CONSTRAINT REORGANIZATION CONSISTENT WITH THE DIRAC PROCEDURE

D.M. GITMAN
Instituto de Física, Universidade de São Paulo,
C.P. 66318, 05315-970 São Paulo, SP, Brasil

I.V. TYUTIN
Lebedev Physics Institute, 53 Leninsky Pr.,
117924 Moscow, Russia

The way of finding all the constraints in the Hamiltonian formulation of singular (in particular, gauge) theories is called the Dirac procedure. The constraints are naturally classified according to the correspondig stages of this procedure. On the other hand, it is convenient to reorganize the constraints such that they are explicitly decomposed into the first-class and second-class constraints. The presence of the first-class constraints is related to the existence of gauge symmetries in the theory. The second-class constraints can be used to formulate the equations of motion and the quantization procedure in an invariant form by means of the Dirac brackets. We discuss the reorganization of the constraints into the first- and second-class constraints that is consistent with the Dirac procedure, i.e., that does not violate the decomposition of the constraints according to the stages of the Dirac procedure. The possibility of such a reorganization is important for the study of gauge symmetries in the Lagrangian and Hamiltonian formulations.

Contents

1 Introduction

It is well known that from the Hamiltonian formulation standpoint, almost all modern physical theories are theories with constraints.[1,2,3] An information about the constraint structure is important for the physical sector identification, for the study of classical and quantum symmetries, for quantization purposes and so on.

The complete set of constraints in the Hamiltonian formulation defines a constraint surface where the dynamics evolves. To describe this surface one can use different sets of equivalent constraints. We call the passage from some set of constraints to another equivalent one the reorganization of constraints. Dirac remarked that it is convenient to reorganize the constraints such that they explicitly split into the first-class constraints (FCC) and the second-class constraints (SCC). The presence of FCC is related to the existence of gauge symmetries in the theory. SCC can be used to formulate the equations of motion and the quantization procedure in an invariant form by means of the Dirac brackets. The way of finding all the constraints in the Hamiltonian formulation is usually called the Dirac procedure (DP). We recall that after the Hamiltonization, a singular Lagrangian theory is described by the Hamilton equations of motion with the primary constraints,[1,2,3]

$$\dot{\eta} = \left\{\eta, H^{(1)}\right\}, \quad \Phi^{(1)}(\eta) = 0, \quad H^{(1)} = H(\eta) + \lambda\Phi^{(1)}(\eta), \quad \dot{\eta} \equiv \frac{d\eta}{dt}. \quad (1)$$

Here $\eta = (q, p)$ are phase-space variables; $\Phi^{(1)}(\eta) = 0$ are the primary constraints (we suppose that all the primary constraints are independent); λ's are the Lagrange multipliers to the primary constraints; $H = H(\eta)$ is the Hamiltonian, and $H^{(1)}$ is the total Hamiltonian. $\{F, G\}$ denotes the Poisson bracket of two functions $F(\eta)$ and $G(\eta)$. Sometimes, the additional variables λ's can be partially or completely eliminated from Eqs. (1). Moreover, some new constraints (additional to the primary ones) may exist in the theory. The way of eliminating λ's and finding new constraints was proposed by Dirac[1] and, as was already said, is called DP. DP is a part of the complete Hamiltonization of a singular Lagrange theory. The procedure is based on the so called consistency conditions $\dot{\Phi} = 0$ which have to hold for any constraint equation $\Phi = 0$. In the general case, the Hamiltonian H and the constraints[a] Φ may depend on time t explicitly. We take such a possibility into account . However, the argument t will not be written explicitly. Using Eqs. (1), we can transform the

[a] We call often the functions Φ constraints as well.

consistency condition to the form

$$\dot{\Phi} = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial \eta}\dot{\eta} = \left\{ \Phi\,, H^{(1)} + \epsilon \right\} = 0\,. \tag{2}$$

Here, ϵ is the momentum conjugate to time t and the Poisson brackets are defined in the extended phase space of the variables $\eta; t, \epsilon$, see for details Ref. 2. Finding the primary constraints can be considered the first stage of DP. At the second stage of DP, we apply the consistency conditions (2) to the primary constraints trying to define some λ's. Those λ's that can be defined here are denoted by $\lambda_1 = \bar{\lambda}_1(\eta)$. In addition, we can reveal some new independent constraints $\Phi^{(2)} = 0$; we call them the second-stage constraints. We can substitute expressions for λ_1 directly in the total Hamiltonian to construct the Hamiltonian $H_1^{(1)} = H^{(1)}\big|_{\lambda_1 = \bar{\lambda}_1}$. At the third stage, we use the consistency conditions for the second-stage constraints to find some λ's (these are denoted by $\lambda_2 = \bar{\lambda}_2(\eta)$) and to reveal some new third-stage constraints $\Phi^{(3)}(\eta) = 0$ independent from the previous ones. We can substitute expressions for $\lambda_{(2)}$ directly in the Hamiltonian $H^{(1)}$ to construct the Hamiltonian $H_2^{(1)} = H^{(1)}\big|_{\lambda_1 = \bar{\lambda}_1, \lambda_2 = \bar{\lambda}_2}$. Continuing DP, we can determine some λ's and obtain some new independent constraints. At the r-th stage, we obtain $\lambda_{r-1} = \bar{\lambda}_{r-1}(\eta)$ and $\Phi^{(r)}(\eta) = 0$ and construct the Hamiltonian $H_{r-1}^{(1)} = H^{(1)}\big|_{\lambda_1 = \bar{\lambda}_1, \ldots, \lambda_{r-1} = \bar{\lambda}_{r-1}}$. Because the number of the degrees of freedom is finite and EM are assumed to be consistent, DP stops at a certain k-th stage, after which new constraints do not appear. We will refer to all the constraints that were obtained by DP and differ from the primary constraints as the secondary constraints, that is, the secondary constraints are the second-, third-, and etc. stage constraints.

