begin{aligned}
P &= \operatorname{diag}(p_1, \ldots, p_n, 0 - p_n, \ldots, -p_1); \\
X &= \begin{vmatrix} A_1 & C_1^T & B_1 \\ \overline{C}_1 & 0 & -\overline{C}_1 \\ -B_1 & -C_1^T & -A_1 \end{vmatrix}, \\
Y &= \begin{vmatrix} A_2 & C_2^T & B_2 \\ -\overline{C}_2 & 0 & -\overline{C}_2 \\ B_2 & C_2^T & A_2 \end{vmatrix}, \\
D &= \operatorname{diag}(d_1, d_2, \ldots, d_n, d_0, d_n, \ldots, d_2, d_1),
\end{aligned} \tag{11}$$

and where A_1, B_1, A_2, B_2 are $n \times n$ matrices:

$$\begin{aligned}
\{A_1\}_{kl} &= \mathrm{i}(1-\delta_{kl})\, gx(q_k - q_l, \\
\{B_1\}_{kl} &= \mathrm{i}g_2 \sqrt{2}\, \delta_{kl}\, x(2q_k) + \mathrm{i}g\,(1-\delta_{kl})\, x(q_k+q_l), \\
\{A_2\}_{kl} &= \mathrm{i}(1-\delta_{kl})\, gy\,(q_k - q_l), \\
\{B_2\}_{kl} &= \mathrm{i}g_2 \sqrt{2}\, \delta_{kl}\, y(2q_k) + \mathrm{i}g\,(1-\delta_{kl})\, y(q_k+q_l), \\
\{C_1\}_k &= \mathrm{i}g_1\, x(q_k), \\
\{C_2\}_k &= \mathrm{i}g_1\, y(q_k) \\
d_k &= \mathrm{i}(e_k - c), \\
e_k &= \sum_{r=1}^{n} g[z(q_k - q_r) + z(q_k + q_r)] + g_1^2 g^{-1}\, z(q_k) + \sqrt{2}\, g_2 z(2q_k), \\
e_0 &= 2g \sum_{k=1}^{n} z(q_k), \quad c = \frac{2}{2n+1}\Big[\sum_{k=1}^{n} e_k + \frac{1}{2}\, e_0\Big].
\end{aligned} \tag{12}$$

These Lax pairs will be valid only if the interaction constants obey the following relations:

$$g_1^2 + \sqrt{2} g_1 g_2 - 2g_2^2 = 0 \quad \text{if} \quad g_1 = 0,$$

$$g_1, g_2 \quad \text{arbitrary if} \quad g_1 = 0.$$

As before, the functions $x(\xi), y(\xi)$ and $z(\xi)$ satisfy (5) and (6) so that the interaction potential has the form (1).

To show that these systems are completely integrable we consider the invariants of L

$$I_k = \frac{1}{k} \operatorname{tr} L_k, \tag{13}$$

which are obviously the integrals of motion. The matrix structure of L implies that these integrals have the form:

$$I_k = \frac{1}{k} \sum p_j^k + \text{ terms of lower degree in the momenta.} \tag{14}$$

Hence the functional independence of the I_k follows from the functional independence of the quantities

$$s_k = \sum_j p_j^k. \tag{15}$$

Let us note that the polynomials s_k are invariants of the corresponding simple (or affine) Lie algebra. The degrees of the basic invariants generating the whole algebra of invariants are well known. For the classical root systems, they are:

$$\begin{aligned} A_{n-1} &\quad : \quad k = 2, 3, \ldots, n, \\ B_n, C_n, BC_n &\quad : \quad k = 2, 4, \ldots, 2n, \\ D_n &\quad : \quad k = 2, 4, \ldots, 2(n-1), n. \end{aligned}$$

The involutivity of the integrals for systems of types I-III and for the non-periodic Toda lattices can be proved by using the argument of Calogero.[14] For this purpose, we consider first the system of type II. Since the potential $U(q)$ is repulsive we can show that $q_a(t) \to \infty$ as $t \to \infty$. (For systems of type A_{n-1}, this means that the particles will move infinitely far away from each other.) In this case, as is clear from the structure of L, the integral I_k (14) tends to ks_k (15), and hence $\{I_k, I_j\} \to kj\{s_k, s_j\} = 0$. Since the Poisson bracket of any two integrals is also an integral, this implies $\{I_k, I_j\} = 0$. Thus both the systems of type II A_{n-1} and II BC_n are completely integrable.

The systems of types I and III are obtained from the systems of type II by taking the limit as $a \to 0$ or by substituting $a \mapsto \mathrm{i}a$, respectively, which implies their integrability.

For the non-periodic Toda lattices, one can show that $q_n(t) \to -\infty$ as $t \to \infty$. In particular, for the ordinary lattice this means that $q_{j+1}(t) - q_j(t) \to \infty$. As before, it follows that $I_k(t) \to s_k$, and the previous reasoning remains valid.

