CLASSICAL DYNAMICAL R-MATRICES AND POISSON HOMOGENEOUS SPACES

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Lu[18] showed that any dynamical r-matrix for the pair $(\mathfrak{g}, \mathfrak{u})$ naturally induces a Poisson homogeneous structure on G/U. She also proved that if $\mathfrak{g}$ is complex simple, $\mathfrak{u}$ is its Cartan subalgebra and r is quasitriangular, then this correspondence is in fact 1–1. In the present paper we find some general conditions under which the Lu correspondence is 1–1. Then we apply this result to describe all triangular Poisson homogeneous structures on G/U for a simple complex group G and its reductive subgroup U containing a Cartan subgroup.

Contents

Foreword by Alexander Stolin

I met Misha Marinov at the Max Born Symposium in Poland in 1996, where Misha presented a group-theoretical approach to construction of dynamical systems.[19] His construction was based on a homogeneous space $M = G/K$ equipped with a Kähler structure. We immediately realized that the construction is related to some solution of the classical Yang-Baxter equation and Misha's quantization of this Kähler structure leads to a solution of the quantum Yang-Baxter equation.

Interplay between physics and geometry interested Misha for a long time.[1,20] We planned further development of ideas about connections between geometry of homogeneous spaces and the Yang-Baxter equation. Unfortunately Misha's unexpected death stopped our plans.

1 Introduction

The notion of a Poisson-Lie group was introduced almost 20 years ago by Drinfeld.[5] Its infinitesimal counterpart, Lie bialgebras, were introduced in the same paper and later it was explained that these objects are in fact quasiclassical limits of quantum groups.[6] Lie bialgebra structures on a Lie algebra $\mathfrak{g}$ are in a natural 1–1 correspondence with Lie algebra structures on the vector space $D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$ with some compatibility conditions. $D(\mathfrak{g})$ with this Lie algebra structure is called the double of the Lie bialgebra $\mathfrak{g}$.

The most popular and important class of Lie bialgebras is the class of quasitriangular Lie bialgebras.[6] They can be defined by an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ (called the classical r-matrix) such that

$$\Omega := r + r^{21}$$

is $\mathfrak{g}$-invariant, and the classical Yang-Baxter equation (CYBE)

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0$$

is satisfied. If r is skew-symmetric, then one says that the corresponding Lie bialgebra is triangular. In general, $\Lambda := r - \frac{\Omega}{2}$ (i.e., the skew-symmetric part of r) satisfies the modified CYBE

$$[\Lambda^{12}, \Lambda^{13}] + [\Lambda^{12}, \Lambda^{23}] + [\Lambda^{13}, \Lambda^{23}] = \frac{1}{4}[\Omega^{12}, \Omega^{23}].$$

It is well known (and can be easily shown) that if $\mathfrak{g}$ is a complex simple finite-dimensional Lie algebra, then any Lie bialgebra structure on $\mathfrak{g}$ is quasitriangular. For the case $\Omega \neq 0$ ("quasitriangular case in the strict sense") they were classified by Belavin and Drinfeld.[2,3] The triangular case was studied by Stolin.[22,23,24]

It was shown[25] that for such $\mathfrak{g}$ there are only two possible structures of the $D(\mathfrak{g})$. In the triangular case $D(\mathfrak{g}) = \mathfrak{g}[\varepsilon] = \mathfrak{g} \oplus \mathfrak{g}\varepsilon$, where $\varepsilon^2 = 0$ and otherwise, $D(\mathfrak{g}) = \mathfrak{g} \times \mathfrak{g}$ (and $\mathfrak{g}$ is embedded diagonally into $\mathfrak{g} \times \mathfrak{g}$). Then it is clear that solutions of the CYBE (resp. the modified CYBE with $\Omega \neq 0$) are in a 1–1 correspondence with Lagrangian subalgebras $\mathfrak{l}$ in $\mathfrak{g}[\varepsilon]$ (resp. in $\mathfrak{g} \times \mathfrak{g}$) such that $\mathfrak{l} \cap \mathfrak{g} = 0$.

Along with the Poisson-Lie groups it is natural to study their Poisson actions, and in particular their Poisson homogeneous spaces. Drinfeld[8] gave a general approach to the classification of Poisson homogeneous spaces. Namely, he showed that if G is a Poisson-Lie group, $\mathfrak{g}$ is the corresponding Lie bialgebra, then Poisson homogeneous G-spaces are essentially in a 1–1 correspondence with G-orbits on the set of all Lagrangian subalgebras in $D(\mathfrak{g})$. A classification of Lagrangian subalgebras in some important cases (including the case $\mathfrak{g}$ is complex simple, $D(\mathfrak{g}) = \mathfrak{g} \times \mathfrak{g}$) was obtained by Karolinsky.[15,16,17]

At the same time an important generalization of the CYBE, the dynamical classical Yang-Baxter equation, was introduced in physics and mathematics. Notice that this equation is defined for a pair $(\mathfrak{g}, \mathfrak{u})$, where $\mathfrak{u}$ is a subalgebra of $\mathfrak{g}$. From the mathematical point of view it was presented by Felder.[12,13] This equation and its quantum analogue were studied in many papers.[9,11,21,26] First classification results for the solutions of the classical dynamical Yang-Baxter equation (dynamical r-matrices) were obtained by Etingof and Varchenko,[11] and Schiffmann.[21]