We use the notation $\Phi^{(i,\ldots,j)} \equiv \left(\Phi^{(i)}, \ldots, \Phi^{(j)}\right)$, $1 \leq i < j \leq k$. The secondary constraints are then $\Phi^{(2,\ldots,k)}$, and the complete set of constraints, both primary and secondary, of the theory is $\Phi = \Phi^{(1,\ldots,k)} = \left(\Phi^{(1)}, \ldots, \Phi^{(k)}\right)$. Schematically, DP can be represented as

$$\begin{array}{ccccccccc} \Phi^{(1)} \rightarrow & \Phi^{(2)} & \rightarrow & \Phi^{(3)} & \rightarrow \cdots \rightarrow & \Phi^{(k)} & \rightarrow & O(\Phi) & . \\ & \bar{\lambda}_1 & & \bar{\lambda}_2 & & \bar{\lambda}_{k-1} & & \bar{\lambda}_k & \end{array} \tag{3}$$

Here, the arrow $\rightarrow$ implies DP, and $O(\Phi)$ denotes the terms proportional to the functions Φ, i.e., $O(\Phi)|_{\Phi=0} = 0$.

The important question is: does a consistent with DP constraint reorganization to the first- and second-class constraints exist, i.e., the reorganization that does not violate the decomposition of the constraints according to their stages in DP. The problem is important for understanding the general structure of singular theories. In particular, the existence of such a reorganization is a crucial point for finding a relation between the constraint structure and the symmetry structure of singular (gauge) theories.[4] This problem was considered by many authors.[5] However, in these publications, either the theories of a particular form were considered or too restrictive assumptions were used.

In the present paper, we are going to demonstrate that a complete set of constraints Φ can be reorganized to the chains of SCC and FCC in consistency with their hierarchy in DP. The possibility of such a constraint reorganization is formulated as the following statement.

It is possible to reorganize the complete set of constraints obtained in DP to the form: $\Phi = \left(\Phi^{(i)}\right) = 0\,,\ \Phi^{(i)} = (\varphi^{(i)};\chi^{(i)}),\ i = 1, ..., k,\ (\chi^{(k)} \equiv 0)$, where $\Phi^{(i)}$ are the constraints of the i-th stage, $\varphi^{(i)}$ are the SCC functions of the i-th stage, $\chi^{(i)}$ are the FCC functions of the i-th stage, and k is the number of all the stages of DP.

The constraints $\chi^{(i)}$ and $\varphi^{(i)}$ are decomposed into the groups

$$\varphi^{(i)} = \left(\varphi^{(i|u)},\ u = 1, ..., k\right),\ \ \chi^{(i)} = \left(\chi^{(i|a)},\ a = 1, ..., k-1\right) \tag{4}$$

such that the total Hamiltonian and the Lagrange multipliers λ have the form

$$H^{(1)} = H + \lambda_{\varphi^u}\varphi^{(1|u)} + \lambda_{\chi^a}\chi^{(1|a)}\,,\ \ \lambda = (\lambda_{\varphi^u}, \lambda_{\chi^a})\ . \tag{5}$$

Each of the groups $\varphi^{(i|u)}$, $\chi^{(i|a)}$, λ_{φ^u}, and λ_{χ^a} may be either empty or contain several functions: $\varphi^{(i|u)} = \left(\varphi^{(i|u)}_{\mu_u}\,,\ \mu_u = 1, ..., r_u\right)$, $\chi^{(i|a)} = \left(\chi^{(i|a)}_{\rho_a}\,,\ \rho_a = 1, ..., s_a\right)$, $\lambda_{\varphi^u} = \left(\lambda^{\mu_u}_{\varphi^u}\right)$, $\lambda_{\chi^a} = (\lambda^{\rho_a}_{\chi^a})$. Each group $\varphi^{(1|u)}$ and $\chi^{(1|a)}$ produces a chain of the groups of constraints of the second, third, and so on stages within DP, $\varphi^{(1|u)} \to \varphi^{(2|u)} \to \varphi^{(3|u)} \to \cdots \to \varphi^{(u|u)} \to \lambda_{\varphi^u} = \bar{\lambda}_u$, $\chi^{(1|a)} \to \chi^{(2|a)} \to \chi^{(3|a)} \to \cdots \to \chi^{(a|a)}$. Here, the indices u and a after the sign of the vertical bar in the superscripts number the constraint chains. All the constraints in a chain are of the same class, and all the groups in the chain have the same number of constraints.

