To the fond memory of Professor M. S. Marinov

NONLOCAL EXTENSION OF THE BORCHERS CLASSES OF QUANTUM FIELDS

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We formulate an equivalence relation between nonlocal quantum fields, generalizing the relative locality which was studied by Borchers in the framework of local QFT. The Borchers classes are shown to allow a natural extension involving nonlocal fields with arbitrarily singular ultraviolet behavior. Our consideration is based on the systematic employment of the asymptotic commutativity condition which, as established previously, ensures the normal spin and statistics connection as well as the existence of PCT symmetry in nonlocal field theory. We prove the transitivity of the weak relative asymptotic commutativity property generalizing Jost-Dyson's weak relative locality and show that all fields in the same extended Borchers class have the same S–matrix.

Contents

1 Introduction

In this article some new results about the possibility of generalizing the *PCT* theorem to nonlocal interactions are reviewed and applied to the corresponding extension of the Borchers classes of quantum fields. Nonlocal QFT's were extensively studied in the 1970s with the hope of overcoming the problems of non-renormalizable Lagrangian field theories. At that time, I was fortunate to work in ITEP together with M. S. Marinov and I never forget the friendly encouragement of this outstanding person. Gauge theories and constrained dynamics were the main subjects of discussions at a seminar organized by M. S. Marinov for our mathematical group. At the same time, I tried extending the scattering theory of particles to nonlocal interactions. This direction of research was initiated by another eminent scientist Professor N. N. Meiman,[1] who also passed away recently.

At present nonlocal QFT models are interesting first of all in connection with string theory and D–brane theory. This connection is best seen from the holographic point of view.[2] Bounds for the S-matrix derivable from nonlocal field theories could provide a way of investigating the bulk locality properties in the AdS/CFT correspondence after extracting flat-space scattering amplitudes from the boundary conformal field theory correlators. On the other hand, the question of a possible *PCT* invariance violation caused by nonlocality is crucial for phenomenological schemes exploiting propagators with nonlocal form-factors suppressing ultraviolet divergences and proposed as an alternative to string theory, see e.g. Ref. 3. It is equally important to characterize the nonlocal theories that have the same S-matrix as usual ones. The present status of this problem first raised in Ref. 4 and related topics is just what we will discuss below.

The article is organized as follows. Section 2 contains a brief sketch of those tools of modern functional analysis that make it possible to extend the basic results of axiomatic approach[5,6] to nonlocal interactions. Next we formulate an asymptotic commutativity condition which replaces local commutativity and is nearer macrocausality. In Section 3, we outline the proof of a new uniqueness theorem[7,8] for distributions which plays a central role in deriving the *PCT* theorem and the spin-statistics relation for nonlocal quantum fields. The significance of the notion of analytic wave front set to these derivations is explained. In Section 4, we describe some important properties of vacuum expectation values of nonlocal fields which follow from the Lorentz covariance, the spectral condition, and from the fact that the complex Lorentz group contains the total space-time inversion. In Section 5, a condition of weak asymptotic commutativity is defined and its equivalence to the existence of

PCT symmetry is demonstrated. In Section 6 we prove the transitivity of the weak relative asymptotic commutativity and show that this property leads to a natural extension of the Borchers classes of quantum fields. In Section 7 we argue that all local and nonlocal fields belonging to the same extended Borchers class have the same S–matrix. Section 8 contains concluding remarks.

2 Carrier cones of analytic functionals and asymptotic commutativity of nonlocal fields

We will consider field theories in which correlation functions can be so singular in their space-time dependence that these singularities violate locality. It is well known[1] that a breakdown of local commutativity occurs if the Fourier transforms of correlation functions have an exponential growth of order ≥ 1 and, among a variety of nonlocal theories, these seem to be most closely related to string theory which is characterized by an exponentially increasing density of states, see Refs. 9 and 10 for more detailed comments. From a technical point of view, this means that we abandon the usual assumption[5,6] that the vacuum expectation values of products of fields are tempered distributions defined on the Schwartz space S consisting of infinitely differentiable functions of fast decrease. The expectation values are supposed instead to be well defined on test functions analytic in coordinate space, i.e., are regarded as analytic functionals. Then observables emerges only above a definite length scale, but an analysis[11,12] shows that a large class of analytic functionals retain a kind of angular localizability. This is just the property that enables one to develop a self-consistent scattering theory for nonlocal interactions. In Refs. 11, 12 we use the spaces S^0_α introduced by Gelfand and Shilov,[13] which are most suitable for nonperturbative formulation of indefinite metric QFT's and particularly, gauge theories. In nonlocal field theories satisfying the positivity condition, another space $S^0 = S^0_\infty$ is commonly employed. The latter is none other than the Fourier transform of the space $\mathcal{D}$ consisting of infinitely differentiable functions of compact support, and this choice of test functions implies that the expectation values have a finite order of singularity in momentum space. The methods and results of Refs. 11, 12 can be extended to cover analytic functionals defined on S^0 but this is not a trivial exercise because of some topological complications. Nevertheless, we outline this extension in view of the important role of the space $\mathcal{D}$ in the theory of distributions.

To each open cone $U \subset \mathbb{R}^n$, we assign a space $S^0(U)$ consisting of those entire analytic functions on $\mathbb{C}^n$, that satisfy the inequalities

$$|f(z)| \leq C_N\,(1+\|x\|)^{-N}\, e^{b\|y\|+bd(x,U)} \quad (N=0,1,\ldots)\,, \tag{1}$$

where b and C_N are positive constants depending on f, and $d(\cdot, U)$ is the distance from the point to the cone U. (The norm in $\mathbb{R}^n$ is assumed to be Euclidean in what follows.) This space can naturally be given a topology by regarding it as the inductive limit of the family of countably normed spaces $S^{0,b}(U)$ whose norms are defined in accordance with the inequalities (1), i.e.,

$$\|f\|_{U,b,N} = \sup_z |f(z)| \left(1 + \|x\|\right)^N e^{-b\|y\| - bd(x,U)} . \tag{2}$$

For each closed cone $K \subset \mathbb{R}^n$, we also define a space $S^0(K)$ by taking another inductive limit through those open cones U that contain the set $K \setminus \{0\}$ and shrink to it. Clearly, $S^0(\mathbb{R}^n) = S^0$. As usual, we use a prime to denote the continuous dual of a space under consideration. A closed cone $K \subset \mathbb{R}^n$ is said to be a *carrier* of a functional $v \in S'^0$ if v has a continuous extension to the space $S^0(K)$, i.e., belongs to $S'^0(K)$. We refer to Refs. 11, 12 for a motivation of this definition and for its connection with the Sato-Martineau theory of hyperfunctions. As is seen from estimate (1), this property may be thought of as a fast decrease (no worse than an exponential decrease of order 1 and maximum type) of v in the complement of K. It should also be emphasized that if v is a tempered distribution with support in K, then the restriction $v|S^0$ is carried by K.

We list the basic facts which allow handling the analytic functionals of class S'^0, in most cases, as easily as tempered distributions.

1. *The spaces $S^0(U)$ are Hausdorff and complete. A set $B \subset S^0(U)$ is bounded if and only if it is contained in some space $S^{0,b}(U)$ and is bounded in each of its norms.*

The proof given in Ref. 14 relies on the acyclicity of the injective sequence of Fréchet spaces $S^{0,b}(U)$.