It is more difficult to prove the involutivity of the integrals of motion for systems of type IV. In this case, it is convenient to take for the basic integrals the eigenvalues of L.[55]

For the systems of types I-IV(1) and the ordinary Toda lattice, it is convenient, in addition to the integrals I_k (13), to consider another set of integrals J_k which are the coefficients of the characteristic polynomial of L:

$$\det (L - \lambda E) = \lambda^n + \sum_{k=1}^{n} J_k \, \lambda^{n-k}. \tag{16}$$

The two integrals I_k and J_k are related by Newton's formula:

$$I_k = I_{k-1} J_1 - I_{k-2} J_2 + \cdots - (-1)^{k-1} \, I_1 J_{k-1} - (-1)^k \, k J_k. \tag{17}$$

2.3 Systems with external harmonic potential

A modification of the Lax equation[54] (see Appendix) allows us also to treat systems with external harmonic potential (systems of type V). In this case, the equations of motion are equivalent to the matrix equations

$$\dot{L}^{\pm} = [L^{\pm}, M] \pm \mathrm{i}\omega L^{\pm}, \tag{1}$$

where M is the same as in the Lax pair for the system of type I, and

$$L^{\pm} = L \pm \mathrm{i}\omega Q \tag{2}$$

with

$$Q = \mathrm{diag}\, [q_1, \ldots, q_n] \tag{3}$$

for the systems of type V. To show the equivalence one uses the relation

$$[Q, M] = X, \tag{4}$$

where X is the off-diagonal part of L.

From (1) it follows that the quantities

$$B_k^{\pm} = \frac{1}{k} \, \mathrm{tr} \, (L^{\pm})^k \tag{5}$$

have a simple time dependence:

$$B_k^{\pm}(t) = B_k^{\pm}(0) \, \exp (\mp \mathrm{i} k \omega t). \tag{6}$$

Using (1) we can also find the integrals of motion. For this purpose, let us consider the matrices

$$N_1 = L^+ L^-, \quad N_2 = L^- L^+. \tag{7}$$

These satisfy the usual Lax equation

$$\dot{N}_j = [N_j, M], \quad j = 1, 2, \tag{8}$$

therefore the traces of powers of these matrices are the integrals of motion.

These are a system of n particles in line, with the Hamiltonian

$$H = \frac{1}{2} \sum_1^k p_j^2 + g^2 \sum_{j<k} \frac{1}{(q_j - q_k)^2}. \tag{9}$$

The phase space in this case is standard:

$$M = \left\{ (\mathbf{p}, \mathbf{q}) \,\middle|\, \mathbf{p} = (p_1, \ldots, p_n) \quad , \qquad \mathbf{q} = (q_1, \ldots, q_n) \right\}. \tag{10}$$

Such a system was considered[25] for $n = 3$ by Jacobi [e], who showed it to be integrable. For arbitrary n, such systems were investigated by Calogero around 1970.[12,14,13] The systems for $n = 4$ and $n = 5$ were shown to be integrable,[53] but I did not succeed at that time to prove that the system is integrable for arbitrary n. This was achieved in 1975 by Ju. Moser[41] (see also Ref. 42) who found the $L - M$ pair for this system:

$$\dot{L} = [L, M]. \tag{11}$$

The quantities

$$I_k = \frac{1}{k} \operatorname{tr} \left(L_1^k \right), \qquad I_1 = \Sigma p_j, \qquad I_2 = H, \ldots \tag{12}$$

are n functionally independent integrals of motion in involution

$$\{I_j, I_k\} = 0 \tag{13}$$

so the Calogero system is a completely integrable Hamiltonian system.

However, not all integrals of motion are quadratic in momenta and, hence, it is not clear how to integrate the equations of motion.

The equations of motion for the systems of types I, II, III, V and for the generalized non-periodic Toda lattices can be solved explicitly by means of the projection method described above. In this section these examples will be extended to include the systems with more than one degree of freedom.

[e] Note that if a system with n degrees of freedom has n functionally independent integrals of motion in involution which depend on momenta quadratically, then it may be solved by the Stäckel separation variable method (see Refs. 66 and 57).

2.4 Integration of equations of motions for rational case

We consider here the systems of types I and V of Sec. *2.2*.[45] The construction makes use of the space $X_n^0 = \{x\}$ of $n \times n$ traceless complex Hermitian matrices which can be viewed as the Euclidean space with a metric given by

$$\mathrm{d}s^2 = \mathrm{tr}\,(\mathrm{d}x)^2. \tag{1}$$

The equations of geodesics in X_n^0 can be written in the Hamiltonian form. This means that the phase space is $T^* X_n^0$, the symplectic structure is determined by the 2-form ω ($\omega = \mathrm{tr}\,(\mathrm{d}x \wedge \mathrm{d}y)$), and the Hamiltonian is $H = \frac{1}{2}\,\mathrm{tr}\,(y^2)$ where y is the momentum variable. The equation of the geodesic is

$$\ddot{x} = 0 \tag{2}$$

and its solutions are the straight lines

$$x(t) = at + b. \tag{3}$$

Clearly, both the metric (1) and the form $\omega = \mathrm{tr}\,(\mathrm{d}x \wedge \mathrm{d}y)$ are invariant under unitary transformations $x \to uxu^+$. Let us reduce the Hermitian matrix $x(t)$ to the diagonal form $Q(t)$

$$x(t) = u(t)\, Q(t)\, u^{-1}(t), \tag{4}$$

where

$$Q(t) = \mathrm{diag}\,[q_1, \ldots, q_n]. \tag{5}$$

We can assume that $q_1 > q_2 > \ldots > q_n$. The map of x to Q is called the *spherical projection.*