The chain of SCC with the number u ends up with the group of the u-th-stage constraints. Their consistency conditions define the $\lambda_{\varphi^u}-$ multipliers. The chain of FCC with the number a ends up with the group of the a-th-stage constraints. The constraints of the last group of any chain of FCC are not involved in determining new constraints.

The Poisson brackets of the SCC constraints from different chains vanish on the constraint surface.

The described hierarchy of the constraints in DP schematically looks as follows:

$$\begin{array}{lllllllll}
\varphi^{(1|1)} & \rightarrow & \bar{\lambda}_1 & & & & & & \\
\varphi^{(1|2)} & \rightarrow & \varphi^{(2|2)} & \rightarrow & \bar{\lambda}_2 & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\
\varphi^{(1|k-1)} & \rightarrow & \varphi^{(2|k-1)} & & \cdots \rightarrow & \varphi^{(k-1|k-1)} & \rightarrow & \bar{\lambda}_{k-1} & \\
\varphi^{(1|k)} & \rightarrow & \varphi^{(2|k)} & & \cdots \rightarrow & \varphi^{(k-1|k)} & \rightarrow & \varphi^{(k|k)} & \rightarrow \quad \bar{\lambda}_k \\
\chi^{(1|k-1)} & \rightarrow & \chi^{(2|k-1)} & & \cdots \rightarrow & \chi^{(k-1|k-1)} & \rightarrow & O(\Phi) & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\
\chi^{(1|2)} & \rightarrow & \chi^{(2|2)} & \rightarrow & O(\Phi^{(1,2,3)}) & & & & \\
\chi^{(1|1)} & \rightarrow & O(\Phi^{(1,2)}) & & & & & &
\end{array} .$$

In addition: [b]

$$\begin{aligned}
& [\lambda_{\varphi^u}] = [\bar{\lambda}_u] = [\varphi^{(i|u)}] = [\varphi^{(1|u)}], \quad [\lambda_{\chi^a}] = [\chi^{(i|a)}] = [\chi^{(1|a)}]\,, \\
& \{\varphi^{(i|u)}, \varphi^{(j|v)}\} = \{\Phi\}, \quad \{\chi^{(i|a)}, \Phi\} = O(\Phi),\ u \neq v\,, \\
& u, v = 1, ..., k\,,\ a = 1, ..., k-1\,.
\end{aligned}$$

In what follows, we present a constructive proof of the statement. Namely, considering a specific version of DP (which we call the refined DP), we construct the above-mentioned set of constraints. It is this set of constraints that we call the constraints consistent with DP.

2 Refined Dirac procedure

We begin with some remarks about the theories under consideration. The only restrictions to be imposed on the theories follow from the requirement of applicability of DP. These requirements will be formulated in terms of the ranks of some Jacobi matrices of the type $\partial\Phi^{(1,\ldots l)}/\partial\eta$ and of the Poisson bracket matrices of the type $\left\{\Phi^{(1)}, \Phi^{(l)}\right\}$. We assume that these matrices are of a constant rank. Literally, this means that they are of a constant rank in a vicinity of the point $\eta = 0$ on the corresponding constraint surface $\Phi^{(1,\ldots l)} = 0$. We also assume that $H = O\left(\eta^2\right)$ and $\Phi^{(1)} = O\left(\eta\right)$. As was already said above, we assume that primary constraint functions $\Phi^{(1)}$ are independent, that is,

$$\operatorname{rank}\frac{\partial\Phi^{(1)}}{\partial\eta} = [\Phi^{(1)}] = \mathrm{m}_1\,. \tag{6}$$

[b] Here and in what follows, we use the notation
$[f]$ = the number of the functions f, i.e., $f = (f_i,\ i = 1, ..., n) \rightarrow\ [f] = n$.

