To memory of my friend Misha Marinov

RENORMALIZATION BY ENFORCING A SYMMETRY

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A new renormalization scheme for theories with nontrivial internal symmetry is proposed. The sheme is regularization independent and respects the symmetry requirements.

Contents

1 Introduction

Renormalization procedure in the theories with nontrivial internal symmetries, like gauge invariant models, is complicated by the necessity to provide the symmetry of the renormalized theory. The crucial role in this procedure is played by the relations between Green functions which are the quantum analogue of a classical symmetry. In the case of Quantum Electrodynamics (QED) these relations are Ward-Takahashi identities (WTI),[1,2] which connect three- and two-point Green functions. For non-Abelian gauge theories corresponding relations (STI) were obtained in Refs. 3 and 4.

There are essentially two approaches to renormalization of gauge invariant theories. The first one uses some intermediate gauge invariant regularization, e.g. dimensional regularization,[5,6] higher covariant derivatives,[7–9] or lattice regularization.[10] Using these regularizations one can prove that the counterterms needed to eliminate ultraviolet divergencies preserve the gauge invariant structure of the renormalized Lagrangian.

In the second approach, known as algebraic renormalization, one firstly defines a finite renormalized theory following the Bogoliubov-Parasiuk R-operation[11] with some particular subtractions.[12] This procedure in general breaks gauge invariance, however one can use a finite renormalization freedom to restore STI for renormalized Green functions[13,14] (for recent development and more complete references see Ref. 15).

Both these approaches have some advantages and disadvantages. Using invariant regularizations allows to prove in a rather simple way gauge independence of observables. Moreover, the dimensional regularization proved to be an efficient way for calculations of Feynman diagrams. On the other hand in this case the procedure is explicitely regularization dependent and, more important, is not directly applicable to some theories including the Standard Model and supersymmetric theories. The invariant regularization procedure for anomaly free theories including chiral fermions, like Salam-Weinberg or unified models was constructed recently,[16,17] and for general Feynman diagrams is still rather complicated.

The algebraic renormalization uses only the algebraic structure of the underlying theory and does not require any particular regularization. So it may be applied to any anomaly free model. However to provide the validity of STI for renormalized Green functions in this approach is a nontrivial problem and calculations are rather clumsy. Note that it refers also to the models without chiral fermions.

In the present paper I propose a new renormalization procedure which shares the advantages of both these approaches. The procedure is regularization independent and at the same time provides automatically STI for renormalized Green functions.[18]

To formulate the main idea of the method I remind some general properties of renormalization procedure which may be found in many textbooks (see e.g. Ref. 19).

In any 4d renormalizable theory ultraviolet divergencies may occur in diagrams with the number of external lines less than five. The vertex functions with three and four external lines may diverge linearly or logarithmically. According to the R-operation[11,19] these divergencies are removed by recursive subtractions of local counterterms (polynomials in momentum space), which may be identified with the values of the vertex functions at some (infrared safe) external momenta. Making these subtractions one firstly introduce some intermediate regularization which makes the integral as a whole convergent by power counting. Such a regularization which is removed after making all necessary subtractions is assumed in the following. Finite renormalized functions obtained in this way have an ambiguity which is a polynomial in external momenta of the degree given by power counting (see Ref. 19).

In what follows we assume that if some intermediate regularization is introduced, it satisfies some natural requirements which usually are tacitly assumed. The most important properties are the following. Regularization does not change the value of integrals convergent by power counting (i.e., in the limit when the regularization is removed one gets the same result as without regularization). Global symmetries are respected by the regularization.

After these preliminaries I formulate the general idea of the method. It follows from the discussion above that ultraviolet divergencies of the vertices with three and four external lines may be removed not necessarily by subtracting the vertex at some fixed external momenta but also by subtracting the values of the corresponding vertices with only one external momentum fixed, in particular the fixed momentum may be equal to zero. In the case of linear divergencies one has to use a properly symmetrized subtraction. Sometimes symmetrization is also useful for subtracting logarithmically divergent integrals.

The subtraction procedure described above makes vertex functions finite but obviously violates locality. To restore the locality one can use the fact that in gauge theories the value of a vertex function at zero momentum of one of the external gauge fields is related to the values of other correlators having at least one external gauge field less. In QED this relation is given by WTI and expresses the electron photon vertex at zero photon momentum in terms of electron polarization. In non-Abelian models the corresponding relations express the vertex functions with three and four external gauge fields in terms of polarization operators and ghost-gauge field vertices. Having this in mind one can define the renormalized proper vertex functions as follows:

$$\Gamma^r(p,q,k) = \Gamma(p,q,k) - \Gamma(p,q,0) + F^r(p,q)\,. \tag{1}$$

Here $\Gamma(p,q,k)$ denotes some proper vertex function with indices suppressed and F^r is a combination of other renormalized correlators, such that for a proper choice of renormalization freedom

$$\Gamma^r(p,q,0) - F^r(p,q) = 0 \tag{2}$$

by virtue of gauge invariance. In other words, Eq. (2) is the renormalized WTI in QED or renormalized STI in non-Abelian models. The notation Γ^r means a renormalized vertex function.