Later Lu[18] found a connection (which is essentially a 1–1 correspondence) between dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$ (where $\mathfrak{u}$ is a Cartan subalgebra of the complex simple finite-dimensional algebra $\mathfrak{g}$), and Poisson homogeneous G-structures on G/U. Here $U \subset G$ are connected Lie groups corresponding to $\mathfrak{u} \subset \mathfrak{g}$, and G is equipped with the standard quasitriangular (with $\Omega \neq 0$) Poisson-Lie structure. Lu also noticed that this connection can be generalized to the case $\mathfrak{u}$ is a subspace in a Cartan subalgebra (with some "regularity" condition). The dynamical r-matrices for the latter case were classified by Schiffmann.[21]

Now let G be a complex connected semisimple Lie group, and let U be its connected subgroup. Suppose $\mathfrak{u} \subset \mathfrak{g}$ are the corresponding Lie algebras. In the present paper we consider connections between Poisson homogeneous structures on G/U related to the triangular Poisson-Lie structures on G (i.e., with $\Omega = 0$), where U is a reductive subgroup containing a Cartan subgroup of G, and triangular dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$.

In fact, our results are based on a general result on relations between dynamical classical r-matrices and Poisson homogeneous structures (see Theorem 12), which is valid also in the quasitriangular case. Notice that the results of Sections 2 and 3 can be used to describe a 1–1 correspondence between dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$, where $\mathfrak{u} \subset \mathfrak{g}$ is a Cartan subalgebra, and Poisson homogeneous G-structures on G/U, where G is equipped with any quasitriangular (with $\Omega \neq 0$) Poisson-Lie structure (of course the latter result is due to Lu). Our approach is based on some strong classification results for dynamical r-matrices given recently by Etingof and Schiffmann.[10]

The paper is organized as follows. In Section 2 we describe a correspondence between the (moduli space of) dynamical r-matrices for a pair $(\mathfrak{g}, \mathfrak{u})$ and Poisson homogeneous G-structures on G/U proving that under certain assumptions it is a bijection. In Section 3 we consider a procedure of twisting for Lie bialgebras and examine its impact on the double $D(\mathfrak{g})$ and Poisson homogeneous spaces for corresponding Poisson-Lie groups. Then we use the twisting to weaken some restrictions needed in Section 2. Finally, in Section 4 we consider the basic example of our paper: $\mathfrak{g}$ is semisimple, $\mathfrak{u} \subset \mathfrak{g}$ is a reductive Lie subalgebra that contains some Cartan subalgebra of $\mathfrak{g}$, and the Lie bialgebra structure on $\mathfrak{g}$ is triangular (i.e., $D(\mathfrak{g}) = \mathfrak{g}[\varepsilon]$).

2 Classical dynamical r-matrices and Poisson homogeneous spaces

In this section we assume $\mathfrak{g}$ to be any finite-dimensional Lie algebra over $\mathbb{C}$. Let G be a connected Lie group such that $\operatorname{Lie} G = \mathfrak{g}$. Let $\mathfrak{u} \subset \mathfrak{g}$ be a Lie subalgebra (not necessary abelian). By U denote the connected subgroup in G such that $\operatorname{Lie} U = \mathfrak{u}$. We propose (under certain conditions) a connection between dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$ and Poisson structures on G/U that make G/U a *Poisson* homogeneous G-space (for certain Poisson-Lie structures on G). Note that this connection was first introduced by Jiang-Hua Lu[18] for the case $\mathfrak{g}$ is simple, $\mathfrak{u}$ is a Cartan subalgebra, and the dynamical r-matrix has non-zero coupling constant.

In order to recall the definition of the classical dynamical r-matrix we need some notation. Let $x_1, ..., x_r$ be a basis of $\mathfrak{u}$. By D denote the formal neighborhood of 0 in $\mathfrak{u}^*$. By functions from D to a vector space V we mean the elements of the space $V[[x_1, ..., x_r]]$, where x_i are regarded as coordinates on D. If $\omega \in \Omega^k(D, V)$ is a k-form with values in a vector space V, we denote by $\overline{\omega} : D \to \wedge^k \mathfrak{u} \otimes V$ the corresponding function. Finally, for an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ we define the classical Yang-Baxter operator

$$\mathrm{CYB}(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}].$$

Recall that a *classical dynamical r-matrix* for the pair $(\mathfrak{g}, \mathfrak{u})$ is an $\mathfrak{u}$-equivariant function $r : D \to \mathfrak{g} \otimes \mathfrak{g}$ that satisfies the classical dynamical Yang-

Baxter equation (CDYBE):

$$\mathrm{Alt}(\overline{dr}) + \mathrm{CYB}(r) = 0, \tag{1}$$

where for $x \in \mathfrak{g}^{\otimes 3}$, we let $\mathrm{Alt}(x) = x^{123} + x^{231} + x^{312}$ (see Etingof and Schiffmann,[9,10] Etingof and Varchenko[11]). Usually one requires also an additional *quasi-unitarity condition*:

$$r + r^{21} = \Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}.$$

Note that if r satisfies the CDYBE and the quasi-unitarity condition then Ω is a constant function.