2. *The space S^0 is dense in every $S^0(U)$ and in every $S^0(K)$.*

As shown in Ref. 7 this follows from an analogous theorem proved for S^0_α in Ref. 12

3. *If a functional $v \in S'^0$ is carried by each of closed cones K_1 and K_2, then it is carried by their intersection.*

Because of this, there is a smallest K such that $v \in S'^0(K)$. The proof is similar to that given for S'^0_α in Ref. 11 but appeals besides to the topological Lemma 5.11 in Ref. 15.

4. *If $v \in S'^0(K_1 \cap K_2)$, then $v = v_1 + v_2$, where $v_j \in S'^0(U_j)$ and U_j are any open cones such that $U_j \supset K_j \setminus \{0\}$, $j = 1, 2$.*

This theorem can also be proved in a manner similar to that of,[11] but using auxiliary conic neighborhoods with additional regularity properties specified in Ref. 15. Within the framework of functional spaces S'^0_α, a stronger decomposition property holds with v_j carried by the cones K_j themselves; such an improvement is also possible for S'^0 if $K_1 \cap K_2 = \{0\}$, see Theorem 5 in Ref. 14. For applications to QFT, the case of a properly convex cone K is of special interest. Then the dual cone $K^* = \{\eta : \eta x \geq 0, \quad \forall x \in K\}$ has a nonempty interior and $e^{i\zeta x} \in S^0(K)$ for all $\operatorname{Im} \zeta \in \operatorname{int} K^*$. Therefore, the Laplace transformation

$$\check{\mathbf{v}}(\zeta) = (2\pi)^{-n}(v, e^{i\zeta x}) \tag{3}$$

is well defined for each $v \in S'^0(K)$. It is easily verified that the function (3) is analytic in the tubular domain $\mathbb{R}^n + i \operatorname{int} K^*$ and satisfies the estimate

$$|\check{\mathbf{v}}(\zeta)| \leq C_{R,V'} |\operatorname{Im} \zeta|^{-N_R} \qquad (\operatorname{Im} \zeta \in V', \ |\zeta| \leq R) \tag{4}$$

for any $R > 0$ for each cone V' such that $\overline{V}' \setminus \{0\} \subset \operatorname{int} K^*$. What is more, the above results make it possible to establish the following theorem of the Paley-Wiener-Schwartz type.

5. *For every properly convex closed cone $K \subset \mathbb{R}^n$, the Laplace transformation isomorphically maps $S'^0(K)$ onto the space of functions analytic in the tube $\mathbb{R}^n + i \operatorname{int} K^*$ and satisfying* (4). *The Fourier transform of v is a boundary value of the function* (3), *i.e., $\check{\mathbf{v}}(\zeta)$ converges to $\check{v}$ in $\mathcal{D}'$ as $\operatorname{Im} \zeta \to 0$, $\operatorname{Im} \zeta \in V'$.*

We will consider a finite family of fields $\{\phi_\iota\}$ that are operator-valued generalized functions defined on the test function space $S^0(\mathbb{R}^4)$ and transform according to irreducible representations of the proper Lorentz group $L_+^\uparrow$ or its covering group $SL(2, \mathbb{C})$. We adopt all the standard assumptions of the Wightman axiomatic approach[5,6] except local commutativity which cannot be formulated in terms of the analytic test functions. It should be noted that, using S^0, we do not impose any restrictions on the high-energy (ultraviolet) behavior of fields because the test functions are of compact support in momentum space. In this sense, the space S^0 is universal for nonlocal fields. We denote by D_0 the minimal common invariant domain, which is assumed to be dense, of the field operators in the Hilbert space $\mathcal{H}$ of states, i.e., the vector subspace of $\mathcal{H}$ that is spanned by the vacuum state Ψ_0 and by various vectors of the form

$$\phi_{\iota_1 \ell_1}(f_1) \ldots \phi_{\iota_n \ell_n}(f_n) \Psi_0 \qquad (n = 1, 2, \ldots),$$

where $f_k \in S^0(\mathbb{R}^4)$ and ℓ_k are the Lorentzian indices. The space S^0, being Fourier-isomorphic to $\mathcal{D}$, is nuclear. Therefore, the n-point vacuum expectation values uniquely determine Wightman generalized functions $\mathcal{W}_{\iota_1 \ell_1, \ldots, \iota_n \ell_n} \in S'^0(\mathbb{R}^{4n})$ and we identify these objects, just as in the standard scheme.[5,6] The property of nuclearity allows us to define as well the expressions

$$\int \phi_{\iota_1 \ell_1}(x_1) \ldots \phi_{\iota_n \ell_n}(x_n) \, f(x_1, \ldots, x_n) \, \mathrm{d}x_1 \ldots \mathrm{d}x_n \Psi_0 \qquad (n = 1, 2, \ldots), \tag{5}$$

where $f \in S^0_\alpha(\mathbb{R}^{4n})$, and to verify that every operator $\phi_{\iota\ell}(f)$ can be extended to the subspace $D_1 \supset D_0$ spanned by vectors (5).

The foregoing inspires the following definition.

Definition 1. The field components $\phi_{\iota\,\ell}$ and $\phi_{\iota'\ell'}$ commute (anticommute) asymptotically for large spacelike separation of their arguments if the functional

$$\langle \Phi,\, [\phi_{\iota\,\ell}(x), \phi_{\iota'\ell'}(x')]_{\substack{-\\(+)}} \Psi\rangle \tag{6}$$

is carried by the cone $\overline{\mathbb{W}} = \{(x,x') \in \mathbb{R}^8 : (x-x')^2 \geq 0\}$ for any vectors $\Phi, \Psi \in D_0$.

We replace the local commutativity axiom by the *asymptotic commutativity* condition which means that any two field components either commute or anticommute asymptotically. This condition is evidently weaker than local commutativity in the sense that it is certainly fulfilled for the restrictions of local tempered distribution fields to $S^0(\mathbb{R}^4)$. The standard considerations of Lorentz covariance imply that the type of the commutation relation depends only on the type of the participating fields, not on their Lorentzian indices, and we drop these indices in what follows.

Lemma 3 in Ref. 8 shows that under the stated condition, the functional

$$\mathcal{W}_{\iota_1,\dots,\iota_n}(x_1,\dots,x_k,x_{k+1},\dots,x_n) - \mathcal{W}_{\iota_1,\dots,\iota_{k+1},\iota_k,\dots,\iota_n}(x_1,\dots,x_{k+1},x_k,\dots,x_n)\,, \tag{7}$$

where the sign $-$ or $+$ corresponds to the type of commutation relation between ϕ_{ι_k} and $\phi_{\iota_{k+1}}$, is carried by the cone $\overline{\mathbb{W}}_{n,k} = \{x \in \mathbb{R}^{4n} : (x_k - x_{k+1})^2 \geq 0\}$. It follows, in particular, that if the asymptotic commutativity condition is satisfied for $\Phi, \Psi \in D_0$, then it is also satisfied for $\Phi, \Psi \in D_1$. Moreover, it is fulfilled for any $\Phi \in \mathcal{H}$, $\Psi \in D_1$. In other words, $\overline{\mathbb{W}}$ is a carrier of the vector-valued functional

$$\Xi(f) = \int [\phi_\iota(x), \phi_{\iota'}(x')]_{\substack{-\\(+)}} f(x,x')\,\mathrm{d}x\mathrm{d}x'\,\Psi \qquad (\Psi \in D_1) \tag{8}$$

defined on $S^0(\mathbb{R}^8)$. Indeed, $\|\Xi(f)\|^2 = \mathcal{B}(\bar{f}, f)$, where $\mathcal{B}$ is a separately continuous bilinear form on $S^0(\mathbb{R}^8)$ with the property that both linear functionals determined by it when one of its two arguments is held fixed are carried by $\overline{\mathbb{W}}$. A consideration analogous to that of Lemma 3 in Ref. 8 shows that any bilinear form possessing this property is identified with an element of the space $S'^0(\overline{\mathbb{W}} \times \overline{\mathbb{W}})$. The main point of the argument is the equality $S^0(U \times \mathbb{R}^d) = S^0(U)\,\hat{\otimes}_i\,S^0(\mathbb{R}^d)$, where the tensor product is equipped with the inductive topology and the hat means completion. If $f_\nu \in S^0(\mathbb{R}^8)$ and $f_\nu \to f \in S^0(\overline{\mathbb{W}})$, then $(\bar{f}_\nu - \bar{f}_\mu) \otimes (f_\nu - f_\mu) \to 0$ in the topology of

$S^0(\overline{\mathbb{W}}\times\overline{\mathbb{W}})$ as $\nu, \mu \to \infty$. Therefore, the sequence $\Xi(f_\nu)$ converges strongly in $\mathcal{H}$ and functional (8) has a continuous extension to $S^0(\overline{\mathbb{W}})$.