We shall derive the equations of motion for $Q(t)$ and $P(t) = \dot{Q}(t)$ on the condition that $x(t)$ satisfies (2). By differentiating (4) with respect to time we obtain

$$u(t)\, L(t)\, u^{-1}(t) = a \tag{6}$$

with

$$L(t) = P(t) + [Q, M], \tag{7}$$

$$M = -u^{-1}(t)\, \dot{u}(t), \tag{8}$$

where L is the Hamiltonian and M is the skew-Hermitian matrix. Differentiating (6) again we obtain

$$\dot{L} = [L, M]. \tag{9}$$

Thus the pair of matrices (7) and (8) satisfies the Lax equation.

On the other hand, one can easily verify that the Lax matrices for the systems of type I satisfy (7). This establishes the equivalence of these two systems.

However, we must note that not every geodesic is mapped by the spherical projection into a trajectory of the system of type I.[f] Indeed, let us consider an "angular momentum" μ of the geodesic (3):

$$\mu = [x, \dot{x}] = [b, a] = u[Q, L]\, u^{-1}. \tag{10}$$

It is easy to see that a commutator $[Q, L]$ has the rather special structure:

$$[Q, L] = g\,(v^T \otimes v - I), \quad v = (1, 1, \ldots, 1), \tag{11}$$

i.e. it has $n-1$ coinciding eigenvalues. (The symmetry group G which determines the momentum map μ is $SU(n)$; the subgroup G_μ which leaves invariant the momentum μ (10) consists of the matrices in $SU(n)$ commuting with $[Q, L]$.)

Now we can obtain the solution of the Cauchy problem for systems of type I. We set $u(0) = I$ and reconstruct parameters of the geodesic from the initial data:

$$a = L(0), \quad b = Q(0). \tag{12}$$

Using (3) we arrive at the important conclusion:
the coordinates $q_j(t)$ are the eigenvalues of the matrix

$$Q(0) + L(0)t. \tag{13}$$

To determine the solutions of the equations of type V we take the same space X_n^0, but instead of free motion we consider the equation of a harmonic oscillator:

$$\ddot{x} + \omega^2 x = 0. \tag{14}$$

Its solutions are

$$x(t) = a\omega^{-1} \sin \omega t + b \cos \omega t, \quad a, b \in X_n^0. \tag{15}$$

As before, taking the spherical decomposition (4) of $x(t)$ and differentiating twice with respect to t we are able to write the dynamical system (9) in the form (14). The equivalence implies the important result:

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[f] From mathematical point of view this means that we should fix the value of moment map, see for details Ref. 26.

the coordinates $q_j(t)$ of the trajectories of the system of type V are the eigenvalues of the matrix

$$Q(0)\,\cos\,\omega t + \omega^{-1}\,L(0)\,\sin\,\omega t. \tag{16}$$

These results can be used to derive many useful corollaries. Here we give just two of them.

First, from them it immediately follows that rational systems are superintegrable, i.e. these systems with n degrees of freedom have not only n integrals of motion, but $2n-1$ integrals of motion.

Second, the quantities I_{jk} defined by the formula

$$I_{jk} = \frac{1}{(j+k)!}\left(\frac{\partial}{\partial\lambda}\right)^k \mathrm{tr}\,(L+\lambda Q)^{j+k}\,|_{\lambda=0}$$

form very interesting infinite-dimensional Lie algebra relative the Poisson brackets. The investigation of this algebra start at Ref. 8.

Third, let us consider the scattering process for systems of type I . Since the potential decays as $|q_j - q_k|^{-2} \to \infty$, we have

$$q_j(t) \sim p_j^{\pm} t + q_j^{\pm}, \quad t \to \pm\infty. \tag{17}$$

If the particles are numbered in such a way that $q_1 < q_2 < \ldots < q_n$, then $p_1^- > p_2^- > \ldots p_n^-$ and $p_1^+ < p_2^+ < \ldots < p_n^+$. The explicit formulae show that the asymptotic values $p_j^{\pm}$ and $q_j^{\pm}$ differ only by a permutation:

$$p_1^- = p_n^+, \quad p_2^- = p_{n-1}^+, \ldots, p_n^- = p_1^+, \tag{18}$$

$$q_1^- = q_n^+, \quad q_2^- = q_{n-1}^+, \ldots, q_n^- = q_1^+. \tag{19}$$

A converse statement by Sinai is also true:

> If we have a one-dimensional n-body problem for which the repulsive pair potential decays as some power of the distance and the asymptotic momenta are related by (18), then the potential will have the form I.