Suppose $\Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$. Let us denote by $\mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$ the set of all classical dynamical r-matrices r for the pair $(\mathfrak{g}, \mathfrak{u})$ such that $r + r^{21} = \Omega$.

Denote by $\mathbf{Map}(D, G)^{\mathfrak{u}}$ the set of all regular $\mathfrak{u}$-equivariant maps from D to G. Suppose $g \in \mathbf{Map}(D, G)^{\mathfrak{u}}$. For any $\mathfrak{u}$-equivariant function $r : D \to \mathfrak{g} \otimes \mathfrak{g}$ set

$$r^g = (\mathrm{Ad}_g \otimes \mathrm{Ad}_g)(r - \overline{\eta_g} + \overline{\eta_g}^{21} + \tau_g),$$

where $\eta_g = g^{-1}dg$, and $\tau_g(\lambda) = (\lambda \otimes 1 \otimes 1)([\overline{\eta_g}^{12}, \overline{\eta_g}^{13}](\lambda))$. Then r is a classical dynamical r-matrix iff r^g is (see Etingof and Schiffmann[10]). The transformation $r \mapsto r^g$ is called a gauge transformation. It is indeed an action of the group $\mathbf{Map}(D, G)^{\mathfrak{u}}$ on $\mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$ (i.e., $(r^{g_1})^{g_2} = r^{g_2 g_1}$). Following Etingof and Schiffmann[10] denote by $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ the moduli space $\mathbf{Map}_0(D, G)^{\mathfrak{u}} \backslash \mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$, where $\mathbf{Map}_0(D, G)^{\mathfrak{u}}$ is the subgroup in $\mathbf{Map}(D, G)^{\mathfrak{u}}$ consisting of maps g satisfying $g(0) = e$.

In what follows we need some notation. Suppose $a \in \mathfrak{g}^{\otimes k}$. By $\overrightarrow{a}$ (resp. $\overleftarrow{a}$) denote the left (resp. right) invariant tensor field on G corresponding to a.

Suppose $\rho \in \mathfrak{g} \otimes \mathfrak{g}$ satisfies the classical Yang-Baxter equation (CYBE), i.e., $\mathrm{CYB}(\rho) = 0$. Assume also that $\rho + \rho^{21} = \Omega$ (i.e., $\rho = \frac{\Omega}{2} + \Lambda$, where $\Lambda \in \wedge^2\mathfrak{g}$). Introduce a bivector field $\pi_\rho = \overrightarrow{\rho} - \overleftarrow{\rho} = \overrightarrow{\Lambda} - \overleftarrow{\Lambda}$ on G. It is well known that (G, π_ρ) is a Poisson-Lie group.

Now let $r \in \mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$. We have $r = \frac{\Omega}{2} + A$, where $A \in \wedge^2\mathfrak{g}$. Set $\tilde{\pi}_r := \overrightarrow{r(0)} - \overleftarrow{\rho} = \overrightarrow{A(0)} - \overleftarrow{\Lambda}$. Consider a bivector field π_r on G/U defined by $\pi_r\left(\underline{g}\right) = p_* \tilde{\pi}_r(g)$, where $p : G \to G/U$ is the natural projection, and $\underline{g} = p(g)$. Note that π_r is well defined since $r(0) \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{u}}$.

The following proposition belongs to Jiang-Hua Lu[18] (note that it is stated there for the case $\mathfrak{g}$ is simple, $\mathfrak{u}$ is a Cartan subalgebra, but the proof fits the general case).

Proposition 1. *The bivector field π_r is Poisson, and $(G/U, \pi_r)$ is a Poisson homogeneous (G, π_ρ)-space.* $\square$

Proposition 2. *Suppose $g \in \mathbf{Map}_0(D,G)^{\mathfrak{u}}$. Then $\pi_r = \pi_{r^g}$.*

Proof. Since $(G/U, \pi_r)$ is a Poisson homogeneous (G, π_ρ)-space, we see that π_r depends only on $\pi_r(\underline{e})$ = the image of $r(0) - \rho$ in $\wedge^2(\mathfrak{g}/\mathfrak{u})$. Thus it is enough to note that $r^g(0) - r(0) \in \mathfrak{u} \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{u}$. □

Corollary 3. *The correspondence $r \mapsto \pi_r$ defines a map from $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ to the set of all Poisson (G, π_ρ)-homogeneous structures on G/U.* □

Suppose now that the following conditions hold:

(a) $\mathfrak{u}$ has an $\mathfrak{u}$-invariant complement $\mathfrak{m}$ in $\mathfrak{g}$ (we fix one).

(b) $\Omega \in (\mathfrak{u} \otimes \mathfrak{u}) \oplus (\mathfrak{m} \otimes \mathfrak{m})$.

(c) $\rho \in \frac{\Omega}{2} + (\wedge^2 \mathfrak{m})^{\mathfrak{u}}$.