3 Analytic wave front set and carriers

The classical derivation of the spin-statistics relation and PCT symmetry in local QFT[5,6] is based on exploiting the analytic properties of vacuum expectation values in x-space and substantially employs a uniqueness theorem of complex analysis which asserts that if a function $\mathbf{w}(z)$ ($z = x + iy \in \mathbb{C}^n$) is analytic on a tubular domain whose basis is an open connected cone and if its boundary value on $\mathbb{R}^n$ vanishes in a nonempty open set, then $\mathbf{w}$ is identically zero. By using this theorem, it is easy to derive another uniqueness theorem which shows that if a tempered distribution $u \in S'$ has support in a properly convex cone V and its Fourier transform $\hat{u}$ vanishes on an open set, then $u \equiv 0$. Indeed, by the Paley-Wiener-Schwartz theorem (see, e.g., Theorem 2.9 in Ref. 5) the support condition implies the existence of the Laplace transform $\hat{\mathbf{u}}(z) = (u, e^{-(\cdot,z)})$ analytic in the tube $\mathbb{R}^n - i\,\mathrm{int}\,V^*$ and whose boundary value is $\hat{u}$. In Refs. 7, 8 a natural analog of the latter uniqueness property is established for distributions in $\mathcal{D}'$, which provides a means for the extension of the spin-statistics and PCT theorems to nonlocal fields with arbitrary high-energy behavior.

Theorem 1. *Let $u \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution whose support is contained in a properly convex cone V. If its Fourier transform $\hat{u} \in S'^0(\mathbb{R}^n)$ is carried by a cone different from $\mathbb{R}^n$, then $u \equiv 0$.*

We will not repeat the proof given in Refs. 7, 8 but we explain the role that is played by the notion of analytic wave front set $WF_A(u)$ in the derivation of this theorem. Recall,[16] that $WF_A(u)$ consists of pairs (p, ξ), where p ranges the smallest closed subset of $\mathbb{R}^n$ outside of which the distribution u is analytic. This subset is denoted by $\mathrm{sing\,supp}_A u$, and for every $p \in \mathrm{sing\,supp}_A u$, the vector ξ ranges the closed cone in $\mathbb{R}^n \setminus \{0\}$ formed by those directions of a "bad" behavior of the Fourier transform $\hat{u}$ at infinity that are responsible for the nonanalyticity of u at the point p. If a closed cone $K \subset \mathbb{R}^n$ is a carrier of $\hat{u}$, then [a]

$$WF_A(u) \subset \mathbb{R}^n \times (K \setminus \{0\})\,. \tag{9}$$

In other words, the directions external to carrier cone cannot be responsible for singularities of u. Formula (9) improves considerably Lemma 8.4.17 in Ref. 16 which asserts that, for each tempered distribution $u \in S'$, the inclusion $WF_A(u) \subset \mathbb{R}^n \times (L \setminus \{0\})$ holds, where L is the limit cone of the set $\mathrm{supp}\,\hat{u}$

[a] It should be noted that the operator $u \to \hat{u}$ is dual of the test function transformation $f(x) \to \int e^{-ipx} f(x)\,dx$, with the minus sign in the exponent, as opposite to $v \to \check{v}$.

at infinity. This cone consists of the limits of various sequences $t_\nu x_\nu$, where $x_\nu \in \operatorname{supp} \hat{u}$ and $0 < t_\nu \to 0$, and it is certainly a carrier of the restriction $\hat{u}|S^0$. The derivation of (9) is based on the employment of the decomposition theorem and the Paley-Wiener-Schwartz-type theorem stated in Section 2. By using this inclusion, we can easily prove Theorem 1 in the simplest case when $0 \in \operatorname{supp} u$. Then every vector in the cone $-V^* \setminus \{0\}$ is an external normal to the support at the point 0. By Theorem 9.6.6 of Ref. 16 all the nonzero elements of the linear span of external normals belong to $WF_A(u)_{p=0}$. Because the cone V is properly convex, the interior of V^* is not empty, and this linear span covers $\mathbb{R}^n$. Therefore, each carrier cone of $\hat{u}$ must coincide with $\mathbb{R}^n$. The general case can be reduced to this special case by considering the series of "contracted" ultradistributions

$$\sum_{\nu=1}^{\infty} c_\nu u_\nu, \qquad (u_\nu, g(p)) \stackrel{\text{def}}{=} \nu^{-n}(u, g(p/\nu)) . \tag{10}$$

As shown in Refs. 7, 8 the coefficients c_ν can be chosen such that this series converges in $\mathcal{D}'$ to a distribution whose support contains the point 0 and whose Fourier transform is carried by the same cones that $\hat{u}$ is.

We present here another result related to Theorem 1 and concerning Fourier hyperfunctions. Recall that this name is used for the continuous linear functionals defined on the Fourier-symmetric space S^1_1 of functions analytic in an complex l–neighborhood of the real space and satisfying the estimate $|f(z)| \le C\, e^{b\|z\|}$, where l, b, and C depend on f. Such functionals, as well as their Fourier transforms, have uniquely defined supports and are used in the most general formulation of local QFT, see Ref. 17.

Theorem 2. *Let* $u \in S'^1_1(\mathbb{R}^n \times \mathbb{R}^m)$ *and* $\operatorname{supp} u \subset V \times \mathbb{R}^m$*, where* V *is a properly convex cone in* $\mathbb{R}^n$*. Then* $\operatorname{supp} \hat{u} = \mathbb{R}^n \times M$*, where* M *is a closed set in* $\mathbb{R}^m$*.*

An analogous theorem holds evidently for tempered distributions. Indeed, if $x \notin \operatorname{supp} \hat{u}$, then there is a neighborhood of x of the form $U_1 \times U_2$ and not meeting $\operatorname{supp} \hat{u}$. Let $u \in S'$ and $f \in \mathcal{D}(U_2)$. Then the distribution $\int \hat{u}(x_1, x_2)\, f(x_2)\, \mathrm{d}x_2$ vanishes in U_1 and its (inverse) Fourier transform has support in the convex cone V. Therefore, this distribution is identically zero and we conclude that $\operatorname{supp} \hat{u}$ does not meet $\mathbb{R}^n \times U_2$ because any test function localized in this region can be approximated by functions belonging to $\mathcal{D}(\mathbb{R}^n) \otimes \mathcal{D}(U_2)$.