The explicit formulae for the solutions show a simple relationship between the trajectories in the systems of both types I and V.[56] Let $q_j(t)$ be the solution for the system of type I, and $\tilde{q}_j(t)$ the solution for the system of type V. Then

$$\tilde{q}_j(t) = q_j\left(\omega^{-1}\,\tan\,(\omega t)\right)\cdot \cos\,\omega t. \tag{20}$$

Finally, we consider the symmetric polynomials of degree k in the coordinates $q_j(t)$. It follows that these will be polynomials of degree k in t if $\omega = 0$, or in both $\sin \omega t$ and $\cos \omega t$ if $\omega \neq 0$.

It is clear that all the above results are valid also for other systems of types I and V which have the Lax representation, i.e. to the general systems I BC_n and V BC_n. For this purpose, one has to consider the configuration space of $(2n+1) \times (2n+1)$ traceless Hermitian matrices satisfying

$$xf + fx^+ = 0, \tag{21}$$

where

$$f = \begin{vmatrix} 0 & 0 & J \\ 0 & 1 & 0 \\ J & 0 & 0 \end{vmatrix}$$

and J is an $n \times n$ matrix with units on the opposite diagonal. The formulae (13) and (16) involve the matrix $L(0)$ and

$$Q = \mathrm{diag}\,(q_1, \ldots, q_n, 0, -q_n, \ldots, -q_1). \tag{22}$$

2.5 Integration of equations of motion for trigonometric case

In this section we construct the solutions of the equations of motion for systems of types II and III.

Let us discuss first the systems of type II. We consider the space X_n^- of $n \times n$ positive-definite Hermitian matrices with determinant 1. This is the homogeneous space: $X_n^- = SL\,(n, \mathbf{C})\,/\,SU(n)$,

$$x \to gxg^+, \quad x \in X_n^-. \tag{1}$$

In particular, any point x can be represented as

$$x = gg^+. \tag{2}$$

The space X_n^- has a metric $\mathrm{d}s^2$ invariant under transformations of type (1)

$$\mathrm{d}s^2 = \mathrm{tr}\left((x^{-1}\,\mathrm{d}x)^2\right). \tag{3}$$

The curvature of X_n^- relative to this metric is non-positive, so we use the minus superscript in our notation. The equation of geodesics may be easily obtained:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x^{-1}(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\,x(t)\right) = 0. \tag{4}$$

Let us consider X_n^- as a submanifold of $SL(n,\mathbf{C})$. The matrices $x^{-1}(t)\,\dot{x}(t)$ and $\dot{x}(t)\,x^{-1}(t)$ are the tangent vectors to $SL(n,\mathbf{C})$ at the identity, but are not tangent to X_n^- in general. However their sum is tangent to X_n^-. Therefore instead of (4) we shall consider the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x^{-1}\dot{x}+\dot{x}x^{-1}}{2}\right)=0. \tag{5}$$

Any geodesic in X_n^- can be written as

$$x(t)=b\,\exp\{2at\}\,b^+, \tag{6}$$

where $b\in SL(n,\mathbf{C}),\ a^+=a,\ \operatorname{tr}a=0.$

From linear algebra it is known that the Hermitian matrix can be reduced to diagonal form by a unitary transformation. Then for any curve $x(t)$ in X_n^- we have

$$x(t)=u(t)\,\exp\{2\gamma Q(t)\}\,u^+(t), \tag{7}$$

where $u\in SU(n)$, γ is a parameter and

$$Q=\operatorname{diag}[q_1,\dots,q_n]. \tag{8}$$

This is the so-called spherical coordinate system on X_n^-. With respect to the parameterization (7) of $x(t)$ we have

$$\frac{1}{2}\,(x^{-1}\dot{x}+\dot{x}x^{-1})=2\gamma uLu^+, \tag{9}$$

where

$$L = P+\frac{1}{4\gamma}\,[\mathrm{e}^{-2\gamma Q}\,M\mathrm{e}^{2\gamma Q}-\mathrm{e}^{2\gamma Q}\,M\mathrm{e}^{-2\gamma Q}], \tag{10}$$

$$M = u^{-1}\dot{u}\quad\text{is the angular velocity,} \tag{11}$$

and $P=\dot{Q}$ is the relative velocity in the coordinate system (7). Differentiating (9) with respect to time we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x^{-1}\dot{x}+\dot{x}x^{-1}}{2}\right)=2\gamma u\,(\dot{L}-[L,M])\,u^+. \tag{12}$$

If $x(t)$ is geodesic, the left-hand side of this equation will vanish. It follows that the geodesic flow on X_n^- has the Lax representation with the matrices L and M given by (10) and (11).

It can be verified directly that the Lax pair for systems of type II satisfies (10), provided that the function $x(\xi)$ entering in L is $x(\xi) = \gamma \coth \gamma\xi$.