Theorem 4. *Under the assumptions above the correspondence $r \mapsto \pi_r$ is a bijection between $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ and the set of all Poisson (G, π_ρ)-homogeneous structures on G/U.*

The rest of this section is devoted to the proof of Theorem 4. First we recall some results belonging to Etingof and Schiffmann.[10] Assume that (a) holds. Set

$$\mathcal{M}_\Omega = \left\{ x \in \frac{\Omega}{2} + (\wedge^2 \mathfrak{m})^{\mathfrak{u}} \;\middle|\; \mathrm{CYB}(x) = 0 \text{ in } \wedge^3 (\mathfrak{g}/\mathfrak{u}) \right\}.$$

Theorem 5 (Etingof, Schiffmann[10]). *1. Any class $\mathcal{C} \in \mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ has a representative $r \in \mathcal{C}$ such that $r(0) \in \mathcal{M}_\Omega$. Moreover, this defines an embedding $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega) \to \mathcal{M}_\Omega$.*

2. Assume that (b) holds. Then the map $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega) \to \mathcal{M}_\Omega$ defined above is a bijection. □

Now suppose $b \in (\wedge^2(\mathfrak{g}/\mathfrak{u}))^{\mathfrak{u}} = (\wedge^2 \mathfrak{m})^{\mathfrak{u}}$. Set $\pi\left(\underline{g}\right) = (L_g)_* b + p_* \pi_\rho(g)$. Since ρ is $\mathfrak{u}$-invariant, we see that $\pi_\rho(g) = 0$ for $g \in U$; therefore π is a well-defined bivector field on G/U.

Proposition 6. *The bivector field π is Poisson iff $\mathrm{CYB}(\rho + b) = 0$ in $\wedge^3(\mathfrak{g}/\mathfrak{u})$.*

Proof. Set $a = \Lambda + b$. Define a bivector field $\tilde{\pi}$ on G by the formula $\tilde{\pi} = \overrightarrow{a} - \overleftarrow{\Lambda}$. Note that $\tilde{\pi} = \overrightarrow{b} + \pi_\rho$, therefore $\pi = p_* \tilde{\pi}$. Let us normalize the Schouten bracket of the bivector fields on G in a way that $[\overrightarrow{x}, \overrightarrow{x}] = \overrightarrow{\mathrm{CYB}(x)}$ for all $x \in \wedge^2\mathfrak{g}$. Then we have

$$[\tilde{\pi}, \tilde{\pi}] = [\overrightarrow{a}, \overrightarrow{a}] - 2[\overrightarrow{a}, \overleftarrow{\Lambda}] + [\overleftarrow{\Lambda}, \overleftarrow{\Lambda}] = \overrightarrow{\mathrm{CYB}(a)} - \overleftarrow{\mathrm{CYB}(\Lambda)}.$$

Since $\rho = \frac{\Omega}{2} + \Lambda$ satisfies the CYBE, we see that $\mathrm{CYB}(\Lambda) = \frac{1}{4}[\Omega^{12}, \Omega^{23}] \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$. Thus

$$[\tilde{\pi}, \tilde{\pi}] = \overrightarrow{\mathrm{CYB}(a) - \frac{1}{4}[\Omega^{12}, \Omega^{23}]} = \overrightarrow{\mathrm{CYB}\left(\frac{\Omega}{2} + a\right)} = \overrightarrow{\mathrm{CYB}(\rho + b)}.$$

To finish the proof it is enough to note that $[\pi, \pi] = p_*[\tilde{\pi}, \tilde{\pi}]$. $\square$

Proof of Theorem 4. Let us construct the inverse map. Suppose $(G/U, \pi)$ is a Poisson homogeneous (G, π_ρ)-space. Set $b = \pi(\underline{e}) \in \wedge^2(\mathfrak{g}/\mathfrak{u}) = \wedge^2\mathfrak{m}$. The condition (c) implies that in fact $b \in (\wedge^2(\mathfrak{g}/\mathfrak{u}))^{\mathfrak{u}} = (\wedge^2\mathfrak{m})^{\mathfrak{u}}$. Furthermore, (c) yields that $\rho + b \in \frac{\Omega}{2} + (\wedge^2\mathfrak{m})^{\mathfrak{u}}$. By Proposition 6, we have $\mathrm{CYB}(\rho + b) = 0$ in $\wedge^3(\mathfrak{g}/\mathfrak{u})$, i.e., $\rho + b \in \mathcal{M}_\Omega$. Then, by Theorem 5, there exists $r \in \mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$ such that $r(0) = \rho + b$, and the image of r in $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ is uniquely determined. It is now easy to verify that $\pi = \pi_r$. $\square$

3 Twisting of Poisson homogeneous structures

Assume again that $\mathfrak{g}$ is an arbitrary finite-dimensional Lie algebra over $\mathbb{C}$. Recall that a Lie bialgebra structure on $\mathfrak{g}$ is a 1-cocycle $\delta : \mathfrak{g} \to \wedge^2\mathfrak{g}$ which satisfies the co-Jacobi identity. Denote by $\mathcal{D}(\mathfrak{g}, \delta)$ the classical double of $(\mathfrak{g}, \delta)$.