The subtraction (1) involves the value of three or four point vertex functions at one of the external momenta equal to zero. Obviously if at least one of the remaining external momenta is different from zero, it serves as the infrared regulator and this function is infrared finite. So our procedure does not meet any infrared problems.

The renormalization procedure works loopwise. One firstly makes arbitrary (infrared safe) subtractions of one loop polarization operators and (in non-Abelian models) ghost-gauge field vertices. Then one defines the renormalized three and four gauge field one loop correlators by the relations of the type (1), where F^r depends only on finite renormalized correlators. The r.h.s. of the Eq. (1) is ultraviolet finite. The WTI and STI are obviously satisfied by the renormalized functions (1). Finally, as was discussed above, the difference

$$\Gamma(p,q,0) - F^r(p,q) \tag{3}$$

in any anomaly free model in the limit of an intermediate regularization removed is a polynomial in p, q (with coefficients divergent in the limit when a regularization is removed). It follows from the fact that $\Gamma^r(p,q,0)$ is obtained from $\Gamma(p,q,0)$ by subtracting a polynomial. It proves the locality of our procedure. Note that if one applies the same procedure to an anomalous model, the difference (3) will not be local anymore, which makes the model inconsistent.

After defining renormalized one loop correlation functions one proceeds defining multiloop correlators according to the standard R-operation. With subgraph divergencies being removed one needs to make a subtraction corresponding to the superficial divergency of the diagram, which is done by the same procedure. There is nothing specific in our method at this point and we refer for further details to textbooks (e.g. Ref. 19).

In the second section I illustrate the method by applying it to QED. In the third section the renormalization of the Yang-Mills theory is constructed. In the fourth section I discuss a simple anomaly free model with chiral fermions.

2 Invariant regularization of QED

In Quantum Electrodynamics formal WT identities for proper Green functions may be written in the form:

$$\Gamma_\mu(p,0) = e\partial_\mu \Sigma(p)\,, \tag{4}$$

where Γ_μ is the proper photon-electron vertex and Σ is the electron self energy. We shall define the renormalized Green functions which satisfy automatically this identity.

The renormalization procedure works loopwise. We start by defining the renormalized one-loop electron self energy. The most general structure of Σ is

$$\Sigma(p) = (\hat{p} - m)\Sigma_1(p^2) + \Sigma_2(p^2)\,. \tag{5}$$

Renormalized electron self energy is given by the folllowing equation,

$$\Sigma^r(p) = (\hat{p} - m)\left[\Sigma_1(p^2) - \Sigma_1(a^2)\right] + \Sigma_2(p^2) - \Sigma_2(m^2)\,. \tag{6}$$

Here a is some arbitrary (infrared finite) normalization point. Equation (6) guarantees that the electron propagator has a pole at $p^2 = m^2$.

The renormalized vertex function is defined by the equation:

$$\Gamma^r_\mu(p,q) = \Gamma_\mu(p,q) - \Gamma_\mu(p,0) + e\partial_\mu \Sigma^r(p)\,. \tag{7}$$

Here Σ^r is the finite renormalized function defined by the Eq. (6). The vertex function diverges logarithmically, so Γ^r_μ is obviously finite and hence regularization independent. By construction it satisfies WTI:

$$\Gamma^r_\mu(p,0) = e\partial_\mu \Sigma^r(p) \tag{8}$$

At first sight the subtraction (7) may seem to be nonlocal. However it is not. Locality of our procedure follows from the fact that in any regularization scheme the difference

$$\Gamma_\mu(p,0) - e\partial_\mu \Sigma^r(p) \tag{9}$$

is a local polynomial. (See the discussion in the Introduction). In a gauge invariant regularization scheme one can rewrite the Eq. (7) in the form

$$\Gamma^r_\mu(p,q) = \Gamma^{inv}_\mu(p,q) + (z_2 - 1)\gamma_\mu - \left[\Gamma^{inv}_\mu(p,0) + (z_2 - 1)\gamma_\mu - e\partial_\mu \Sigma^{inv,r}(p)\right]\,. \tag{10}$$

In an invariant regularization scheme the term in square bracketts vanishes by virtue of WTI, and $\Gamma^r_\mu(p,q) = \Gamma^{inv,r}_\mu(p,q)$. In a gauge invariant regularization the Green functions $\Gamma^{inv,r}_\mu$ and $\Sigma^{inv,r}$, renormalized by the counterterms obeying the relation $z_1 = z_2$, satisfy WT identity. The locality of the procedure is manifest.