Thus if the geodesic $x(t)$ is chosen in such a way that under the spherical parameterization (7) its angular velocity (11) has standard form then the evolution of the radial part $Q(t)$ will have the Lax representation. As was shown before, this is equivalent to the original Hamiltonian system.

The special choice of geodesics has a simple mechanical interpretation. Let us consider the "angular momentum" $\mu = \frac{1}{2}(x^{-1}\dot{x} - \dot{x}x^{-1})$ of the curve $x(t)$ relative to the Hamiltonian action of the group $SU(n)$ on the phase space. It turns out that if the "angular velocity" M has the standard form, the "angular momentum" will be time-independent and will be given by

$$\mu = -2i\gamma^2 g\,(v \otimes v - I), \quad v = (1, \ldots, 1). \tag{13}$$

On the other hand, the "angular momentum" of the geodesic (6) is

$$\mu = -(bab^{-1} - (b^+)^{-1}ab^+). \tag{14}$$

By comparing these two expressions we obtain the constraints for the parameters (a, b) of the geodesics which are projected onto the trajectories of the systems of type II. Thus the geodesic flow (5) is reduced, by the spherical projection at the fixed "angular momentum" (13), to the Sutherland system.

The above construction yields immediately an explicit expression for the trajectories. Let us indicate the solution of the Cauchy problem. Setting $u(0) = I$ we find from (6) and (7)

$$b = \exp\{\gamma Q(0)\}, \tag{15}$$

$$a_{jk} = \gamma p_j(0)\,\delta_{jk} + \mathrm{i}\gamma^2 g\,\sinh^{-1}\big(\gamma(q_j(0) - q_k(0))\big)\,(1 - \delta_{jk}). \tag{16}$$

So we arrive at the following result:
the quantities $2\gamma q_j(t)$ (i.e. the solutions of the equations of motion for a system of type II) are the logarithms of the eigenvalues of the matrix

$$x(t) = b\exp\{2at\}\,b^+, \tag{17}$$

where the matrices a and b are given by (15), (16).

The transition to the solutions for systems of type III is performed formally by substituting $\mathrm{i}\gamma$ for the parameter γ in the above formulae. Geometrically, this substitution represents the transition from the space X_n^- of the positive Hermitian matrices to the space of the unitary matrices $X_n^+ = SU(n)$. The group $SU(n)$ is a Riemannian symmetric space of nonnegative curvature with respect to the invariant metric (3). The relationship between X_n^- and X_n^+ is the special case of duality established by E. Cartan for a large class of symmetric spaces which generalizes the well-known relationship between hyperbolic and spherical geometries.

Analogously, the more complicated systems of type III BC_n result from geodesic flows on the complex Grassmann manifolds $SU(2n+1)\,/\,SU(n) \otimes U(n+1)$ while the systems of type II BC_n are related to the dual noncompact symmetric spaces $SU(n+1,n)\,/\,SU(n) \otimes U(n+1)$. In this case, the solutions are also obtained by diagonalizing the matrix of form (6) where $b \in SU(n+1,n)$ and a satisfies the condition

$$af + fa^+ = 0, \tag{18}$$

$$f = \begin{pmatrix} 0 & 0 & J \\ 0 & 1 & 0 \\ J & 0 & 0 \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix},$$

and

$$Q = \mathrm{diag}\,[q_1, \dots, q_n, 0, -q_n, \dots, -q_1]. \tag{19}$$

2.6 Systems with two types of particles.

A simple change of variables proposed by F. Calogero[14] gives the generalization of the systems of type II. We shall consider the change of variables

$$q_j \to q_j + \mathrm{i}\frac{\pi}{2\gamma}, \quad 0 < n_1 < j \le n. \tag{1}$$

Then the potential

$$U(q) = g^2 \sum_{k<j} \gamma^2 \sinh^{-2} \gamma(q_j - q_k) \tag{2}$$

takes the form:

$$U(q) = g^2 \sum_{\substack{j<i\le n_1,\\ n_1<j<i\le n}} \gamma^2 \sinh^{-2} \gamma(q_i - q_j) - g^2 \sum_{\substack{i>n_1,\\ j\le n_1}} \gamma^2 \cosh^{-2} \gamma(q_i - q_j). \tag{3}$$

So we may consider this system as containing n_1 particles of the same sign and $n_2 = n - n_1$ particles of the opposite sign. To be more precise we assume that $n_1 \geq n_2$. Obviously, the substitution (1) does not affect the main results of the previous Section. In particular, the quantities $\exp 2\gamma q_j(t)$ are the eigenvalues of the matrix $x(t)$. However, this prescription does not tell us whether the system with the potential (3) has bound states or not. Using the geometric approach one can show that the bound states always exist but are unstable except in the case $n_1 = n_2 = 1$.[51]