This argument is unapplicable to Fourier hyperfunctions because S^1_1 does not contain any functions of compact support. However, we may make use of the inclusions

$$N(\operatorname{supp} \hat{u}) \subset WF_A(\hat{u}) \subset \mathbb{R}^{n+m} \times (V \setminus \{0\}) , \tag{11}$$

where $N(\cdot)$ is the set of normals. The first of inclusions (11) holds according to,[16] and the second can be proved in complete analogy to the derivation of (9), because a suitable decomposition theorem and a Paley-Wiener-Schwartz-type theorem are well known for Fourier hyperfunctions, see Propositions 2.7 and 2.9 in Ref. 17. The set $N(\cdot)$ includes both external and internal normals, whereas the cone V is properly convex and does not contain a straight line. Therefore, (11) implies that any normal to the closed set $\operatorname{supp}\hat{u}$ has zero projection on $\mathbb{R}^n$ and so this set must be of the form indicated above.

A few words of explanation concerning the last conclusion are, perhaps, necessary. Let X be a closed set in $\mathbb{R}^n \times \mathbb{R}^m$, $y \notin X$, and let z be a point in X with minimal distance to y. Then $y - z \in N(X)|_z$ by the definition.[16] We know that for any y, the projection of $y - z$ on R^n is zero and we need to show that then $x = (x_1, x_2) \in X$ implies $x' = (x'_1, x_2) \in X$ for any $x'_1 \in \mathbb{R}^n$. Assume contrarily that there is a point $x' \notin X$ of this form. For convenience, we suppose that $x_2 = 0$ and identify $\mathbb{R}^n$ with its canonical image in $\mathbb{R}^n \times \mathbb{R}^m$. Let B be the open ball in $\mathbb{R}^n$ which is centred at x'_1 and whose radius is equal to the distance from x'_1 to $X \cap \mathbb{R}^n$. For every $y \in B$, let $r(y) > 0$ be the squared distance of y from X. By our assumption, any point $z \in X$ such that $\|y-z\|^2 = r(y)$ is of the form $z = (y, z_2)$ and hence $\|z_2\|^2 = r(y)$. Let $z' \in X$ be such that $\|y' - z'\|^2 = r(y')$. Then $r(y) \le \|y - z'\|^2 = \|y - y'\|^2 + r(y')$. Analogously, $r(y') \le \|y - y'\|^2 + r(y)$. Therefore, the function $r(y)$ is differentiable and its derivative is identically zero, i.e., this function is constant, but this contradicts that it must tends to zero as y nears a boundary point of B belonging to X. This completes the proof.

4 Jost points in nonlocal QFT

In local field theory, Jost points are real points of the extended domain of analyticity of the Wightman functions $\mathcal{W}(x_1, \dots, x_n)$. The Bargman-Hall-Wightman theorem shows that this extension is obtained by applying various complex Lorentz transformations to the primitive domain of analyticity determined by the spectral condition. We recall[5,6] that $x \in \mathbb{R}^{4n}$ belongs to the extended domain if and only if the convex cone generated in $\mathbb{R}^4$ by the points $\xi_k = x_k - x_{k+1}$, $k = 1, \dots, n-1$, contains only spacelike vectors. In other words, the set of Jost points is the open cone

$$\mathcal{J}_n = \left\{ x \in \mathbb{R}^{4n} : \ \left(\sum_{k=1}^{n-1} \lambda_k (x_k - x_{k+1}) \right)^2 < 0 \quad \forall\, \lambda_k \ge 0,\ \sum_{k=1}^{n-1} \lambda_k > 0 \right\}. \quad (12)$$

The covariance with respect to the action of the complex Lorentz group $L_+(\mathbb{C})$ implies the following transformation rule for the analytic Wightman functions under the total space-time inversion

$$\mathcal{W}_{\iota_1\ldots\iota_n}(z_1,\ldots,z_n) = (-1)^{2J}\mathcal{W}_{\iota_1\ldots\iota_n}(-z_1,\ldots,-z_n)\,, \tag{13}$$

where J is the total number of unpointed indices of the fields involved in the vacuum expectation value. If the expectation values grow in momentum space faster than exponentially of order 1, then the analyticity domain in coordinate space is empty because the Laplace transformation does not exist for such functions. However, as first observed by Lücke,[18] the symmetry (13) acts in a hidden manner even in this essentially nonlocal case and leads to important consequences.

Theorem 3. *Let $\{\phi_\iota\}$ be a family of fields defined on the test function space $S^0(\mathbb{R}^4)$ and satisfying all Wightman axioms except possibly locality. Assume that they transform according to the irreducible representations (j_ι, k_ι) of the group $SL(2,\mathbb{C})$ and let $\mathcal{W}_{\iota_1\ldots\iota_n}$ be the generalized function determined by the n-point vacuum expectation value $\langle\Psi_0,\,\phi_{\iota_1}(x_1)\ldots\phi_{\iota_n}(x_n)\Psi_0\rangle$. Then the complement $\mathbf{C}\mathcal{J}_n$ of Jost cone is a carrier of the functional*

$$\mathcal{W}_{\iota_1\ldots\iota_n}(x_1,\ldots,x_n) - (-1)^{2J}\mathcal{W}_{\iota_1\ldots\iota_n}(-x_1,\ldots,-x_n)\,, \tag{14}$$

where $J = j_{\iota_1} + \ldots + j_{\iota_n}$.

Proof. We pass to the difference variables ξ_k and to the generalized functions $W_{\iota_1\ldots\iota_n}$ connected with $\mathcal{W}_{\iota_1\ldots\iota_n}$ by the relation

$$(\mathcal{W},\,f) = \left(W,\,\int f(t^{-1}\xi)\,\mathrm{d}\xi_n\right), \tag{15}$$

where $t\,:\,(x_1,\ldots,x_n)\to(\xi_1 = x_1 - x_2,\ldots,\xi_{n-1} = x_{n-1}-x_n, \xi_n = x_n)$. The correspondence $W\to\mathcal{W}$ defined by (15) is an injective mapping $S'^0(\mathbb{R}^{4(n-1)})\to S'^0(\mathbb{R}^{4n})$ which is evidently continuous under the weak topologies of these spaces. It is easily verified that every translation-invariant functional belongs to its range. Furthermore, Lemma 4 in Ref. 8 shows that $W\in S'^0(U)$, with U an open cone in $\mathbb{R}^{4(n-1)}$, if and only if $\mathcal{W}\in S'^0(\mathcal{U})$, where $\mathcal{U} = \{x\in\mathbb{R}^{4n} : (x_1-x_2,\ldots,x_{n-1}-x_n)\in U\}$. We regularize the ultraviolet behavior of W by multiplying its Fourier transform $\check{W}$ with $\omega_M(p) = \omega((P\cdot P)/M^2)$, where $P = \sum_{k=1}^n p_k$, the momentum-space variables p_k are conjugates of ξ_k, the inner product is Minkowskian, and $\omega(t)$ is a smooth function with support in the interval $(-1,1)$ and identically equal to 1 for $|t|\le 1/2$. Clearly ω_M is a multiplier for $S_0 = \mathcal{D}$ and, for every $u\in\mathcal{D}'$, $u\,\omega_M$ tends to u in $\mathcal{D}'$ as $M\to\infty$. From the spectral condition, it follows that $\check{W}_M = \check{W}\,\omega_M$ has a continuous extension to the Schwartz space S, i.e., is a tempered distribution. Namely, let us denote the closed forward light cone by $\overline{\mathbb{V}}_+$ and prove the following lemma:

Lemma 1. *Let $u \in \mathcal{D}'(\mathbb{R}^{4n})$ be a Lorentz-covariant distribution whose support is contained in the cone $\overline{\mathbb{V}}_+ \times \ldots \times \overline{\mathbb{V}}_+ = \overline{\mathbb{V}}_+^n$. If $\omega \in \mathcal{D}(\mathbb{R})$ and $P = \sum_{k=1}^n p_i$, then $u\,\omega(P\cdot P) \in S'(\mathbb{R}^{4n})$.*

Proof. We denote by T the representation according to which u transforms under the action of $L_+^\uparrow$. With a basis fixed in the representation space, u can be identified with the system of distributions $u^i \in \mathcal{D}'$; the number of these is equal to the dimension of the representation. A distribution belonging to $\mathcal{D}'$ has a continuous extension to S if and only if its convolution with any test function $g \in \mathcal{D}$ has no worse than a power growth at infinity. (This simple and convenient criterion of extendability is proved in Ref. 19 § II.10.7.) The value of the convolution $(u^i\,\omega{*}g)$ at a point q is the value taken by the distribution u^i on the shifted function $g(p-q)$. We can assume, without loss of generality, that $\operatorname{supp} g$ lies in the set $\{p \in \mathbb{R}^{4n} : \|p\| < 1/n\}$, where $\|\cdot\|$ is the Euclidean norm. Let $Q = \sum_{k=1}^n q_k$. We need only consider the shifts along the surface $Q\cdot Q = 0$ because for the other directions, $(u^i\,\omega * g)(q)$ vanishes for sufficiently large $\|q\|$. It can be additionally assumed that $Q^2 = Q^3 = 0$, because any vector in $\mathbb{R}^4$ is converted to this form by an appropriate spatial rotation. We now use the light-cone variables $Q^\pm = (Q^0 \pm Q^1)/\sqrt{2}$ and set $Q^- = 0$, $Q^+ \to +\infty$ for definiteness. Let Λ be the transformation $p_k^+ \to p_k^+/Q^+$, $p_k^- \to Q^+ p_k^-$, $k = 1, \ldots, n$. In view of the Lorentz covariance of u and invariance of ω, we have

$$(u^i\,\omega{*}g)(q) = \sum_j T^i_j(\Lambda)(u^j, g_q), \quad \text{where} \quad g_q(p) = \omega(P\cdot P)\, g(q-\Lambda^{-1}p)\,. \tag{16}$$

The points of $\operatorname{supp} g_q$ satisfy the inequalities $|P\cdot P| < 1$ and $(P^2)^2 + (P^3)^2 < 1$ by construction and hence $|P^+P^-| < 1$. Furthermore $|Q^+ - Q^+P^+| < 1$. Therefore, if Q^+ is sufficiently large, $\operatorname{supp} g_q$ is contained in the set $\|P\| < 2$. We fix a neighborhood $\mathcal{V}$ of the support of u by taking the union of a neighborhood of the origin with the product V^n, where V is an open properly convex cone in $\mathbb{R}^4$ containing $\overline{\mathbb{V}}_+ \setminus \{0\}$. For the points of V^n, the inequality $\|p\| < \theta\|P\|$ holds with some constant $\theta > 0$, because otherwise we could find a sequence of points $p_{(\nu)} \in V^n$ such that $\|p_{(\nu)}\| = 1$ and $\|P_{(\nu)}\| < 1/\nu$. Then we could choose a convergent subsequence whose limit $\bar{p}$ is a nonzero vector in $\overline{V}^n$ such that $|\bar{P}| = 0$, which contradicts the assumption that the cone V is properly convex. Therefore, the set $\operatorname{supp} g_q \cap \mathcal{V}$ lies in the ball of radius 2θ and we have the estimate

$$|(u^j,\, g_q)| \le C_j \|g_q\|_{2\theta, N_j}\,, \tag{17}$$

where $\|g_q\|_{2\theta,N} = \max_{|\kappa|\leq N} \sup_{\|p\|\leq 2\theta} |\partial^\kappa g_q(p)|$ and N_j has the meaning of the singularity order of the distribution u^j in this ball. The transformation Λ^{-1} contracts the graph of g by Q^+ times with respect to every variable p_k^+. Therefore,

$$\sup_p |\partial^\kappa g(q - \Lambda^{-1}p)| = \sup_p |\partial^\kappa g(\Lambda^{-1}p)| \leq C_\kappa Q^{+|\kappa|}$$

and, consequently,

$$\|g_q\|_{2\theta,N} \leq C_N\,(1 + \|q\|)^N. \tag{18}$$

Taking into account that the representation matrix elements $T_k^j(\Lambda)$ are rational functions of the boost parameter Q^+ and combining (16)–(18), we conclude that the function $(u^i\omega * g)(q)$ is polynomially bounded. Lemma 1 is thus proved.

We return to the proof of Theorem 3 and consider the (inverse) Laplace transform $\mathbf{W}_M$ of the distribution $\check{W}_M$. It is holomorphic on the usual tube $\mathbb{T}_{n-1} = \mathbb{R}^{4(n-1)} - i\mathbb{V}_+^{(n-1)}$ and W_M is its boundary value. Since the regularization preserves the Lorentz covariance, we can apply the Bargman-Hall-Wightman theorem,[5,6] which shows that $\mathbf{W}_M$ allows an analytic continuation into the extended domain $\mathbb{T}_{n-1}^{\mathrm{ext}}$ and the continued function is covariant under the complex Lorentz group $L_+(\mathbb{C})$. Therefore, the function $\mathcal{W}_M(z_1,\ldots,z_n) = \mathbf{W}_M(z_1 - z_2,\ldots,z_{n-1} - z_n)$ satisfies (13) in the corresponding analyticity domain. As a consequence, the tempered distribution

$$F_M \stackrel{\text{def}}{=} \mathcal{W}_M(x_1,\ldots,x_n) - (-1)^{2J}\mathcal{W}_M(-x_1,\ldots,x_n)$$

vanishes in the Jost cone and its restriction to S^0 is carried by $\mathbf{C}\mathcal{J}_n$. Moreover, it has a continuous extension to the space $S^0(U)$ associated with the open cone $U = \mathbf{C}\overline{\mathcal{J}}_n$, i.e., with the interior of the cone complementary to the Jost cone. In fact, this extension $\tilde{F}_M$ can be defined by $(\tilde{F}_M, f) = (F_M, \chi f)$, where χ is a multiplier for the Schwartz space S which is identically equal to 1 in an ϵ-neighborhood of $\mathbf{C}\mathcal{J}_n$ and vanishes outside the 2ϵ-neighborhood. Such a multiplier satisfies the estimate $|\partial^q\chi(x)| \leq Ch^{|q|}$ and, for any function $f \in S^0(U)$, we have the inequalities $|\partial^q f(x)| \leq C'\,\|f\|_{b,N}\, b^{|q|}(1+\|x\|)^{-N}$ which hold on $\operatorname{supp}\chi$. Hence, the multiplication by χ continuously maps $S^0(U)$ into S. It is important that the extensions $\tilde{F}_M$ are compatible with each other if M and M' are large enough compared to b, namely,

$$\tilde{F}_M | S^{0,\,b}(U) = \tilde{F}_{M'} | S^{0,\,b}(U)\,. \tag{19}$$

To prove (19), we use the density theorem mentioned in Section 2. Its more detailed formulation given in Refs. 7, 12 shows that there is a constant c such that for $b' \geq cb$, the space $S^{0,\,b'}$ is dense in $S^{0,\,b}(U)$ in the topology of $S^{0,\,b'}(U)$. Let $M, M' > 2cb$, $f \in S^{0,\,b}(U)$, $f_\nu \in S^{0,\,cb}$, and let $f_\nu \to f$ in $S^{0,\,cb}(U)$. Then $f_\nu \in S^{0,\,M/2} \cap S^{0,\,M'/2}$ and we have the estimate $|f_\nu(z)| \leq C_{\nu,N} e^{\min(M,M')\|y\|/2}(1 + \|x\|)^{-N}$, which implies that supp $\check{f}_\nu$ are contained in the ball $\|p\| \leq \min(M, M')/2$, where both regularizing multipliers ω_M, $\omega_{M'}$ are equal to 1 by construction. Therefore, $(\tilde{F}_M, f) = (\tilde{F}_{M'}, f) = \lim_{\nu\to\infty}(F, f_\nu)$, where $F = \mathcal{W}(x) - (-1)^{2J}\mathcal{W}(-x)$. Thus, there exists a continuous extension of the functional F to $S^0(U)$. This completes the proof.