2.1 First stage

Consider the antisymmetric matrix $C^{(1)}_{\alpha_0\beta_0} = \{\Phi^{(1)}_{\alpha_0}, \Phi^{(1)}_{\beta_0}\}$. This matrix appears in consistency conditions (2) for the primary constraints,

$$\{\Phi^{(1)}_{\alpha_0}, H^{(1)} + \epsilon\} = \{\Phi^{(1)}_{\alpha_0}, H + \epsilon\} + C^{(1)}_{\alpha_0\beta_0}\lambda^{\beta_0} = 0\,. \tag{7}$$

We suppose that the matrix $C^{(1)}$ has a constant rank, $\operatorname{rank} C^{(1)} = r_1$. Therefore, there exists a submatrix of size $r_1 \times r_1$, that is located on the diagonal and is also antisymmetric, we let denote it by $M^{(1)}_{\mu_1\nu_1}$, $[\nu_1] = [\mu_1] = r_1$, $\det M^{(1)} \neq 0$. The equation

$$C^{(1)}Z^{(1)} = O\left(\Phi^{(1)}\right)$$

has $m_2' = m_1 - r_1$ linearly independent solutions $Z^{(1)}_{\alpha_1}{=}Z^{(1)\alpha_0}_{\alpha_1}$, $[\alpha_1] = m_2'$, such that $\det Z^{(1)\beta_1}_{\alpha_1} \neq 0$, $\alpha_0 = (\mu_1, \beta_1)$. Together with the vectors $Z^{(1)}_{\mu_1} = Z^{(1)\alpha_0}_{\mu_1} = \delta^{\alpha_0}_{\mu_1}$, these solutions form a set of m_1 linearly independent vectors (see, for example, Ref. 2, p. 27). Using a nonsingular matrix Z_1, we reorganize the primary constraints:

$$\Phi^{(1)} \to Z_1\Phi^{(1)} = \left(\begin{array}{l} \varphi^{(1|1)}_{\mu_1} = \Phi^{(1)}_{\mu_1} \\ \phi^{(1|1)}_{\alpha_1} = Z^{(1)\alpha_0}_{\alpha_1}\Phi^{(1)}_{\alpha_0} \end{array}\right),\ \alpha_0 = (\mu_1, \alpha_1)\ ,$$

$$Z_1 = Z^{\beta_0}_{1\alpha_0} = \left(\begin{array}{cc} \delta^{\nu_1}_{\mu_1} & 0 \\ Z^{(1)\nu_1}_{\alpha_1} & Z^{(1)\beta_1}_{\alpha_1} \end{array}\right),\ \beta_0 = (\nu_1, \beta_1)\ .$$

We thus have

$$\Phi^{(1)} \xrightarrow{Z} \left(\begin{array}{c} \varphi^{(1|1)} \\ \phi^{(1|1)} \end{array}\right),\quad H^{(1)} = H + \lambda_{\varphi^1}\varphi^{(1|1)} + \lambda_{\phi^1}\phi^{(1|1)}\,. \tag{8}$$

We call such a kind of reorganization the $Z-$reorganization.

We remark that any reorganization of primary constraints is always accompanied by the corresponding λ-multiplier reorganization.[2] These new λ's appear as the multipliers in front of the reorganized primary constraints.

The new primary constraints satisfy the properties

$$\left\{\varphi^{(1|1)}, \varphi^{(1|1)}\right\} = M^{(1)},\ \det M^{(1)} \neq 0\,, \tag{9}$$

$$\left\{\phi^{(1|1)}, \varphi^{(1|1)}\right\} = O\left(\Phi^{(1)}\right),\ \left\{\phi^{(1|1)}, \phi^{(1|1)}\right\} = O\left(\Phi^{(1)}\right)\ . \tag{10}$$

Considering the consistency conditions (7) for the primary constraints $\varphi^{(1|1)}$, we determine the Lagrange multipliers λ_{φ^1} :

$$\lambda_{\varphi^1} = \bar{\lambda}_1 \equiv -\left[M^{(1)}\right]^{-1}\left\{\varphi^{(1|1)},\, H + \epsilon\right\}\,. \tag{11}$$

It is convenient to represent the Hamiltonian (8) as

$$H^{(1)} = H_1^{(1)} + \Lambda_1 \varphi^{(1|1)},\ \Lambda_1 = \lambda_{\varphi^1} - \bar{\lambda}_1,$$
$$H_1^{(1)} = H_1 + \lambda_{\phi^1}\phi^{(1|1)},\ H_1 = H + \bar{\lambda}_1 \varphi^{(1|1)}\,. \tag{12}$$

The properties

$$\left\{\phi^{(1|1)}, H^{(1)}+\epsilon\right\} = \left\{\phi^{(1|1)}, H_1^{(1)}+\epsilon\right\} + O\left(\Phi^{(1)}\right) = \left\{\phi^{(1|1)}, H_1+\epsilon\right\} + O\left(\Phi^{(1)}\right),$$
$$\left\{\varphi^{(1|1)}, H_1+\epsilon\right\} = O\left(\Phi^{(1)}\right) \tag{13}$$

are valid.