2.7 Integration of equations of motion for non-periodic Toda lattice

Let us consider first the ordinary non-periodic Toda lattice with the potential

$$U(q) = \sum_{j=1}^{n-1} \exp 2(q_j - q_{j+1}). \tag{1}$$

Let us remind first that this system was considered first by Toda.[64] The Lax representation for standard Toda system was found by Flaschka[17] and Manakov[40] and for general case by Bogoyavlensky.[9] The explicit integration of equations of motion was achieved in Refs. 48, 30 and 50. Below we follow Ref. 48. According to the general idea of the projection method we should indicate the larger phase space $\tilde{\mathcal{M}}$ whose projection would be the phase space of the dynamical system with potential (1). In our case, this is the phase space of the geodesic flow on the space of the symmetric positive-definite matrices with determinants equal to 1

$$X_n = SL(n, \mathbf{R}) \,/\, SO(n).$$

The Riemannian metric and geodesics are given by the same formulae as in the the case of the Sutherland system (the matrices are real).

We shall now define the projection. Let Z be the subgroup of $SL(n, \mathbf{R})$ consisting of the upper triangular matrices with units on the principal diagonal, and let H be the subgroup of the diagonal matrices. Any matrix x in X_n can be written uniquely as

$$x = z(x)\, h^2(x)\, z^t(x), \quad z(x) \in Z, \quad h(x) \in H, \tag{2}$$

where z^t is the matrix transposed to the matrix z, The pair $(h(x), z(x))$ is called the *horospherical coordinates* in the space X_n. For $n = 2$, X_2 is the Lobachevsky plane and (h, z) are the standard horospherical coordinates. The diagonal matrix $h(x) = \operatorname{diag}[h_1, \ldots, h_n]$ is called the horospherical projection.

Let $\Delta_j(x)$ be the j-th lower principal minor of x. Then the horospherical projection is given by formula

$$h_1 = \frac{1}{\Delta_{n-1}}, \quad h_2 = \frac{\Delta_{n-1}}{\Delta_{n-2}}, \dots, h_{n-1} = \frac{\Delta_2}{\Delta_1}, \quad h_n = \Delta_1. \tag{3}$$

We have the two following propositions:

1. Under the condition

$$2(bab^{-1})_{j,j-k} = \begin{cases} 1, & k = 1, \\ 0, & k > 1 \end{cases} \tag{4}$$

the horospherical projection (2) maps the geodesic flow into the motion defined by the potential (1).

2. Let

$$(q^0, p^0) = (q_1^0, \dots, q_n^0, p_1^0, \dots, p_n^0)$$

be the initial data for the non-periodic Toda lattice. We define the $n \times n$ matrix a by

$$a_{jk} = p_j^0\, \delta_{jk} + e^{q_{j-1}^0 - q_j^0}\, \delta_{j,k+1} + e^{q_j^0 - q_{j+1}^0}\, \delta_{j,k-1}, \tag{5}$$

and let Δ_j be the j-th lower principal minor of $\exp\{2at\}$. Then the solution of the equations of motion is given by the formula:

$$q_j(t) = q_j^0 + \log \frac{\Delta_{n+1-j}}{\Delta_{n-j}}. \tag{6}$$

To prove this, we consider the horospherical decomposition (2) of the geodesic $x(t)$:

$$x(t) = z(x)\, \mathrm{e}^{2Q(t)}\, z^t(x). \tag{7}$$

Then

$$\dot{x}x^{-1} = z\,\{2P + e^{2Q}\, M^t\, \mathrm{e}^{-2Q} + M\}\, z^{-1}, \tag{8}$$

where $P = \dot{Q}$, and

$$M = z^{-1}\dot{z}. \tag{9}$$

Let us denote

$$L = P + \frac{1}{2}\, M + \frac{1}{2}\, \mathrm{e}^{2Q}\, M^t\, e^{-2Q}. \tag{10}$$

Then

$$\dot{x}x^{-1} = 2zhz^{-1}. \tag{11}$$

Differentiating this again we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\dot{x}x^{-1} = 2z\left(\dot{L} - [L, M]\right)z^{-1}. \tag{12}$$

Since $x(t)$ is geodesic, the left-hand side of (12) vanishes. Thus the geodesic flow has the Lax representation with matrices (9), (10). Note that M lies in the Lie algebra of the group Z.

On the other hand, the Lax pair of the Toda lattice also obeys the relation (10). Moreover, the matrix M is triangular and has a very special form. This condition singles out the particular class of geodesics. It can be put in a more natural form by using the moment map μ for the action of Z:

$$\mu = (\dot{x}x^{-1})_{-}, \tag{13}$$

where the minus subscript means that all entries lying on and above the main diagonal of the matrix in the brackets, are replaced by zeros. Equation (10) shows that the "angular momentum" of the geodesics related to the Toda lattice is

$$\{\mu\}_{jk} = \delta_{j,k+1}. \tag{14}$$

Calculating the "angular momentum" (13) of the geodesic and using (14) we obtain (4) which proves the first proposition.