5 Generalization of the PCT theorem

We now formulate an analog of the Jost-Dyson weak locality condition.

Definition 2. Let $\{\phi_\iota\}$ be a family of quantum fields defined on the test function space $S^0(\mathbb{R}^4)$, and with a common invariant domain in the Hilbert space of states. We say that $\{\phi_\iota\}$ satisfies the weak relative asymptotic commutativity condition if for each system of indices $\iota_1, \ldots, \iota_n$, the functional

$$\langle \Psi_0,\, \phi_{\iota_1}(x_1)\ldots\phi_{\iota_n}(x_n)\Psi_0\rangle - i^F \langle \Psi_0,\, \phi_{\iota_n}(x_n)\ldots\phi_{\iota_1}(x_1)\Psi_0\rangle\,, \tag{20}$$

where F is the number of fields with half-integer spin in the monomial $\phi_{\iota_1}\ldots\phi_{\iota_n}$, is carried by the complement $\mathbf{C}\mathcal{J}_n$ of Jost cone.

This condition is certainly fulfilled in theories with the normal asymptotic commutation relations because Jost points are totally spacelike, i.e, $(x_1, \ldots, x_n) \in \mathcal{J}_n$ implies $(x_j - x_k)^2 < 0$ for all $j \neq k$.

Theorem 4. *Assume we are dealing with quantum fields $\{\phi_\iota\}$ defined on $S^0(\mathbb{R}^4)$ and satisfying all Wightman axioms except possibly locality. Let ϕ_ι transform according to the irreducible representation (j_ι, k_ι) of the group $SL(2,\mathbb{C})$. Then the weak relative asymptotic commutativity condition is equivalent to the existence of an antiunitary PCT-symmetry operator Θ which leaves the vacuum invariant and acts on the fields according to the rule*

$$\Theta\,\phi_\iota(x)\,\Theta^{-1} = (-1)^{2j_\iota}\, i^{F_\iota}\, \phi_\iota(-x)^*\,, \tag{21}$$

where F_ι is the spin number of ϕ_ι.

Proof. Let the PCT symmetry hold and an operator Θ with the listed properties exist. Let us consider the vacuum expectation value of the monomial $\phi_{\iota_1}(x_1)\cdots\phi_{\iota_n}(x_n)$. Applying (21) and using the invariance of Ψ_0 and the antiunitary property of Θ^{-1}, we obtain the relation

$$\mathcal{W}_{\iota_1\ldots\iota_n}(x_1, \ldots, x_n) = (-1)^{2J}\, i^F\, \mathcal{W}_{\iota_n\ldots\iota_1}(-x_n, \ldots, -x_1)\,, \tag{22}$$

where $J = j_{\iota_1} + \ldots + j_{\iota_n}$, $F = F_{\iota_1} + \ldots + F_{\iota_n}$. In deriving (22), it should be kept in mind that the expectation value is zero for odd F and that $(-i)^F = i^F$

for even F. Subtracting the functional $(-1)^{2J}\,\mathcal{W}_{\iota_1\dots\iota_n}(-x_1,\dots,-x_n)$ from the left-hand and right-hand sides of (22) and applying Theorem 3, we conclude that the weak relative asymptotic commutativity condition is fulfilled. Conversely, if $\{\phi_\iota\}$ satisfies this condition, then the difference $\mathcal{W}_{\iota_1\dots\iota_n}(x_1,\dots,x_n)-(-1)^{2J}\,i^F\,\mathcal{W}_{\iota_n\dots\iota_1}(-x_n,\dots,-x_1)$ is representable as a sum of two functionals carried by the cone $\mathbf{C}\mathcal{J}_n \neq \mathbb{R}^{4n}$. Its Fourier transform has support in the properly convex cone

$$\left\{p\in\mathbb{R}^{4n} : \sum_{k=1}^{n} p_k = 0, \quad \sum_{k=1}^{l} p_k \in \overline{\mathbb{V}}_+, \quad l=1,\dots,n-1\right\}$$

determined by the spectral condition. Therefore, equality (22) holds identically by Theorem 1. The operator Θ can now be constructed in the ordinary way. First we define it on those vectors that are obtained by applying monomials in fields to the vacuum. Namely, we set

$$\Theta\Psi_0=\Psi_0, \quad \Theta\phi_{\iota_1}(f_1)\dots\phi_{\iota_n}(f_n)\Psi_0 = (-1)^{2J}\,i^F\,\phi_{\iota_1}(f_1^-)^*\dots\phi_{\iota_n}(f_n^-)^*\Psi_0\,,$$

where $f^-(x)=f(-x)$. It easily seen that Θ is well defined. In fact, taking into account that ϕ_ι^* transforms according to the conjugate representation (k_ι, j_ι), we see that (22) implies the relation $\langle\Theta\Phi,\,\Theta\Psi\rangle = \overline{\langle\Phi,\,\Psi\rangle}$ for vectors of this special form. Therefore, if a vector Ψ is generated by different monomials $M_1(f_1)$, $M_2(f_2)$, then the scalar product $\langle\Theta M_1\Psi_0,\,\Theta M_2\Psi_0\rangle$ is equal to squared length of each of the vectors $\Theta M_1\Psi_0$, $\Theta M_2\Psi_0$, i.e., these vectors coincide. Analogously, if $\Psi=\Psi_1+\Psi_2$, where all vectors are obtained by applying monomials to Ψ_0, then $\Theta\Psi=\Theta\Psi_1+\Theta\Psi_2$. Therefore, Θ can be extended to D_0 by antilinearity. A further extension by continuity yields an antiunitary operator on $\mathcal{H}$. This completes the proof.

Theorem 15 in Ref. 8 shows that the asymptotic commutativity condition stated in Section 2 guarantees the existence of a Klein transformation reducing the commutation relations to the normal form. Because of this, the PCT symmetry holds in the nonlocal theories satisfying this condition. The proof given above shows also that this symmetry holds even if the difference (20) is carried by the complement of a cone generated by an arbitrarily small real neighborhood of a Jost point and that then this functional is necessarily carried by $\mathbf{C}\mathcal{J}_n$.