We say that two Lie bialgebra structures δ_1, δ_2 on $\mathfrak{g}$ are in the same class if there exists a Lie algebra isomorphism $f : \mathcal{D}(\mathfrak{g}, \delta_1) \to \mathcal{D}(\mathfrak{g}, \delta_2)$ which intertwines the canonical forms Q_i on $\mathcal{D}(\mathfrak{g}, \delta_i)$, and such that the following diagram is commutative:

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[dr] & \mathcal{D}(\mathfrak{g}, \delta_1) \arrow[d, "f"] \\ & \mathcal{D}(\mathfrak{g}, \delta_2). \end{tikzcd}$$

Theorem 7. *Two Lie bialgebra structures δ, δ' on $\mathfrak{g}$ are in the same class if and only if $\delta' = \delta + ds$, where $s \in \wedge^2\mathfrak{g}$ and*

$$\mathrm{CYB}(s) = \mathrm{Alt}(\delta \otimes \mathrm{id})(s). \tag{2}$$

Proof. Let us consider $\mathcal{D}(\mathfrak{g},\delta)=\mathfrak{g}\oplus\mathfrak{g}^*$. Notice that Lie bialgebra structures on $\mathfrak{g}$ that are in the same class with δ are in a 1–1 correspondence with Lagrangian subalgebras $\mathfrak{l}\subset\mathcal{D}(\mathfrak{g},\delta)$ such that $\mathfrak{l}\cap\mathfrak{g}=0$.

Any $s=\sum_i s_i'\otimes s_i''\in\wedge^2\mathfrak{g}$ defines a linear map $S:\mathfrak{g}^*\to\mathfrak{g}$, $S(l)=\sum_i\langle l,s_i'\rangle s_i''$. Let $\mathfrak{l}\subset\mathcal{D}(\mathfrak{g},\delta)$ be the graph of S. Clearly, $\mathfrak{l}\cap\mathfrak{g}=0$, and $\mathfrak{l}$ is a Lagrangian subspace because s is skew-symmetric.

It is easy to verify that for any $l_1,l_2,l_3\in\mathfrak{g}^*$,

$$\langle l_1\otimes l_2\otimes l_3,\mathrm{CYB}(s)-\mathrm{Alt}(\delta\otimes\mathrm{id})(s)\rangle=Q([l_1+S(l_1),l_2+S(l_2)],l_3+S(l_3)),$$

where $\langle\cdot\,,\cdot\rangle$ is the canonical pairing, and Q is the canonical bilinear form on $\mathcal{D}(\mathfrak{g},\delta)$. Therefore $\mathfrak{l}$ is a subalgebra if and only if (2) holds.

Suppose $s\in\wedge^2\mathfrak{g}$ satisfies (2). Let δ' be the Lie bialgebra structure on $\mathfrak{g}$ defined by s. It is straightforward to verify that $\delta'=\delta+ds$. This completes the proof of the theorem. $\square$

Remark 8. If we consider our Lie bialgebra $(\mathfrak{g},\delta)$ as a Lie quasibialgebra, then $(\mathfrak{g},\delta+ds)$ is called "twisting via s". The notions of Lie quasibialgebra and twisting via s was introduced by Drinfeld.[7] The theorem above can be also deduced from the general results on Lie quasibialgebras (see Drinfeld[7]).

Further, we are going to examine the effect of the twisting on Poisson homogeneous spaces. First we recall some definitions and rather well-known results.

Let G be a connected complex Poisson-Lie group, $(\mathfrak{g},\delta)$ its Lie bialgebra, and $\mathcal{D}(\mathfrak{g})=\mathcal{D}(\mathfrak{g},\delta)$ the corresponding classical double of $\mathfrak{g}$ with the canonical invariant form Q.

Recall that an action of G on a Poisson manifold M is called Poisson if the defining map $G\times M\to M$ is a Poisson map, where $G\times M$ is equipped with the product Poisson structure. If the action is transitive, we say that M is a *Poisson homogeneous G-space*.

Let M be a homogeneous G-space, and let π be any bivector field on M. For any $x\in M$ let us consider the map

$$\pi_x:T_x^*M\to T_xM,\ \ \pi_x(l)=(l\otimes\mathrm{id})(\pi(x)).$$

On the other hand, $M\cong G/H_x$ and $T_xM=\mathfrak{g}/\mathfrak{h}_x$, $T_x^*M=(\mathfrak{g}/\mathfrak{h}_x)^*=\mathfrak{h}_x^\perp\subset\mathfrak{g}^*$, where H_x is the stabilizer of x, and $\mathfrak{h}_x=\mathrm{Lie}\,H_x$. Therefore we can consider π_x as a map $\pi_x:\mathfrak{h}_x^\perp\to\mathfrak{g}/\mathfrak{h}_x$.

Now let us consider the following set of subspaces in $\mathcal{D}(\mathfrak{g})=\mathfrak{g}\oplus\mathfrak{g}^*$:

$$\mathfrak{l}_x=\{a+l\,|\,a\in\mathfrak{g},\,l\in\mathfrak{h}_x^\perp,\,\pi_x(l)=\overline{a}\},\tag{3}$$

where $\overline{a}$ is the image of a in $\mathfrak{g}/\mathfrak{h}_x$. Observe that $\mathfrak{l}_x$ are Lagrangian subspaces, and $\mathfrak{l}_x \cap \mathfrak{g} = \mathfrak{h}_x$. The following result was obtained by Drinfeld.[8]

Theorem 9 (Drinfeld[8]). *(M, π) is a Poisson homogeneous G-space if and only if for any $x \in M$ the subspace $\mathfrak{l}_x$ is a subalgebra of $\mathcal{D}(\mathfrak{g})$, and $\mathfrak{l}_{gx} = \mathrm{Ad}_g \mathfrak{l}_x$ for all $g \in G$.* □

Now set $\delta' = \delta + ds$, where $s \in \wedge^2\mathfrak{g}$ satisfies (2). Then we have two Poisson-Lie groups, (G, π_δ) and $(G, \pi_{\delta'})$, whose Lie bialgebras are $(\mathfrak{g}, \delta)$ and $(\mathfrak{g}, \delta')$ respectively. Let (M, π) be a Poisson homogeneous (G, π_δ)-space. Consider a bivector field ξ on M defined by the formula $\xi(x) =$ the image of s in $\wedge^2(\mathfrak{g}/\mathfrak{h}_x) = \wedge^2 T_x M$. Set $\pi' = \pi - \xi$.