Gauge invariant renormalization of the vacuum polarization makes no problems. The polarization tensor has the structure:

$$\Pi_{\mu\nu}(p) = (g_{\mu\nu}p^2 - p_\mu p_\nu)\,\Pi_1(p^2) + g_{\mu\nu}\Pi_2(p^2)\,. \tag{11}$$

The renormalized polarization tensor is defined by the equation

$$\Pi^r_{\mu\nu} = P_{\mu\alpha}\,[\Pi_{\alpha\nu}(p) - \Pi_{\alpha\nu}(0)] - p^2 P_{\mu\nu}\,\Pi_1(b^2)\,. \tag{12}$$

Here b is again some arbitrary normalization point and $P_{\mu\nu} = g_{\mu\nu} - p_\mu p_\nu p^{-2}$ is the transversal projection operator. The function $\Pi^r_{\mu\nu}$ is finite and obviously satisfies the transversality condition

$$p_\mu \Pi^r_{\mu\nu}(p) = 0 \tag{13}$$

required by gauge invariance.

To complete the one-loop renormalization we have to consider the diagram with four photon external lines. The differential WTI for the function $\Pi_{\mu\nu\rho\sigma}(p,q,k)$ reads:

$$\Pi_{\mu\nu\rho\sigma}(p,q,0) = 0\,. \tag{14}$$

This equation is preserved in any gauge invariant regularization scheme. As was discussed in the introduction the result obtained in any other regularization scheme may differ by a polynomial. As $\Pi_{\mu\nu\rho\sigma}$ is presented by a logarithmically divergent integral in arbitrary scheme after removal of an intermediate regularization $\Pi_{\mu\nu\rho\sigma}(p,q,0)$ is a zero order polynomial (constant tensor).

We define the renormalized four-point function by the equation:

$$\Pi^r_{\mu\nu\rho\sigma}(p,q,k) = \Pi_{\mu\nu\rho\sigma}(p,q,k) - \Pi_{\mu\nu\rho\sigma}(p,0,0)\,. \tag{15}$$

As the function $\Pi_{\mu\nu\rho\sigma}$ diverges logarithmically, the renormalized function defined by the Eq. (15) is obviously finite. The external momentum p serves as the infrared regulator, so no infrared problems arise. It also satisfies WTI (14). Locality of the procedure holds for the same reasons as above.

Renormalization of the Green functions according to Eqs. (6, 7, 12, 15) is equivalent to introducing the following one-loop counterterms to the QED Lagrangian:

$$\begin{aligned} L^r = &-\frac{z_3}{4}F_{\mu\nu}F_{\mu\nu} + iz_2\bar{\psi}(\hat{\partial} - m)\psi \\ &+ \delta m\,\bar{\psi}\psi + ez_1\bar{\psi}\gamma_\mu\psi A_\mu + z_4(A_\mu A_\mu)^2 + \delta\mu\,A^2_\mu\,. \end{aligned} \tag{16}$$

Here

$$1 - z_2 = \Sigma_1(a^2); \quad \delta m = -\Sigma_2(m^2); \quad (1 - z_1)\gamma_\mu = \Gamma_\mu(p, 0) - e\partial_\mu \Sigma^r(p);$$
$$1 - z_3 = \Pi_1(b^2); \quad \delta\mu = -\Pi_2(0); \quad z_4 = -\frac{1}{3}\Pi_{0000}(p, 0, 0)\,. \tag{17}$$

In an invariant regularization scheme

$$\delta\mu = 0; \quad z_4 = 0; \quad z_1 = z_2\,. \tag{18}$$

Assuming that the one loop counterterms are introduced according to Eqs. (16, 17), we define the renormalized two loop functions by Eqs. (6, 7, 12, 15). The divergencies in subgraphs of the diagrams corresponding to these functions are killed by the one-loop counterterms (16), therefore they may diverge only superficially. Repeating literally the discussion given above, we prove the locality and gauge invariance of our procedure at two-loop level. Extension to higher loops is straightforward. To avoid misunderstanding I emphasize that our renormalization is formulated as the recursive substraction procedure and does not require explicit introduction of counterterms. Equations (16-18) are presented to establish the connection with more conventional approach.

3 Renormalization of Yang-Mills theory

In this section we apply the same idea to renormalization of Yang- Mills theory. The two-point Green functions will be renormalized with the help of local subtractions compatible with gauge invariance, and renormalized vertex functions will be defined by means of nonlocal subtractions supplemented by addition of explicitly known finite terms restoring the locality and gauge invariance.

In a covariant α-gauge the effective action of Yang-Mills theory looks as follows

$$S = \int d^4x \left[-\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2\alpha}(\partial_\mu B_\mu)^2 + + \bar{c}^a \partial_\mu(\delta^{ab}\partial_\mu - g t^{abc} B^c_\mu) c^b \right]. \tag{19}$$

We want to define renormalized Green functions so that they satisfy automatically ST identities. For the Yang-Mills self energy these identities reduce to the condition of transversality . So the one-loop renormalized Yang-Mills field self energy may be defined in the same way as in QED, by the Eq. (12).

Next identity relates the three point function for the Yang-Mills field with the ghost-gauge interaction vertex and ghost and gauge fields propagators.