6 Transitivity of weak relative asymptotic commutativity

Theorem 5. *Let $\{\phi_\iota\}$ be a family of fields satisfying the assumptions of Theorem 4 and Θ be the corresponding PCT-symmetry operator. Let ψ be another* *field with the same domain of definition and transforming according to the representation (j,k) of $SL(2,\mathbb{C})$. Suppose that the joined family $\{\phi_\iota,\, \psi\}$ satisfies all Wightman axioms except locality and that for any system $\iota_1,\dots,\iota_n$ of indices and for every m, the functionals*

$$\langle\Psi_0,\, \phi_{\iota_1}(x_1)\dots\phi_{\iota_m}(x_m)\psi(x)\phi_{\iota_{m+1}}(x_{m+1})\dots\phi_{\iota_n}(x_n)\Psi_0\rangle$$

$$-\, i^F\langle\Psi_0,\, \phi_{\iota_n}(x_n)\dots\phi_{\iota_{m+1}}(x_{m+1})\psi(x)\phi_{\iota_m}(x_m)\dots\phi_{\iota_1}(x_1)\Psi_0\rangle\,, \qquad (23)$$

where F is the number of spinor fields in the monomial $\phi_{\iota_1}\cdots\phi_{\iota_n}\psi$, are carried by the cone $\mathbf{C}\mathcal{J}_{n+1}$. Then Θ implements the PCT symmetry for ψ as well and the joined family of fields satisfies the weak asymptotic commutativity condition.

Proof. Applying Theorems 1 and 3 as above, we find that

$$\langle\Psi_0,\, \phi_{\iota_1}(x_1)\dots\phi_{\iota_m}(x_m)\psi(x)\phi_{\iota_{m+1}}(x_{m+1})\dots\phi_{\iota_n}(x_n)\Psi_0\rangle \qquad (24)$$

$$= (-1)^{2J}\, i^F\langle\Psi_0,\, \phi_{\iota_n}(-x_n)\dots\phi_{\iota_{m+1}}(-x_{m+1})\psi(-x)\phi_{\iota_m}(-x_m)\dots\phi_{\iota_1}(-x_1)\Psi_0\rangle,$$

where $J = j_{\iota_1} + \ldots + j_{\iota_n} + j$. After averaging with the test function $f(x_1)\dots f(x_n)f(x)$, the relation (24) takes the form

$$\langle\Theta^{-1}\Phi,\, \psi(f)\Psi\rangle = (-1)^{2j}\,(-i)^{F(\psi)}\langle\Theta\Psi,\, \psi(f^-)\Phi\rangle, \qquad (25)$$

where the following designations are used:

$$\Phi = \phi_{\iota_m}(f_m^-)\dots\phi_{\iota_1}(f_1^-)\Psi_0, \quad \Psi = \phi_{\iota_{m+1}}(f_{m+1})\dots\phi_{\iota_n}(f_n)\Psi_0\,. \qquad (26)$$

Performing the complex conjugation and using the antiunitary property of Θ, we obtain

$$\langle\Phi,\, \Theta\psi(f)\Psi\rangle = (-1)^{2j}\, i^{F(\psi)}\langle\Phi,\, \psi(f^-)^*\Theta\Psi\rangle\,,$$

Because of the cyclicity of the vacuum with respect to $\{\phi_\iota\}$, the subspaces spanned by vectors Φ and Ψ of the form (26) are dense in the Hilbert space, and we conclude that

$$\Theta\,\psi(f)\,\Theta^{-1} = (-1)^{2j}\, i^{F_\psi}\,\phi(f^-)^*\,. \qquad (27)$$

Thus, the operator Θ transforms ψ correctly, as was to be proved.

Corollary. *The weak relative asymptotic commutativity property is transitive in the sense that if each of fields ψ_1,ψ_2 satisfies the assumptions of Theorem 5, then this property holds for $\{\psi_1,\psi_2\}$.*

Indeed, then there is a PCT-symmetry operator common to the fields $\{\phi_\iota, \psi_1, \psi_2\}$ and by Theorem 4, the weak relative asymptotic commutativity condition is satisfied not only for $\{\psi_1, \psi_2\}$ but also for the whole family $\{\phi_\iota, \psi_1, \psi_2\}$.

The developed technique can also be used in deriving the transitivity of relative locality in hyperfunction QFT. Specifically, let us prove the following proposition.

Let $\{\phi_\iota\}$ be a family of local quantum fields defined on the test function space $S^1_1(\mathbb{R}^4)$, D_0 be its cyclic domain in the Hilbert space, and assume the standard spin-statistics relation holds. Let $\{\psi_1, \psi_2\}$ be a pair of fields with the same domain of definition and such that the joined family $\{\phi_\iota, \psi_1, \psi_2\}$ satisfies all Wightman axioms except possibly locality which is replaced by the assumption that

$$[\phi_\iota(x), \psi_1(x')]_{\underset{(+)}{-}} \Psi = 0, \quad [\phi_\iota(x), \psi_2(x')]_{\underset{(+)}{-}} \Psi = 0 \quad \text{for } (x-x')^2 < 0,\ \Psi \in D_0, \tag{28}$$

where the commutation relations are normal. Then

$$[\psi_1(x), \psi_2(x')]_{\underset{(+)}{-}} \Psi = 0 \quad \text{for } (x-x')^2 < 0,\ \Psi \in D_0, \tag{29}$$

and this commutation relation is also normal.

In fact, let us consider the vacuum expectation value

$$\langle \Psi_0,\, \phi_{\iota_1}(x_1)\dots\phi_{\iota_m}(x_m)\psi_1(x)\psi_2(x')\phi_{\iota_{m+1}}(x_{m+1})\dots\phi_{\iota_n}(x_n)\Psi_0\rangle \tag{30}$$

at real points of analyticity $(x_1,\dots,x_m,x,x',x_{m+1}\dots,x_n) \in \mathcal{J}_{n+2}$. Proposition 9.15 in Ref. 6 showing the transitivity of weak relative locality for tempered distribution fields is directly extendable to Fourier hyperfunction QFT. Therefore, the fields $\phi_\iota, \psi_1, \psi_2$ are weakly relatively local and we can invert the order of field operators in (30), which results in appearance of the factor i^F, where F is the number of spinor fields in the monomial under consideration. Next we use locality of ϕ_ι and the assumption of relative locality (28) to restore the initial order of field operators in the expectation value. Then the factor is changed to -1 if both fields ψ_1, ψ_2 have half-integer spin and to 1 in the other cases. So we have

$$\langle \Psi_0,\, \phi_{\iota_1}(x_1)\dots\phi_{\iota_m}(x_m)[\psi_1(x),\, \psi_2(x')]_{\underset{(+)}{-}}\phi_{\iota_{m+1}}(x_{m+1})\dots\phi_{\iota_n}(x_n)\Psi_0\rangle = 0 \tag{31}$$

for all Jost points. In momentum space, the generalized function (31) has support in the wedge determined by

$$\sum_{k=1}^{l} p_k \in \overline{\mathbb{V}}_+, \quad l = 1, \dots, m; \quad \sum_{k=1}^{n} p_{m+k} \in \overline{\mathbb{V}}_-, \quad l = 1, \dots, n-m\,.$$

It remains to note that any pair of points (x, x') such that $(x - x')^2 < 0$ enters in a Jost point $(x_1, \dots, x_m, x, x', x_{m+1} \dots, x_n) \in \mathcal{J}_{n+2}$ and apply Theorem 2.