2.2 Second stage

The consistency conditions (7) for the primary constraints $\phi^{(1|1)}$ result in the secondary constraints (the second-stage constraints)

$$\left\{\phi_{\alpha_1}^{(1|1)}, H_1 + \epsilon\right\} \equiv \phi_{\alpha_1}^{(2|1)} = 0\,,$$

which obey the relations

$$\left\{\varphi^{(1,1)}, \phi^{(2|1)}\right\} = \left\{\varphi^{(1,1)}, \left\{\phi^{(1|1)}, H_1 + \epsilon\right\}\right\} \tag{14}$$
$$= \left\{\left\{\varphi^{(1,1)}, \phi^{(|1)}\right\},\ H_1 + \epsilon\right\} + \left\{\phi^{(1|1)}, \left\{\varphi^{(1,1)}, H_1 + \epsilon\right\}\right\} = O\left(\Phi^{(1)}, \phi^{(2|1)}\right)\,.$$

Together with the primary constraints they may form a dependent set of constraints. We suppose that the matrix $\partial\left(\Phi^{(1)}, \phi^{(2|1)}\right)/\partial\eta$ has a constant rank,

$$\mathrm{rank}\frac{\partial\left(\Phi^{(1)}, \phi^{(2|1)}\right)}{\partial\eta} = \mathrm{m}_1 + \mathrm{m}_2 \le \left[\Phi^{(1)}\right] + \left[\phi^{(2|1)}\right]\,. \tag{15}$$

We now consider the consistency conditions for the constraints $\phi^{(2|1)}$ (one can use the Hamiltonian $H_1^{(1)}$ instead of $H^{(1)}$ in DP)

$$\{\phi^{(2|1)}, H_1^{(1)} + \epsilon\} = \{\phi^{(2|1)}, H_1 + \epsilon\} + C^{(2)}\lambda_{\phi^1} = 0\,,$$
$$C^{(2)} = C^{(2)}_{\alpha_1\beta_1} = \left\{\phi_{\alpha_1}^{(2|1)}, \phi_{\beta_1}^{(1|1)}\right\}\,. \tag{16}$$

We can see that the matrix $C^{(2)}$ obeys the relation

$$
\begin{aligned}
C^{(2)}_{\alpha_1\beta_1} &= \left\{\left\{\phi^{(1|1)}_{\alpha_1}, H_1+\epsilon\right\}, \phi^{(1|1)}_{\beta_1}\right\} \\
&= \left\{\phi^{(1|1)}_{\alpha_1}, \left\{H_1+\epsilon, \phi^{(1|1)}_{\beta_1}\right\}\right\} + \left\{\left\{\phi^{(1|1)}_{\alpha_1}, \phi^{(1|1)}_{\beta_1}\right\}, H_1+\epsilon\right\} \\
&= \left\{\phi^{(2|1)}_{\beta_1}, \phi^{(1|1)}_{\alpha_1}\right\} + O\left(\Phi^{(1)}, \phi^{(2|1)}\right) = C^{(2)}_{\beta_1\alpha_1} + O\left(\Phi^{(1)}, \phi^{(2|1)}\right) ,
\end{aligned}
$$

this means that $C^{(2)}$ is symmetric on the constraint surface $\Phi^{(1)} = \phi^{(2|1)} = 0$ (we say that $C^{(2)}$ is $O\left(\Phi^{(1)}, \phi^{(2|1)}\right)$-symmetric). We suppose that the matrix $C^{(2)}$ has a constant rank, $\operatorname{rank} C^{(2)} = r_2$. Therefore, there exists a submatrix of size $r_2 \times r_2$, which is located on the diagonal and is also symmetric, we let denote it by $M^{(2)}_{\mu_2\nu_2}$, $[\nu_2] = [\mu_2] = r_2$, $\det M^{(2)} \neq 0$. We now apply the $Z-$reorganization to the constraints $\phi^{(1|1)}, \phi^{(2|1)}$. The equation

$$
C^{(2)} Z^{(2)} = O\left(\Phi^{(1)}, \phi^{(2|1)}\right)
$$

has $m''_3 = m'_2 - r_2$ linearly independent solutions $Z^{(2)}_{\sigma_1} = Z^{(2)\alpha_1}_{\sigma_1}$, $[\sigma_1] = m''_3$, such that $\det Z^{(2)\sigma'_1}_{\sigma_1} \neq 0$ ($\alpha_1 = (\mu_2, \sigma_1)$, $[\mu_2] = r_2$). Together with the vectors $Z^{(2)}_{\mu_2} = Z^{(2)\alpha_1}_{\mu_2} = \delta^{\alpha_1}_{\mu_2}$, these solutions form a set of m'_2 linearly independent vectors. Using a nonsingular matrix Z_2, we reorganize the constraints $\phi^{(1|1)}, \phi^{(2|1)}$:

$$
\begin{aligned}
&\phi^{(1|1)} \xrightarrow{Z} Z_2\phi^{(1|1)} = \left(\begin{array}{l} \varphi^{(1|2)}_{\mu_2} = \phi^{(1|1)}_{\mu_2} \\ \Psi^{(1|1)}_{\sigma_1} = Z^{(2)\alpha_1}_{\sigma_1}\phi^{(1|1)}_{\alpha_1} \end{array}\right), \\
&\phi^{(2|1)} \xrightarrow{Z} Z_2\phi^{(2|1)} = \left(\begin{array}{l} \varphi'^{(2|2)}_{\mu_2} = \phi^{(2|1)}_{\mu_2} \\ \Psi'^{(2|1)}_{\sigma_1} = Z^{(2)\alpha_1}_{\sigma_1}\phi^{(2|1)}_{\alpha_1} \end{array}\right), \\
&Z^{\beta_1}_{2\alpha_1} = \left(\begin{array}{ll} \delta^{\nu_2}_{\mu_2} & 0 \\ Z^{(2)\nu_2}_{\sigma_1} & Z^{(2)\sigma'_1}_{\sigma_1} \end{array}\right).
\end{aligned}
$$

The new reorganized constraints have the properties

$$
\begin{aligned}
&\left\{\varphi'^{(2|2)}, \varphi^{(1|2)}\right\} = M^{(2)}, \quad \det M^{(2)} \neq 0 , \\
&\left\{\Psi'^{(2|1)}, \varphi^{(1|2)}\right\} = O\left(\Phi^{(1)}, \phi^{(2|1)}\right), \quad \left\{\Psi'^{(2|1)}, \Psi^{(1|1)}\right\} = O\left(\Phi^{(1)}, \phi^{(2|1)}\right), \\
&\left\{\varphi'^{(2,2)}, \Psi^{(1|1)}\right\} = O\left(\Phi^{(1)}, \phi^{(2|1)}\right) . \qquad (17)
\end{aligned}
$$

In addition, we reorganize the second-stage constraints adding some terms proportional to the first-stage constraints::

$$\varphi'^{(2|2)}_{\mu_2} \to \varphi^{(2|2)}_{\mu_2} = \varphi'^{(2|2)}_{\mu_2} - \frac{1}{2}\varphi^{(1|2)}\left[M^{(2)}\right]^{-1}\left\{\varphi'^{(2|2)}, \varphi'^{(2|2)}_{\mu_2}\right\},$$
$$\Psi'^{(2|1)}_{\sigma_1} \to \Psi^{(2|1)}_{\sigma_1} = \Psi'^{(2|1)}_{\sigma_1} - \varphi^{(1|2)}\left[M^{(2)}\right]^{-1}\left\{\varphi'^{(2|2)}, \Psi'^{(2|1)}_{\sigma_1}\right\}. \quad (18)$$

We thus have

$$\begin{pmatrix} \phi^{(1|1)} \\ \phi^{(2|1)} \end{pmatrix} \xrightarrow{Z} \begin{pmatrix} \varphi^{(1|2)} \\ \Psi^{(1|1)} \\ \varphi'^{(2|2)} \\ \Psi'^{(2|1)} \end{pmatrix} \to \begin{pmatrix} \varphi^{(1|2)} \\ \Psi^{(1|1)} \\ \varphi^{(2|2)} \\ \Psi^{(2|1)} \end{pmatrix}.$$

The Poisson brackets for the reorganized second-stage constraints $\varphi^{(2|2)}, \Psi^{(2|1)}$ are

$$\left\{\varphi^{(2|2)}, \varphi^{(2|2)}\right\} = O\left(\Phi^{(1)}, \phi^{(2|1)}\right),\ \left\{\varphi^{(2|2)}, \Psi^{(2|1)}\right\} = O\left(\Phi^{(1)}, \phi^{(2|1)}\right),$$
$$\left\{\varphi^{(2,2)}, \Psi^{(1|1)}\right\} = O\left(\Phi^{(1)}, \phi^{(2|1)}\right),\ \left\{\varphi^{(2|2)}, \varphi^{(1|2)}\right\} = M^{(2)} + O\left(\Phi^{(1)}, \phi^{(2|1)}\right),$$
$$\left\{\Psi^{(2|1)}, \varphi^{(1|2)}\right\} = O\left(\Phi^{(1)}, \phi^{(2|1)}\right),\ \left\{\Psi^{(2|1)}, \Psi^{(1|1)}\right\} = O\left(\Phi^{(1)}, \phi^{(2|1)}\right). \quad (19)$$

At the same time, the reorganized constraints are related to the primary constraints by

$$\varphi^{(2|2)} = \left\{\varphi^{(1|2)}, H_1 + \epsilon\right\} + O\left(\Phi^{(1)}\right) = \left\{\varphi^{(1|2)}, H^{(1)} + \epsilon\right\} + O\left(\Phi^{(1)}\right),$$
$$\Psi^{(2|1)} = \left\{\Psi^{(1|1)}, H_1 + \epsilon\right\} + O\left(\Phi^{(1)}\right) = \left\{\Psi^{(1|1)}, H^{(1)} + \epsilon\right\} + O\left(\Phi^{(1)}\right).$$

One can see that the constraints $\Phi^{(1)}, \varphi^{(2|2)}$ are independent.