It follows from the structure of ghost interaction vertex and Lorentz invariance, that the ghost field self energy has the form

$$\Pi_G^{ab}(p^2) = \delta^{ab} p^2 \Pi(p^2)\,. \tag{20}$$

We renormalize it by making a subtraction at arbitrary infrared finite point

$$\Pi^r(p^2) = \Pi(p^2) - \Pi(c^2)\,. \tag{21}$$

The ghost-gauge vertex also may be renormalized at will. It diverges logarithmically, and it follows from the structure of interaction that the corresponding local structure is proportional to $t^{abc}k_\mu$, where k_μ is the ghost field momentum. Therefore we can perform the renormalization by the following prescription:

$$\Gamma_\mu^{abc,r}(k,p) = \Gamma_\mu^{abc}(k,p) - k_\mu \Gamma^{abc}(b^2)\,, \tag{22}$$

$$\Gamma_\mu^{abc}(k,p)_{p=0} = k_\mu \Gamma^{abc}(k^2)\,. \tag{23}$$

In this equation p stands for the gauge field momentum.

The subtractions (12, 21, 22) are equivalent to introducing to the Lagrangian the following counterterms

$$\begin{aligned} &- \frac{z_2 - 1}{4}\left(\partial_\mu B_\nu^a - \partial_\nu B_\mu^a\right)\left(\partial_\mu B_\nu^a - \partial_\nu B_\mu^a\right) \\ &+ (\tilde{z}_2 - 1)\,\bar{c}^a \,\Box\, c^a - (\tilde{z}_1 - 1)\, g\, t^{acb}\, \bar{c}^a\, \partial_\mu (B_\mu^c c^b)\,. \end{aligned} \tag{24}$$

Now we should define the renormalized three-point Yang-Mills field vertex, shown in Fig. 1, in such a way it satisfies the ST identities. We firstly assume

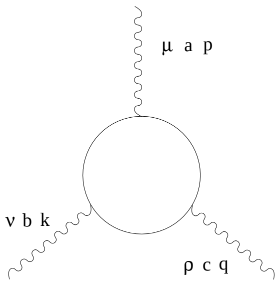

Figure 1: Three-point vertex function for Yang-Mills fields.

that some gauge invariant regularization is introduced. Then one can get in a usual way the following identity for renormalized correlators

$$\frac{i}{\alpha}\left\langle B^a_\mu(x)B^b_\nu(y)\partial_\rho B^c_\rho(z)\right\rangle^r = \tilde{z}_2\left\langle \partial_\mu c^a(x)\bar{c}^c(z)B^b_\nu(y)\right\rangle^r$$
$$-\tilde{z}_2\,\tilde{g}\,t^{ade}\left\langle B^d_\mu(x)c^e(x)\bar{c}^c(z)B^b_\nu(y)\right\rangle^r + (x\to y, a\to b, \mu\to\nu)\,. \quad (25)$$

Here the parameter $\tilde{g}$ is the effective coupling constant which enters the gauge transformations in the renormalized theory

$$\delta B^a_\mu = \partial_\mu\alpha^a - \tilde{g}t^{abc}B^b_\mu\alpha^c\,. \quad (26)$$

I recall that in this derivation we assume that some gauge invariant regularization is present, so that Eq. (25) and the following equations make sense. However the final definition of the renormalized vertices will be regularization independent.

Our definition of the renormalized three-point Yang-Mills vertex is based on the identity (25). So we firstly transform the Eq. (25) to a more convenient form. In this transformation we shall use the quantum equations of motoin for the ghost field c. These equations look as follows

$$\int\left[\Box c^a(x) - \tilde{g}t^{aed}\partial_\mu(B^e_\mu(x)c^d(x)) + \tilde{z}_2^{-1}\eta^a(x)\right]e^{iL_R}\,dB_\mu d\bar{c}dc = 0\,. \quad (27)$$

Differentiating this equation with respect to $\eta^b(y)$ and $J^c_\mu(z)$ we have

$$\left\langle\Box c^a(x)\,\bar{c}^b(y)\right\rangle^r - \tilde{g}\,t^{aed}\left\langle\partial_\mu(B^e_\mu(x)c^d(x))\,\bar{c}^b(y)\right\rangle^r + \tilde{z}_2^{-1}\,\delta^{ab}\delta(x-y) = 0 \quad (28)$$

and

$$\left\langle\Box c^a(x)\,\bar{c}^c(z)B^b_\nu(y)\right\rangle^r - \tilde{g}\,t^{aed}\left\langle\partial_\mu(B^e_\mu(x)c^d(x))\,\bar{c}^c(z)B^b_\nu(y)\right\rangle^r = 0 \quad (29)$$