7 S–equivalence of nonlocal fields

The Haag-Ruelle scattering theory shows that the existence of S–matrix is derivable from the general principles of local QFT complemented by some technical assumptions about the energy momentum spectrum and the structure of the single particle space, see Ref. 6. Under these assumptions, we can assign to each field ϕ_ι an operator $\varphi_\iota(g, t)$ depending on the parameter $t \in \mathbb{R}$ and the function $g(\mathbf{p}) \in \mathcal{D}(\mathbb{R}^3)$ so that in the Hilbert space of states, there exist the strong limits

$$\Phi^{\substack{\mathrm{in}\\ \mathrm{out}}}_{\iota_1,\dots,\iota_n}(g_1, \dots, g_n) = \lim_{t\to\mp\infty} \varphi_{\iota_1}(g_1, t) \cdots \varphi_{\iota_n}(g_n, t)\Psi_0\,, \tag{32}$$

describing the incoming and outgoing scattering states of n particles of the kind $\iota_1, \dots, \iota_n$ with the momentum space wave packets $g_1, \dots, g_n$. Moreover, the linear operators $\phi^{\mathrm{ex}}_\iota(g)$ ($\mathrm{ex} = \mathrm{in}, \mathrm{out}$) determined by

$$\phi^{\mathrm{ex}}_\iota(g)\Phi^{\mathrm{ex}}_{\iota_1,\dots,\iota_n}(g_1, \dots, g_n) = \Phi^{\mathrm{ex}}_{\iota,\iota_1,\dots,\iota_n}(g, g_1, \dots, g_n)\,, \tag{33}$$

with the subsequent extension by linearity, are well defined and represent free fields averaged with the corresponding test functions. In this construction, the local commutativity axiom is used only in deriving the strong cluster decomposition property of vacuum expectation values. This property plays the central role in the proof of the existence of limits (32). However, the asymptotic commutativity condition ensures the strong cluster property as well if the usual assumption is made that the theory has a finite mass gap between the one-particle state and the continuum. Namely, if $\mathcal{W}^T_n$ is the truncated part of the n–point vacuum expectation value and $f \in S^0(\mathbb{R}^{4n})$, then the convolution $(\mathcal{W}^T_n * f)(x)$ considered at equal times as a function of the difference variables $\mathbf{x}_i - \mathbf{x}_j$ belongs to the space $S^0(\mathbb{R}^{3(n-1)})$. Because of this, the Haag–Ruelle construction admits a direct generalization to the nonlocal theories under study, see Refs. 20, 21. The scattering matrix is defined by

$$S\Phi^{\mathrm{out}}_{\iota_1,\dots,\iota_n}(g_1, \dots, g_n) = \Phi^{\mathrm{in}}_{\iota_1,\dots,\iota_n}(g_1, \dots, g_n)$$

and is an isometric operator with the domain $\mathcal{H}^{\rm out}$ and the range $\mathcal{H}^{\rm in}$, where $\mathcal{H}^{\rm ex}$ are the closures of linear spans $D^{\rm ex}$ of vectors (32) and Ψ_0. The asymptotic free fields are connected by the relation

$$\phi_\iota^{\rm out}(x) = S^{-1}\phi_\iota^{\rm in}(x)S\,. \tag{33}$$

The results of Section 6 show that Theorem 4.20 in Ref. 5 on the S–equivalence of quantum fields can be generalized in the following way:

Theorem 6. *Let $\phi_1(x)$ be a field defined on the space $S^0(\mathbb{R}^4)$ and satisfying all Wightman axioms with the asymptotic commutativity substituted for locality. Assume that a field $\phi_2(x)$ transforms according to the same representation of the Lorentz group and the pair $\{\phi_1, \phi_2\}$ satisfies the weak asymptotic commutativity condition. If the "in" and "out" limits exist and $\phi_1^{\rm in}(x) = \phi_2^{\rm in}(x)$, then $\phi_1^{\rm out}(x) = \phi_2^{\rm out}(x)$.*

Proof. Indeed, by Theorem 5 the fields ϕ and ψ have a PCT operator which transforms "in" fields into "out" fields according to the formula

$$\Theta\,\phi_\iota^{\rm in}(x)\,\Theta^{-1} = (-1)^{2j_\iota}\, i^{F_\iota}\,\phi_\iota^{\rm out}(-x)^*\,, \tag{34}$$

precisely as in local field theory.

In particular, a field $\phi(x)$ with nontrivial S–matrix cannot be weakly asymptotically commuting with its associated free fields $\phi^{\rm in}(x)$ and $\phi^{\rm out}(x)$.

If in addition the asymptotic completeness $\mathcal{H}^{\rm ex} = \mathcal{H}$ is assumed, then S–matrix can be expressed through the operator Θ and the PCT operator of the free fields. Specifically, applying $\Theta_{\rm out}$ to (34) and using the relation

$$\Theta_{\rm out}^{-1}\,\phi_\iota^{\rm out}(-x)^*\,\Theta_{\rm out} = (-1)^{2j_\iota}\,(-i)^{F_\iota}\,\phi_\iota^{\rm out}(x)\,,$$

we obtain

$$S = \Theta^{-1}\Theta_{\rm out}\,. \tag{35}$$

The same result follows directly from the definition[6] of action of the operators Θ and $\Theta_{\rm ex}$ on $D^{\rm ex}$. Relation (35) implies the unitary property of the S–matrix and, taking into account the equality $\Theta^2 = \Theta^2_{\rm out}$, the PCT invariance:

$$\Theta_{\rm out} S\,\Theta_{\rm out}^{-1} = S^*\,.$$

Thus, the asymptotic commutativity condition stated in Section 2 guarantees the fulfillment of all properties required for the physical interpretation of nonlocal QFT and established previously by the Haag–Ruelle theory for local fields.

8 Concluding remarks

The Borchers class of a massive scalar free field ϕ in three and more dimensional space-time was first determined by Epstein[22] under the assumption that the vacuum expectation values are tempered distributions, and it turned out to be just the Wick polynomials including derivatives of ϕ. Under the same assumption of temperedness, Baumann[23] has proved that if a massless scalar field in $3+1$ dimensional space-time has a trivial S–matrix, then this field is relatively local to the free field. In Ref. 24 infinite series of Wick powers of a massive free field were studied. These series were shown to be well defined in the sense of hyperfunction and to extend the Borchers class in the same sense if they are of order < 2 or of order 2 and type 0. The Wick–ordered entire functions with an arbitrary order of growth are considered in Ref. 25. A relatively straightforward application of the Cauchy-Poincaré theorem shows that these functions satisfy the asymptotic commutativity condition in both the massive and massless cases and for any space-time dimension. It is natural to suppose that the convergent series of Wick powers acted upon by differential operators of an infinite order exhaust the nonlocal extension of Borchers class of free field.

In order to construct the scattering states and asymptotic free fields, it suffices to use the above-mentioned version of cluster property which means that the truncated Wightman functions decrease faster than any inverse power of the variable $\max_{i,j} |\mathbf{x}_i - \mathbf{x}_j|$. As shown in Ref. 26 this decrease is actually exponential in theories with a mass gap regardless their local properties, and this is essential to the derivation of bounds on growth of scattering amplitudes in nonlocal QFT's. A serious difficulty in dealing with analytic functionals is that the limit of functionals carried by a closed cone is not necessarily carried by the same cone, in contrast to the customary notion of support of a distribution. Because of this, the domains of nonlocal field operators in the Hilbert space need much more attention compared to the standard local QFT. It should be emphasized that questions of this kind are important even for formulating hyperfunction QFT, where the closure of Hermitian field operators can destroy localization properties, see Ref. 27. In this paper, we have made no attempt to derive an analog of Theorem 2 for $u \in \mathcal{D}'$, which is desirable to accomplish the suggested generalization of Borchers classes. It would be also worthwhile to consider possible links of the developed scheme with the modern derivation of the spin-statistics and PCT theorems in the framework of algebraic QFT,[28,29] where the modular covariance plays a significant role and the localization of charges in spacelike cones instead of in compact regions is admitted.

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