Proposition 10. *(M, π') is a Poisson homogeneous $(G, \pi_{\delta'})$-space, and thus one obtains a bijection between the sets of all Poisson (G, π_δ)- and $(G, \pi_{\delta'})$-homogeneous structures on M.*

Proof. Theorem 7 allows one to identify $\mathcal{D}(\mathfrak{g}, \delta)$ and $\mathcal{D}(\mathfrak{g}, \delta')$. It is easy to verify that under this identification the sets of Lagrangian subspaces that correspond to (M, π) and (M, π') are the same. This completes the proof, according to Theorem 9. □

Finally, we are going to generalize the main result of the previous section to the twisted case. Assume that $(\mathfrak{g}, \delta)$ is a quasitriangular Lie bialgebra, i.e., $\delta = d\rho$, where $\rho \in \mathfrak{g} \otimes \mathfrak{g}$ and $\mathrm{CYB}(\rho) = 0$. It is easy to verify that the condition (2) for an element $s \in \wedge^2\mathfrak{g}$ is equivalent to

$$\mathrm{CYB}(s) + [\![\rho, s]\!] + [\![s, \rho]\!] = 0, \tag{4}$$

where for $a, b \in \mathfrak{g}^{\otimes 2}$ we set $[\![a, b]\!] = [a^{12}, b^{13}] + [a^{12}, b^{23}] + [a^{13}, b^{23}] \in \mathfrak{g}^{\otimes 3}$ (i.e., $\mathrm{CYB}(a) = [\![a, a]\!]$).

Fix $\Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$ and assume that $\rho \in \frac{\Omega}{2} + \wedge^2\mathfrak{g}$. As before, consider the Poisson-Lie group (G, π_δ), where $\pi_\delta = \pi_\rho = \overrightarrow{\rho} - \overleftarrow{\rho}$. Suppose $s \in \wedge^2\mathfrak{g}$ satisfies (4). Set $\delta' = \delta + ds = d(\rho + s)$; then $\pi_{\delta'} = \pi_{\rho,s} := \overrightarrow{\rho + s} - \overleftarrow{\rho + s}$, and $(G, \pi_{\rho,s})$ is a Poisson-Lie group.

Let U be a connected Lie subgroup in G, and $\mathfrak{u} = \mathrm{Lie}\, U$. Consider $r \in \mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$. As usually, set $\tilde{\pi}_r = \overrightarrow{r(0)} - \overleftarrow{\rho}$ and denote by π_r the natural projection of $\tilde{\pi}_r$ on G/U. By Proposition 1, $(G/U, \pi_r)$ is a Poisson homogeneous (G, π_ρ)-space. Set also $\tilde{\pi}_{r,s} = \tilde{\pi}_r - \overleftarrow{s} = \overrightarrow{r(0)} - \overleftarrow{\rho + s}$ and denote by $\pi_{r,s}$ its projection on G/U. According to Proposition 10, $(G/U, \pi_{r,s})$ is a Poisson homogeneous $(G, \pi_{\rho,s})$-space.

Moreover, if we combine Theorem 4 and Proposition 10, we get the following

Theorem 11. *Assume that* $\mathfrak{u}$, Ω, *and* ρ *satisfy the conditions (a), (b), and (c) from the previous section. Then the correspondence* $r \mapsto \pi_{r,s}$ *is a bijection between* $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ *and the set of all Poisson* $(G, \pi_{\rho,s})$*-homogeneous structures on* G/U*.* □

Clearly, this can be reformulated as follows:

Theorem 12. *Assume that* $\mathfrak{u}$ *and* Ω *satisfy the conditions (a) and (b) from the previous section. Suppose also that there exists* $s \in \wedge^2\mathfrak{g}$ *such that (4) holds, and* $\rho + s \in \frac{\Omega}{2} + (\wedge^2\mathfrak{m})^{\mathfrak{u}}$*. Then the correspondence* $r \mapsto \pi_r$ *is a bijection between* $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ *and the set of all Poisson* (G, π_ρ)*-homogeneous structures on* G/U*.*
□

Let us apply our previous results to the triangular case.

Corollary 13. *Assume that* $\mathfrak{u}$ *satisfies the condition (a) from the previous section. Set* $\Omega = 0$*. Consider any* $\rho \in \wedge^2\mathfrak{g}$ *that satisfies the CYBE. Then the correspondence* $r \mapsto \pi_r$ *is a bijection between* $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ *and the set of all Poisson* (G, π_ρ)*-homogeneous structures on* G/U*.*

Proof. Set $s = -\rho$ and apply Theorem 12. □

4 Poisson homogeneous structures in the triangular case

Now assume that $\mathfrak{g}$ is semisimple. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and denote by $\mathbf{R}$ the corresponding root system. In this section we apply the results of the previous sections to the case $\mathfrak{u}$ is a reductive Lie subalgebra in $\mathfrak{g}$ containing $\mathfrak{h}$, $\Omega = 0$, and $\rho \in \wedge^2\mathfrak{g}$ such that $\mathrm{CYB}(\rho) = 0$.