Note that Eqs. (27, 28, 29) are valid also in the "partially renormalized" theory, described by the Lagrangian, including the counterterms (24) (no counterterms for three- and four-point gauge field vertices). It follows from the Eq. (29) that the sum of the first two terms in the l.h.s. of identity (25) is transversal with respect to differentiation over x_μ and remaining terms are transversal with respect to differentiation over y_ν. Using this observation we can rewrite the identity (25) as follows. We firstly perform a Fourier transformation of Eq. (25). Then cutting the Yang-Mills field propagators corresponding to the external lines of the three-point correlator at the l.h.s. of the Eq. (25) and taking into account the transversality of Yang-Mills field self energy, we can write for the proper vertex function $\Gamma^{abc,r}_{\mu\nu\rho}(p,q)$ the following identity:

$$q_\rho\Gamma^{abc,r}_{\mu\nu\rho}(p,q) - q^2\left[(G^{-1}_{tr})^r_{\mu\alpha}(p)G^r(q)\Gamma^{abc,r}_{\alpha\nu}(p,q)\right.$$
$$\left.+(\mu\to\nu,\ a\to b,\ p\to -p-q)\right] = 0\,. \quad (30)$$

In deriving this equation we used the transversality of the r.h.s. of Eq. (25) discussed above, and replaced the inverse gauge field propagator $G_{\mu\nu}$ by its transversal part. After such projection the first term in the r.h.s. of Eq. (25), which is longitudinal, does not contribute. In this equation

$$\begin{aligned} \delta^{ab}(G_{tr}^{-1})_{\mu\nu}^{r} &= P_{\mu\alpha}(G^{-1})_{\alpha\nu}^{ab,r}\,, \\ \delta^{ab}G^{r}(p) &= G^{ab,r}(p)\,, \end{aligned} \tag{31}$$

$G^{ab,r}(p)$ is the renormalized ghost field propagator. The function $\Gamma_{\mu\nu}^{abc,r}(p,q)$ is the Fourier transform of the expectation value of the composite operator

$$\tilde{\Gamma}_{\mu\nu}^{abc,r} = (G^{-1})_{\nu\alpha}^{r}(y)(G^{-1})^{r}(z)\left\langle \tilde{z}_2\tilde{g}t^{aed}B_{\mu}^{e}(x)c^{d}(x)\bar{c}^{c}(z)B_{\nu}^{b}(y)\right\rangle^{r}\,. \tag{32}$$

Differentiating the equality (30) with respect to q_ρ and putting $q=0$ we get the differential identity, which will be used for the definition of renormalized three-point Yang-Mills field vertex

$$\begin{aligned} \Gamma_{\mu\nu\rho}^{abc,r}(p,0) - \frac{\partial}{\partial q_\rho}\big[q^2(G_{tr}^{-1})_{\mu\alpha}^{r}(p)G^{r}(q)\Gamma_{\alpha\nu}^{abc,r}(p,q) \\ +(\mu\to\nu,\ a\to b,\ p\to -p-q)\big]_{q=0} = 0\,. \end{aligned} \tag{33}$$

The renormalized proper vertex function for gauge fields is defined by the equation

$$\begin{aligned} &\Gamma_{\mu\nu\rho}^{abc,r}(p,q) = \Gamma_{\mu\nu\rho}^{abc}(p,q) - \Big\{\Gamma_{\mu\nu\rho}^{abc}(p,0) + \Gamma_{\mu\nu\rho}^{abc}(0,q) \\ &-\frac{\partial}{\partial q^\rho}\left[q^2(G_{tr}^{-1})_{\mu\alpha}^{r}(p)G^{r}(q)\Gamma_{\alpha\nu}^{abc,r}(p,q) + (\nu\to\mu,\ a\to b,\ p\to -p-q)\right]_{q=0} \\ &-\frac{\partial}{\partial p^\mu}[p^2(G_{tr}^{-1})_{\rho\beta}^{r}(q)G^{r}(p)\Gamma_{\beta\nu}^{cba,r}(q,p) + (\nu\to\rho,\ c\to b,\ q\to -p-q)]_{p=0}\Big\}\,. \end{aligned} \tag{34}$$

The divergent part of $\Gamma_{\mu\nu\rho}^{abc}(p,q)$ is the first order polynomial in p,q. Therefore the sum of the three first terms in the r.h.s. of Eq. (34) is finite. Below we shall prove that the remaining terms in the r.h.s. of Eq. (34) are also finite.

As it was mentioned above the r.h.s. of Eq. (34) includes the expectation value of the composite operator $\Gamma_{\mu\nu}^{abc}$. So we have to prove that the counterterms (24) make it finite. In this proof we again shall use the quantum equations of motion for gauge fields (27, 28, 29). The function $\tilde{\Gamma}_{\mu\nu}$ may be separated into one particle reducible and irreducible parts as follows (see Fig. 2):

$$\tilde{\Gamma}_{\mu\nu}^{abc,r}(x,y,z) = \int \tilde{G}_{\mu}^{ad}(x-x')\tilde{\Gamma}_{\nu}^{dbc,r}(x',y,z)dx' + (\tilde{\Gamma}^{ir})_{\mu\nu}^{abc}(x,y,z)\,. \tag{35}$$

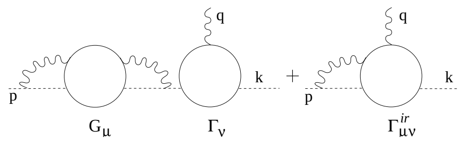

Figure 2: Decomposition of $\Gamma_{\mu\nu}$ into one-particle reducible and irreducible parts.