To prove the second proposition we compare the expression for the geodesics

$$x(t) = b\,\exp\left(2at\right)b^{+}, \tag{15}$$

and the horospherical decomposition (7). Taking $z(0) = I$ we find

$$b = \exp\,Q(0) = \mathrm{diag}\,[e^{q_1^0}, \ldots, e^{q_n^0}]. \tag{16}$$

Equation (11) gives the expression (5) for the matrix a in terms of the initial data. From (15) and (7) we find

$$\mathrm{e}^{2Q(t)} = z^{-1}b\,\exp\{2at\}\,b^{t}\,(z^{t})^{-1}. \tag{17}$$

Since b is the diagonal matrix, we have $z^{-1}b = bz_1$ where z_1 is some element of Z. Therefore (17) can be written as

$$\exp\,2\big(Q(t) - Q(0)\big) = z_1\,\exp\{2at\}\,z_1^{t}. \tag{18}$$

Combining this with the definition of horospherical projection ((2) and (3)) we obtain (6).

To conclude, we note that in the case under consideration, all calculations are reduced to taking the matrix exponent, with a subsequent computation of minors, i.e. to pure algebraic operations.

2.8 Elliptic case

Let us remind first that for N-body elliptic Calogero system two Lax representations are known: one without spectral parameter[14] and another with spectral parameter.[31] This gives the possibility to apply methods of algebraic geometry for integration of equations of motion.[31] For this system recently was found the generalization of projection method. Namely, in Ref. 23 it was proved that every symmetric elliptic function in coordinates $q_1, q_2, ..., q_N$ is a meromorphic function of time t. This gives the possibility to obtain the explicit formulae for solution of equation of motion in terms of genus $N-1$ theta functions. The proof of these statements and other details may be found in Ref. 23.

Appendix: Completely integrable classical systems connected with semi-simple Lie algebras. II. Preprint ITEP – 27 (1976)

Abstract

The complete integrability of a class of dynamical systems of the type considered in the preceding paper[44] but with the potential $V(q) = q^{-2} + cq^2$ *is proved.*

Let us consider the classical dynamical system with n degrees of freedom that is described by the Hamiltonian

$$H(p_k, q_l) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + U(q_1, \dots, q_n). \tag{1}$$

The preceding paper by Olshanetsky and Perelomov[44] (see also preceding papers[41,13,14]) considers the systems of the type (1) with a potential

$$U_0(q_1, \dots, q_n) = g^2 \sum_{k<l} [V(q_k - q_l) + \varepsilon V(q_k + q_l)] + g_1^2 \sum_{k=1}^{n} V(q_k) + g_2^2 \sum_{k=1}^{n} V(2q_k), \tag{2}$$

where constants ε, g, g_1 and g_2 satisfy the condition

$$\varepsilon = 0 \text{ or } 1 \text{ for } g_1 = g_2 = 0; \quad \varepsilon = 1, \text{ for } g_1^2 + g_2^2 > 0$$

and the potential $V(q)$ has one of the following forms

$$\begin{aligned}
&\mathbf{I}. && V(q) = q^{-2}\,, \\
&\mathbf{II}. && V(q) = a^2 \sinh^{-2} aq\,, \\
&\mathbf{III}. && V(q) = a^2 \sin^{-2} aq\,, \\
&\mathbf{IV}. && V(q) = a^2\, \wp(aq)
\end{aligned} \tag{3}$$

($\wp(q)$ is the Weierstrass function). In Ref. 44 it was also shown that these systems are related to the semi-simple Lie algebras.

The aim of the present paper is to extend results of Ref. 44 to the systems of the type (1) with the potential (2), where $V(q) = q^{-2} + cq^2$.[g] Here, besides the integrals of motion I_k, we obtain also the explicit expressions for quantities B_k^* and B_k which are a classical analog of quantum operators of creation and annihilation B_k^+ and B_k introduced in Ref. 53. Note that in Ref. 53 the algorithm was given for finding of the operators B_k^+ and B_k, and the explicit form of the first three of them was found.[h]

Note first of all that under the fulfillment of condition $\sum_{j=1}^n q_j = 0$ the system of interest described by the Hamiltonian (1) with the potential (2) and $V(q) = q^{-2} + \alpha^2 q^2$ is equivalent to the system with the potential

$$U = U_0 + \frac{\omega^2}{2} \sum_{k=1}^{n} q_k^2, \tag{4}$$

where U_0 is given by the formula (2) with $V(q) = q^{-2}$.

In order to prove the complete integrability of the system under consideration and to find explicit form of B_k, B_k^* and I_k we generalize the Moser method[41] which in turn applies the Lax trick.[34] Namely, we will show that the Hamilton equations

$$\dot{q}_k = \frac{\partial H}{\partial p_k} = p_k, \ \dot{p}_k = -\frac{\partial H}{\partial q_k} \tag{5}$$

are equivalent to the matrix equation

$$\frac{d\tilde{L}}{dt} = [\tilde{L}, M] - i\omega \tilde{L}, \tag{6}$$

where matrices M and L depend on p_k and q_l, and the matrix iM is Hermitian.