To be more precise, consider $\mathbf{U} \subset \mathbf{R}$, and suppose $\mathfrak{u} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \mathbf{U}} \mathfrak{g}_\alpha)$ is a reductive Lie subalgebra in $\mathfrak{g}$. If this is the case, then we say that a subset $\mathbf{U} \subset \mathbf{R}$ is *reductive* (i.e., $(\mathbf{U} + \mathbf{U}) \cap \mathbf{R} \subset \mathbf{U}$ and $-\mathbf{U} = \mathbf{U}$; see Gorbatsevich, Onishchik, and Vinberg[14]). Condition (a) is satisfied since $\mathfrak{m} = \bigoplus_{\alpha \in \mathbf{R} \setminus \mathbf{U}} \mathfrak{g}_\alpha$ is an $\mathfrak{u}$-invariant complement to $\mathfrak{u}$ in $\mathfrak{g}$.

Applying Corollary 13 (and results of Etingof and Schiffmann cited in Section 2), we get:

1. Any structure of a Poisson homogeneous (G, π_ρ)-space on G/U is of the form $p_*(\overrightarrow{x} - \overleftarrow{\rho})$, where $x \in \mathcal{M}_\Omega$.

2. If $x \in \mathcal{M}_\Omega$, then there exists (a unique up to the $\mathbf{Map}_0(D, G)^{\mathfrak{u}}$-action) $r \in \mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$ such that $r(0) = x$.

Let us now describe $\mathcal{M}_\Omega$ and thus get an explicit description of all Poisson (G, π_ρ)-homogeneous structures on G/U. Recall that in our case by definition

$$\mathcal{M}_\Omega = \left\{ x \in (\wedge^2 \mathfrak{m})^{\mathfrak{u}} \;\middle|\; \mathrm{CYB}(x) = 0 \text{ in } \wedge^3 (\mathfrak{g}/\mathfrak{u}) \right\}.$$

We need to fix some notation. Fix a nondegenerate invariant bilinear form (*invariant scalar product*) $\langle \cdot , \cdot \rangle$ on $\mathfrak{g}$. For any $\alpha \in \mathbf{R}$ choose $E_\alpha \in \mathfrak{g}_\alpha$ such that $\langle E_\alpha, E_{-\alpha} \rangle = 1$. Further, suppose $\mathbf{N}$ is a reductive subset which contains $\mathbf{U}$. We say that $h \in \mathfrak{h}$ is $(\mathbf{N}, \mathbf{U})$-*regular* if $\alpha(h) = 0$ for all $\alpha \in \mathbf{U}$, and $\alpha(h) \neq 0$ for all $\alpha \in \mathbf{N} \setminus \mathbf{U}$.

Proposition 14. *$x \in \mathcal{M}_\Omega$ iff*

$$x = x_{\mathbf{N},h} = \sum_{\alpha \in \mathbf{N} \setminus \mathbf{U}} \frac{1}{\alpha(h)} E_\alpha \otimes E_{-\alpha}, \tag{5}$$

where $\mathbf{N}$ is a reductive subset in $\mathbf{R}$ containing $\mathbf{U}$, and $h \in \mathfrak{h}$ is $(\mathbf{N}, \mathbf{U})$-regular.

Proof. One can easily prove the following lemmas:

Lemma 15. *$x \in (\wedge^2 \mathfrak{m})^{\mathfrak{u}}$ iff*

$$x = \sum_{\alpha \in \mathbf{R} \setminus \mathbf{U}} x_\alpha \cdot E_\alpha \otimes E_{-\alpha},$$

where $x_{-\alpha} = -x_\alpha$, and for all $\alpha, \beta \in \mathbf{R} \setminus \mathbf{U}$ such that $\alpha + \beta \in \mathbf{U}$, we have $x_\alpha + x_\beta = 0$. □

Lemma 16. *Suppose*

$$x = \sum_{\alpha \in \mathbf{R} \setminus \mathbf{U}} x_\alpha \cdot E_\alpha \otimes E_{-\alpha} \in (\wedge^2 \mathfrak{m})^{\mathfrak{u}}.$$

Then $x \in \mathcal{M}_\Omega$ iff the following condition holds: for all $\alpha, \beta, \gamma \in \mathbf{R} \setminus \mathbf{U}$, $\alpha + \beta + \gamma = 0$, we have $x_\alpha x_\beta + x_\beta x_\gamma + x_\gamma x_\alpha = 0$. □

Now consider the following properties of the function $\mathbf{R} \setminus \mathbf{U} \to \mathbb{C}$, $\alpha \mapsto x_\alpha$:

(d) $x_{-\alpha} = -x_\alpha$ for all $\alpha \in \mathbf{R} \setminus \mathbf{U}$.

(e) If $\alpha, \beta \in \mathbf{R} \setminus \mathbf{U}$, $\alpha + \beta \in \mathbf{U}$, then $x_\alpha + x_\beta = 0$.