Here $\tilde{\Gamma}^{ir}_{\mu\nu}$ denotes the strongly connected part of $\tilde{\Gamma}_{\mu\nu}$, and Γ^r_ν is the renormalized ghost-gauge field vertex. Finally

$$\tilde{G}^{ab}_\mu(x-x') = \left\langle \tilde{z}_2 \tilde{g} t^{aed} B^e_\mu(x) c^d(x) \bar{c}^b(x') \right\rangle . \tag{36}$$

By virtue of Lorentz invariance $G^{ab}_\mu(p)$ is proportional to p_μ. Then the Eq. (28) gives

$$G^{ab}_\mu(p) = \tilde{z}_2 p_\mu G^{ab}(p) + \delta^{ab} p_\mu p^{-2} \tag{37}$$

Substituting the decomposition (35, 37) into Eq. (29) we get

$$\Gamma^{abc,r}_\nu(p,q) + ip_\mu (\Gamma^{ir})^{abc}_{\mu\nu}(p,q) = 0 \tag{38}$$

This equation shows that the function $p_\mu(\Gamma^{ir})^{abc}_{\mu\nu}$ is finite. As the corresponding integral diverges logarithmically, Lorentz invariance implies that $(\Gamma^{ir})^{abc}_{\mu\nu}(p,q)$ is also finite. The Eq. (38) expresses the well known fact, firstly established in Ref. 3, that renormalization of the composite operator (32) is related to the renormalization of the ghost-gauge field vertex. It can be taken as a definition of the renormalized composite operator (32) in terms of the renormalized ghost-gauge field vertex, or vice versa.

The function $\Gamma^{abc}_{\mu\nu}(p,q)$ enters into Eq. (34) being multiplied by the transverse projector. As the Eqs. (35-37) show the reducible part of $\Gamma_{\mu\nu}$ is longitudinal and hence only irreducible part of $\Gamma_{\mu\nu}$ contributes to Eq. (34). It proves the finiteness of the r.h.s. of Eq. (34). Therefore the Eq. (34) indeed defines the finite function $\Gamma^{abc,r}_{\mu\nu\rho}(p,q)$ which does not depend on a particular regularization scheme used for its calculation provided this regularization satisfies the natural conditiona formulated earlier.

Now we shall prove that the definition (34) corresponds to subtraction from $\Gamma^{abc}_{\mu\nu\rho}(p,q)$ a local polynomial and that $\Gamma^{abc,r}_{\mu\nu\rho}(p,q)$ satisfies ST identities.

In arbitrary regularization preserving a global symmetry of the model the divergent part of the three point vertex has the following structure

$$i(z_1-1)t^{abc}\left[(p-k)_\rho g_{\mu\nu} + (k-q)_\mu g_{\nu\rho} + (q-p)_\nu g_{\mu\rho}\right]. \tag{39}$$

Having this in mind we rewrite the sum of the three first terms in the r.h.s. of Eq. (34) in the form

$$
\begin{aligned}
&\Big(\Gamma^{abc}_{\mu\nu\rho}(p,q) + i(z_1-1)t^{abc}[(p-k)_\rho g_{\mu\nu} + (k-q)_\mu g_{\nu\rho} + (q-p)_\nu g_{\mu\rho}]\Big) \\
&-\Big(\Gamma^{abc}_{\mu\nu\rho}(p,0) + i(z_1-1)t^{abc}[2p_\rho g_{\mu\nu} - p_\mu g_{\nu\rho} - p_\nu g_{\mu\rho}]\Big) \\
&-\Big(\Gamma^{abc}_{\mu\nu\rho}(0,q) + i(z_1-1)t^{abc}[q_\rho g_{\mu\nu} - 2q_\mu g_{\nu\rho} + q_\nu g_{\mu\rho}]\Big)\,. \qquad (40)
\end{aligned}
$$

The constant z_1 is defined by the condition

$$
\begin{aligned}
&\Gamma^{abc}_{\mu\nu\rho}(p,0) + i(z_1-1)t^{abc}(2p_\rho g_{\mu\nu} - p_\mu g_{\nu\rho} - p_\nu g_{\mu\rho}) \qquad (41) \\
&-\frac{\partial}{\partial q_\rho}\left\{q^2 (G^{-1}_{tr})^r_{\mu\alpha}(p) G^r(q)\Gamma^{abc,r}_{\alpha\nu} + (\mu \to \nu,\ a \to b,\ p \to -p-q)\right\}_{q=0} = 0\,.
\end{aligned}
$$

Equation (41) is nothing but the differential STI, hence in any anomaly free model in the limit when intermediate regularization is removed it may be satisfied for a proper choice of z_1. After such transformation the expression in the curly bracketts in the r.h.s. of Eq. (34) vanishes in the limit when intermediate regularization is removed and we get

$$
\begin{aligned}
&\Gamma^{abc,r}_{\mu\nu\rho}(p,q) = \Gamma^{abc}_{\mu\nu\rho}(p,q) \\
&+ i(z_1-1)t^{abc}[(p-k)_\rho g_{\mu\nu} + (k-q)_\mu g_{\nu\rho} + (q-p)_\nu g_{\mu\rho}]\,. \qquad (42)
\end{aligned}
$$

So we proved that in the limit when intermediate regularization is removed our renormalization procedure coincides with the usual local subtraction. The validity of STI is obvious.