Let us first of all note that as it is shown in Refs. 44, 41, 13 and 14 for the system described by the Hamiltonian $H = \frac{1}{2}\sum_{k=1}^n p_k^2 + U_0$, where $V(q)$ is given by the formula (3), the Hamilton equations are equivalent to the matrix equation

$$\frac{dL}{dt} = [L, M] \tag{7}$$

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[g] After this paper has been performed we got the possibility to acquaint with Adler's preprint[3] in which the similar results were obtained.

[h] Note that analogously to Ref. 14 it may be shown that in the quantum case ($p_k \to -i\,\partial/\partial q_k$) operators B_k and B_k^+ are well-defined. Thereby, we get for them the explicit expressions.

(the explicit form for the matrices $iM = (-D + Y)$ and $L = P + iX$ is given in the above mentioned papers). In the case of interest we take

$$\tilde{L} = L - i\,\omega Q, \tag{8}$$

where the matrix Q is obtained from the matrix P replacing p_k by q_k and correspondingly the matrix $\tilde{L}$ is obtained from the matrix L replacing p_k by $b_k = p_k - i\,\omega q_k$.

Then, as it can be easily checked, equations (6) are a consequence of the equation (8) and of identities

$$\frac{d\tilde{L}}{dt} = [L, M] - i\omega(P - i\omega Q), \qquad L = P + [M, Q]. \tag{9}$$

The check of the first of these identities is very simple while for the check of the second one it is necessary to make use of explicit expressions for the matrices L and M from Refs. 44, 41, 13 and 14.

Let us now turn to obtaining of consequences from the equation (6). Form the quantities

$$B_k = \operatorname{tr}(\tilde{L}^k) \quad \text{and} \quad B_k^* = \operatorname{tr}\left(\left(\tilde{L}^+\right)^k\right). \tag{10}$$

Then, as it follows from (6),

$$\dot{B}_k(t) = -\,ik\omega\, B_k(t), \qquad B_k(t) = \exp\left(-\,ik\omega t\right) B_k(0). \tag{11}$$

Thus, the quantities $B_k(t)$, not being the integrals of motion, possess, however, a very simple time dependence.[i]

Let us now find the integrals of motion. It is easy to see that these are both $B_k^* B_k$ and $\{B_k, B_k^*\}$. Thus the knowledge of quantities B_k is sufficient to find the invariants. It will be more convenient for us to use another set of invariants. Let us consider matrices $N_1 = \tilde{L}^+\tilde{L}$ and $N_2 = \tilde{L}\tilde{L}^+$. It follows from (6) that they satisfy the equation

$$\frac{dN_k}{dt} = [N_k, M], \qquad k = 1, 2. \tag{12}$$

Consequently, the matrix

$$N = \frac{1}{2}\left(\tilde{L}^+\tilde{L} + \tilde{L}\tilde{L}^+\right) = L^2 + \omega^2 Q^2 \tag{13}$$

[i] The quantities $B_k(t)$ completely determine the evolution of considered system. They are rational functions of variables p_k and q_l. Expressing the coordinates and momenta through $B_k(t)$ and $B_l^*(t)$ we get an explicit expression for $p_k(t)$ and $q_l(t)$ and thereby completely integrate the system of Hamilton's equations (5).

also satisfies the equation

$$\frac{dN}{dt} = [N, M]. \tag{14}$$

Therefore, the eigenvalues of matrix N do not change with time and consequently the quantities

$$I_k = \mathrm{tr}\,(N^k) \tag{15}$$

are the integrals of motion.

In conclusion, let us prove that the quantities B_k are in involution, i.e., $\{B_k, B_l\} = 0$. Here

$$\{A, B\} = \sum_{k=1}^{n} \left(\frac{\partial A}{\partial p_k} \frac{\partial B}{\partial q_k} - \frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} \right)$$

is the Poisson bracket for A and B.

Indeed, the quantities B_k are obtained from integrals of motion $I_k^{(0)}$ for the system described by Hamiltonian H_0 replacing in them p_k by $b_k = p_k - i\,\omega q_k$. With this replacement

$$\frac{\partial I_k^{(0)}}{\partial p_j} \to \frac{\partial B_k}{\partial b_j}, \qquad \frac{\partial I_k^{(0)}}{\partial q_j} \to \frac{\partial B_k}{\partial q_j} - i\omega\, \frac{\partial B_k}{\partial b_j}.$$

Thus,

$$\{B_k, B_l\} = \sum_j \left(\frac{\partial B_k}{\partial p_j} \frac{\partial B_l}{\partial q_j} - \frac{\partial B_k}{\partial q_j} \frac{\partial B_l}{\partial p_j} \right) = \sum_j \left(\frac{\partial B_k}{\partial b_j} \frac{\partial B_l}{\partial q_j} - \frac{\partial B_k}{\partial q_j} \frac{\partial B_l}{\partial b_j} \right) = 0, \tag{16}$$

since as it follows from Refs. 44, 41, 13 and 14,

$$\{I_k^{(0)}, I_l^{(0)}\} = 0.$$

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