(f) If $\alpha, \beta, \gamma \in \mathbf{R} \setminus \mathbf{U}$, $\alpha + \beta + \gamma = 0$, then $x_\alpha x_\beta + x_\beta x_\gamma + x_\gamma x_\alpha = 0$.

It is also straightforward to prove the following:

Lemma 17. *x_α satisfies (d)–(f) iff*

$$x_\alpha = \begin{cases} 1/\alpha(h), & \text{if } \alpha \in \mathbf{N} \setminus \mathbf{U} \\ 0, & \text{if } \alpha \in \mathbf{R} \setminus \mathbf{N}, \end{cases}$$

for a certain reductive subset $\mathbf{N} \subset \mathbf{R}$ such that $\mathbf{N} \supset \mathbf{U}$, and $(\mathbf{N}, \mathbf{U})$-regular element $h \in \mathfrak{h}$. □

The last lemma proves the proposition. □

Remark 18. We note that Lemmas 15, 16, and 17 are essentially contained in the paper by Donin, Gurevich, and Shnider.[4]

In that paper, among other results, the *symplectic* G-invariant structures on G/U are classified if U is a Levi subgroup of G. Actually, in this case there exists a G-equivariant symplectomorphism from G/U to a semisimple coadjoint G-orbit equipped with the Kirillov-Kostant-Souriau bracket.

Moreover, it is easy to show that if G/U has a G-invariant symplectic structure, then U is a Levi subgroup. Indeed, let $p_*\overrightarrow{x_{\mathbf{N},h}}$ (where $x_{\mathbf{N},h}$ is defined by (5)) be a G-invariant Poisson structure on G/U. Obviously, it is symplectic iff $\mathbf{N} = \mathbf{R}$. Since h is $(\mathbf{R}, \mathbf{U})$-regular, i.e., $\alpha(h) = 0$ for all $\alpha \in \operatorname{Span}\mathbf{U}$ and $\alpha(h) \neq 0$ for all $\alpha \in \mathbf{R} \setminus \mathbf{U}$, we see that $(\operatorname{Span}\mathbf{U}) \cap \mathbf{R} = \mathbf{U}$. It is well known that the latter condition is equivalent to the fact that U is a Levi subgroup.

Let us also remark that the existence of reductive non-Levi subgroups is the main difference between the triangular and the strictly quasitriangular cases. Indeed, suppose U is a Cartan subgroup. Then in the triangular case the Poisson homogeneous structures on G/U relate to *all* reductive subgroups of G, while in the strictly quasitriangular case they relate to the Levi subgroups only (see Karolinsky[17] and Lu[18]).

Finally we are going to describe the Lagrangian subalgebras corresponding to the Poisson (G, π_ρ)-homogeneous structures on G/U. Since the Lie bialgebras corresponding to (G, π_ρ) are all in the same class, we may assume without loss of generality that $\rho = 0$. It is clear that the corresponding Manin triple is $(\mathfrak{g}[\varepsilon], \mathfrak{g}, \mathfrak{g}\varepsilon)$, where $\mathfrak{g}[\varepsilon] = \mathfrak{g} \oplus \mathfrak{g}\varepsilon$, $\varepsilon^2 = 0$, and the canonical form is given by the formula

$$Q(a + b\varepsilon, c + d\varepsilon) = \langle a, d \rangle + \langle b, c \rangle.$$

Suppose $\rho = 0$. Assume that $\mathbf{N}$ and h are as in Proposition 14. Set $\pi_{\mathbf{N},h} = p_*\overrightarrow{x_{\mathbf{N},h}}$, where $x_{\mathbf{N},h}$ is defined by (5). By $\mathfrak{l}_{\mathbf{N},h}$ denote the Lagrangian subalgebra corresponding to $(G/U, \pi_{\mathbf{N},h})$ at the base point $\underline{e}$.

Proposition 19. $\mathfrak{l}_{\mathbf{N},h} = \mathfrak{u} \oplus \left(\bigoplus_{\alpha\in\mathbf{R}\setminus\mathbf{N}} \varepsilon\mathfrak{g}_\alpha\right) \oplus \left(\bigoplus_{\alpha\in\mathbf{N}\setminus\mathbf{U}} (1-\alpha(h)\varepsilon)\mathfrak{g}_\alpha\right)$

Proof. By definition (see (3)),

$$\mathfrak{l}_{\mathbf{N},h} = \{a + b\varepsilon \mid a \in \mathfrak{g}, b \in \mathfrak{u}^\perp = \mathfrak{m}, (b\otimes 1)(x_{\mathbf{N},h}) = \overline{a}\},$$

where $\overline{a}$ is the image of a in $\mathfrak{g}/\mathfrak{u} = \mathfrak{m}$. Suppose $b = E_\alpha$, where $\alpha \in \mathbf{R}\setminus\mathbf{U}$. Then

$$(b\otimes 1)(x_{\mathbf{N},h}) = \begin{cases} -\frac{1}{\alpha(h)}E_\alpha, & \text{if } \alpha \in \mathbf{N}\setminus\mathbf{U} \\ 0, & \text{if } \alpha \in \mathbf{R}\setminus\mathbf{N}. \end{cases}$$

This completes the proof. □

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