The last one-loop diagram to be analyzed is the proper four-point Yang-Mills vertex. It can be analyzed in exact analogy with the three point function. However it also can be done in an easier way, which does not require the analysis of composite operators. Below we present the corresponding construction.

In any regularization this function has the following structure

$$
\Pi^{abcd}_{\mu\nu\rho\sigma}(p,q,k) = P\left[A t^{abe} t^{cde} g_{\mu\rho} g_{\nu\sigma} + B\delta^{ab}\delta^{cd} g_{\mu\nu} g_{\rho\sigma}\right] + O(p,q,k)\,. \qquad (43)
$$

Here P is the symmetrization operator with respect to the pairs $(a,\mu; b,\nu; c,\rho; d,\sigma)$, A and B are some (divergent when regularization is removed) constants, $O(p,q,k)$ is a finite function.

It follows from Eq. (43) that

$$
B = \frac{1}{3}p^{-4} p_\mu p_\nu p_\rho p_\sigma \Pi^{aaaa}_{\mu\nu\rho\sigma}(p,0,0) + O_1 \qquad (44)
$$

(no summation over a),

$$A = \frac{1}{3Np^2} P_{\nu\rho}(p) p_\mu p_\sigma \Pi^{abab}_{\mu\nu\rho\sigma}(p,0,0) + O_2 \tag{45}$$

(no summation over a, and $a \neq b$). Here $N = \sum_{b,e} t^{abe} t^{abe}$, $P_{\mu\nu}$ is the transversal projector operator and O_1, O_2 are some finite functions.

We define the renormalized four-point function as follows:

$$\begin{aligned} &\Pi^{abcd,r}_{\mu\nu\rho\sigma}(p,q,k) = \Pi^{abcd}_{\mu\nu\rho\sigma}(p,q,k) \\ &-P(t^{abe}t^{cde} g_{\mu\rho} g_{\nu\sigma})(3Np^2)^{-1} P_{\beta\gamma}(p) p^\alpha p^\delta (\Pi_{con})^{mnmn}_{\alpha\beta\gamma\delta}(p,0,0) \\ &-P(g_{\mu\nu} g_{\sigma\rho} \delta^{ab} \delta^{cd}) 3^{-1} p^{-4} p^\alpha p^\beta p^\gamma p^\delta (\Pi_{con})^{mnmn}_{\alpha\beta\gamma\delta}(p,0,0) \,. \end{aligned} \tag{46}$$

Note that the subtracted terms at the r.h.s. of this equation include the functions $\Pi_{con}(p,0,0)$ which represent all connected one loop diagrams with four external gauge field lines. They include apart from the proper four-point vertex also weakly connected diagrams shown at Fig. 3. The divergent subgraphs in these diagrams are assumed to be renormalized as described above and do not introduce either ultraviolet or infrared singularities.

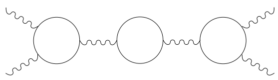

Figure 3: Weakly connected diagrams contributing to the l.h.s. of Eq. (46).

By virtue of Eqs. (44, 45) the renormalized function $\Pi^{abcd,r}_{\mu\nu\rho\sigma}(p,q,k)$ defined by the Eq. (46) is finite.

To prove the locality and gauge invariance we use again differential STI. The differential STI for the four point function imply the relation

$$\begin{aligned} &\left\langle \partial_\mu B^a_\mu(x) \partial_\nu B^b_\nu(y) \partial_\rho B^c_\rho(z) \partial_\sigma B^d_\sigma(u) \right\rangle \\ &= \delta(x-u)\delta(y-z)\delta^{ad}\delta^{bc} + (\{x,a\} \to \{y,b\}) + (\{x,a\} \to \{z,c\}) \,. \end{aligned} \tag{47}$$

This equation shows that the connected part of the correlator (47) is equal to zero. The corresponding differential identity for the Fourier transform of the four point vertex looks as follows

$$p_\mu p_\nu (\Pi^r_{con})^{abcd}_{\mu\nu\rho\sigma}(p,0,0) = 0 \,. \tag{48}$$

Let us replace all the functions $\Pi^{abcd}_{\mu\nu\rho\sigma}$ at the r.h.s. of Eq. (46) by the functions renormalized with the help of local subtractions

$$\Pi^{abcd}_{\mu\nu\rho\sigma} \to \Pi^{abcd}_{\mu\nu\rho\sigma} + (z_4 - 1) P(t^{abe} t^{cde} g_{\mu\nu} g_{\rho\sigma}) + (\tilde{z}_4 - 1) P\left(g_{\mu\nu} g\rho\sigma \delta^{ab} \delta^{cd}\right) . \tag{49}$$

The constants z_4 and $\tilde{z}_4$ are defined by the condition that in the limit when a regularization is removed the functions (49) satisfied the differential identity (48). It is always possible in an anomaly free model. Such a substitution does not change the Eq. (46) as the terms $\sim (z_4 - 1)$ and $\sim (\tilde{z}_4 - 1)$ in the first the second and the third lines cancel. We see that the function renormalized according to our prescription in the limit when a regularization is removed coincides with the corresponding function renormalized with the help of local subtractions. The differential STI are fullfilled by construction.

Extension to the higher loops is done exactly as in the case of QED. Introducing the one-loop counterterms we are left with the two-loop diagrams which diverge only superficially. Using the standard R-operation we can define the renormalized two- and higher loop correlation functions by the same equations as above. They are obviously finite. The proof of locality is identical to the one loop-case.

4 A model with chiral fermions

As the last example we consider a simple model which includes chiral fermions. We choose the Abelian sector of Salam-Weinberg model with the Higgs interaction switched off. This model is known to be anomaly free but not a vectorlike one.

The Lagrangian looks as follows:

$$
\begin{aligned}
L = -\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + i\sum_{q\pm} \bar{\psi}_{q\pm}\gamma_\mu(\partial_\mu - ig_{q\pm}B_\mu)\psi_{q\pm} \\
+ i\sum_{l\pm} \bar{\psi}_{l\pm}\gamma_\mu(\partial_\mu - ig_{l\pm}B_\mu)\psi_{l\pm}\,. \qquad (50)
\end{aligned}
$$

Here $\psi_{q\pm}$ and $\psi_{l\pm}$ represent left (right) handed fields of quarks and leptons respectively. The corresponding charges are denoted by $g_{q\pm}$ and $g_{l\pm}$. Due to the condition

$$
\sum_{q\pm} g^3_{q\pm} + \sum_{l\pm} g^3_{l\pm} = 0 \qquad (51)
$$

the triangle anomaly is absent.

Renormalized gauge field and fermion propagators are defined by the Eqs. (6, 12) of Sec. 2. The renormalized gauge field-fermion vertices are defined as in Eq. (7),

$$
\Gamma^{\mu,r}_{q\pm,l\pm} = \Gamma^\mu_{q\pm,l\pm}(p,q) - \Gamma^\mu_{q\pm,l\pm}(p,0) + g_{q\pm,l\pm}\partial_\mu\Sigma^r_{q\pm,l\pm}(p)\,. \qquad (52)
$$

The four-point gauge field correlator can be renormalized according to Eq. (15), but in this case the subtracted term has to include all connected diagrams with four external gauge field lines.

The only essentially new moment is a possible renormalization of the three-point gauge field vertex. In QED it is zero due to Furry theorem, but in our case it does not vanish.

WTI for this function have the same form as for the four-point function

$$p_\mu \Gamma_{\mu\nu\rho}(p,q) = 0; \quad \Gamma_{\mu\nu\rho}(0,q) = 0\,. \tag{53}$$

Accordingly we choose the following definition of renormalized vertex function

$$\Gamma^r_{\mu\nu\rho}(p,q) = \Gamma_{\mu\nu\rho}(p,q) - \Gamma_{\mu\nu\rho}(p,0) - \Gamma_{\mu\nu\rho}(0,q)\,. \tag{54}$$

This function is finite and can be calculated in any regularization scheme. In a gauge invariant scheme one has due to WTI (53)

$$\Gamma^r_{\mu\nu\rho}(p,q) = \Gamma^{inv}_{\mu\nu\rho}(p,q)\,. \tag{55}$$

5 Discussion

The renormalization procedure presented above combines the advantages of algebraic renormalization and invariant regularization schemes. It is regularization independent and may be applied to any anomaly free model. At the same time it preserves gauge invariance at all stages and does not require additional fine tuning of counterterms to restore the symmetry. It also avoids the problem of infrared singularities in renormalization of Green functions.

From the point of view of practical calculations it may be more complicated than dimensional regularization if one is interested only in some particular diagram. Our procedure is recurrent and it requires knowleadge of lower order diagrams for some particular configurations of external momenta. However it seems well suited for systematic calculation of Feynman diagrams up to a given order. It is important to emphasise that our renormalization procedure may be applied directly to the models where dimensional regularization fails, in particular to the Standard Model and supersymmetric theories.

It is worth to mention that to make practical calculations easier one can combine our renormalization procedure with the dimensional regularization. For example in chiral fermion models one may define[5,6] the γ_5 matrix as the product $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$, and then apply the dimensional regularization. For diagrams with virtual fermion lines this definition is known to break chiral invariance. However the symmetry may be easily restored with the help of subtractions described above.

It is worth to stress that our renormalization is defined as the subtraction procedure using the combinatorics of the standard R-operation. It does not require explicit calculation of counterterms. The expressions for counterterms presented in the paper serve the goal to establish the connection with more standard approaches and to facilitate the proofs.

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