Taking (15) into account, we can reorganize the constraints $\Psi^{(2|1)}$ as follows: $\Psi^{(2|1)}_{\sigma_1} \to \left(\phi^{(2|2)}_{\alpha_2} = \Psi^{(2|1)}_{\alpha_2}, \chi^{(2|1)}_{\rho_1} = U^{(2)\sigma_1}_{\rho_1}\Psi^{(2|1)}_{\sigma_1}\right)$, $\sigma_1 = (\alpha_2, \rho_1)$, $\left[\phi^{(2|2)}\right] = m_2 - r_2 \equiv m'_3$, where $U^{(2)}_{\rho_1}$ are $m''_3 - m'_3 \equiv s_1$ independent vectors such that the constraints $\Phi^{(1,2)}$ are independent, and $\chi^{(2|1)} = O\left(\Phi^{(1)}, \varphi^{(2|2)}\right)$, where $\Phi^{(2)} = \left(\varphi^{(2|2)}, \phi^{(2|2)}\right)$, $\left[\Phi^{(2)}\right] = m_2$; we then reorganize the constraints $\Psi^{(1|1)}_{\sigma_1}$: $\Psi^{(1|1)}_{\sigma_1} \to \left(\phi^{(1|2)}_{\alpha_2} = \Psi^{(1|1)}_{\alpha_2}, \chi^{(1|1)}_{\rho_1} = U^{(2)\sigma_1}_{\rho_1}\Psi^{(1|1)}_{\sigma_1}\right)$, $\left[\chi^{(1|1)}\right] = s_1$,

$m_1 = r_1 + r_2 + s_1 + m_3'$, $m_2 = r_2 + m_3'$. The new constraints obey the relations

$$\begin{aligned}
&\phi^{(2|2)} = \left\{\phi^{(1|2)}, H_1 + \epsilon\right\} + O\left(\Phi^{(1)}\right) = \left\{\phi^{(1|2)}, H^{(1)} + \epsilon\right\} + O\left(\Phi^{(1)}\right), \\
&\left\{\chi^{(1|1)}, H^{(1)} + \epsilon\right\} = \chi^{(2|1)} + O\left(\Phi^{(1)}\right) = O\left(\Phi^{(1)}, \varphi^{(2|2)}\right) . \qquad (20)
\end{aligned}$$

Thus, the consistency conditions for the constraints $\chi^{(1|1)}$ do not lead to any new constraints. The consistency conditions for the constraints $\phi^{(2|2)}$ allow us to find the Lagrange multipliers λ_{φ^2} ,

$$\lambda_{\varphi^2} = -\left[M^{(2)}\right]^{-1}\left\{\phi^{(2|2)},\, H_1 + \epsilon\right\} \equiv \bar{\lambda}_2 \,. \qquad (21)$$

It is now useful to represent the Hamiltonian $H^{(1)}$ as

$$\begin{aligned}
&H^{(1)} = H_2^{(1)} + \lambda_{\chi^1}\chi^{(1|1)} + \Lambda_1\varphi^{(1|1)} + \Lambda_2\varphi^{(1|2)}, \\
&H_2^{(1)} = H_2 + \lambda_{\phi^2}\phi^{(1|2)}, \;\; \Lambda_2 = \lambda_{\varphi^2} - \bar{\lambda}_2\,, \\
&H_2 = H_1 + \bar{\lambda}_2\varphi^{(1|2)} = H + \bar{\lambda}_1\varphi^{(1|1)} + \bar{\lambda}_2\varphi^{(1|2)} \,. \qquad (22)
\end{aligned}$$

Finally, after the two first stages, we have the following picture:

$$\begin{array}{lcl}
\varphi^{(1|1)} & \rightarrow & \bar{\lambda}_1 \\
\varphi^{(1|2)} & \rightarrow & \varphi^{(2|2)} \rightarrow \bar{\lambda}_2 \\
\phi^{(1|2)} & \rightarrow & \phi^{(2|2)} \\
\chi^{(1|1)} & \rightarrow & O\left(\Phi^{(1)}, \varphi^{(2|2)}\right)
\end{array} .$$

The reorganized constraints

$$\begin{aligned}
&\Phi^{(1)} = \left(\varphi^{(1|1)}, \varphi^{(1|2)}, \phi^{(1|2)}, \chi^{(1|1)}\right) - \text{primary constraints}\,, \\
&\Phi^{(2)} = \left(\varphi^{(2|2)}, \phi^{(2|2)}\right) - \text{second} - \text{stage constraints} \qquad (23)
\end{aligned}$$

are independent. They obey the relations:

$$
\begin{aligned}
&\left\{\varphi^{(1|1)}, \varphi^{(1|1)}\right\} = M^{(1)} + O\left(\Phi^{(1)}\right), \\
&\left\{\varphi^{(1|2)}, \varphi^{(2|2)}\right\} = M^{(2)} + O\left(\Phi^{(1,2)}\right), \\
&\left\{\varphi^{(1|2)}, \varphi^{(1|2)}\right\} = O\left(\Phi^{(1)}\right), \left\{\varphi^{(2,2)}, \varphi^{(2|2)}\right\} = O\left(\Phi^{(1,2)}\right), \\
&\left\{\varphi^{(1|1)}, \varphi^{(1|2)}\right\} = O\left(\Phi^{(1)}\right), \left\{\varphi^{(1|1)}, \varphi^{(2,2)}\right\} = O\left(\Phi^{(1,2)}\right), \\
&\left\{\varphi^{(1|1)}, \phi^{(1|2)}\right\} = O\left(\Phi^{(1)}\right), \left\{\varphi^{(1|1)}, \chi^{(1|1)}\right\} = O\left(\Phi^{(1)}\right), \\
&\left\{\varphi^{(1|2)}, \Phi^{(1)}\right\} = O\left(\Phi^{(1)}\right), \left\{\phi^{(1|2)}, \Phi^{(1)}\right\} = O\left(\Phi^{(1)}\right), \\
&\left\{\chi^{(1|1)}, \Phi^{(1)}\right\} = O\left(\Phi^{(1)}\right), \\
&\left\{\varphi^{(2,2)}, \phi^{(1|2)}\right\} = O\left(\Phi^{(1,2)}\right), \left\{\varphi^{(2,2)}, \chi^{(1|1)}\right\} = O\left(\Phi^{(1,2)}\right), \\
&\left\{\varphi^{(2,2)}, \phi^{(2|2)}\right\} = O\left(\Phi^{(1,2)}\right), \left\{\phi^{(2,2)}, \Phi^{(1)}\right\} = O\left(\Phi^{(1,2)}\right),
\end{aligned} \tag{24}
$$

and

$$
\begin{aligned}
&\left\{\varphi^{(2|2)}, H_2 + \varepsilon\right\} = O\left(\Phi^{(1,2)}\right), \\
&\left\{\phi^{(2|2)}, H^{(1)} + \varepsilon\right\} = \left\{\phi^{(2|2)}, H_2 + \varepsilon\right\} + O\left(\Phi^{(1,2)}\right), \\
&\left\{\varphi^{(1|1)}, H_i + \varepsilon\right\} = O\left(\Phi^{(1)}\right), \\
&\left\{\chi^{(1|1)}, H_i + \varepsilon\right\} = O\left(\Phi^{(1)}, \varphi^{(2|2)}\right), \\
&\left\{\varphi^{(1|2)}, H_i + \varepsilon\right\} = \varphi^{(2,2)} + O\left(\Phi^{(1)}\right), \\
&\left\{\phi^{(1|2)}, H_i + \varepsilon\right\} = \phi^{(2,2)} + O\left(\Phi^{(1)}\right), \ i = 1, 2,
\end{aligned}
$$

where the Hamiltonians H_i , are given by Eqs. (12) and (22). The commutation relations $\left\{\phi^{(2,2)}, \phi^{(2|2)}\right\}$ remain unknown at this stage.

It follows from (24) that $\varphi^{(1|2)}, \varphi^{(2|2)}$ are SCC.

Considering the consistency conditions for the constraints $\phi^{(2|2)}$, we can use the Hamiltonian H_2 instead of $H^{(1)}$. These consistency conditions result in the third-stage constraints,

$$\phi^{(3|2)} \equiv \left\{\phi^{(2|2)}, H_2 + \varepsilon\right\} = 0. \tag{25}$$

The functions $\phi^{(3|2)}$ must be analyzed similarly to the previous consideration.

2.3 p-th stage (induction hypothesis)

We are now going to prove that the above-refined DP formulated for two stages can be continued producing similar structures for any-stage constraints. The proof is by induction. The induction hypothesis is formulated as follows.

Suppose that after any $l \leq p$ stages of the refined DP, the constraints $\Phi^{(1,\ldots,l)}$ and the total Hamiltonian $H^{(1)}$ can be reorganized as

$$\begin{aligned}
&\Phi^{(i)} = \left(\varphi_{\mu_u}^{(i|u)},\, \phi_{\alpha_l}^{(i|l)},\, \chi_{\rho_a}^{(i|a)}\right);\ \left[\Phi^{(i)}\right] = m_i = \sum_{j=i}^{l} r_j + \sum_{a=i}^{l-1} s_a + m'_{l+1}\,,\\
&1 \leq i \leq l,\ i \leq u \leq l,\ i \leq a \leq l-1\,,\\
&\left[\varphi^{(i|u)}\right] = r_u\,,\ \left[\chi^{(i|a)}\right] = s_a\,,\ \left[\phi^{(i|l)}\right] = m'_{l+1}\,,\\
&H^{(1)} = H_l^{(1)} + \sum_{u=1}^{l} \Lambda_u \varphi^{(1|u)} + \sum_{a=1}^{l-1} \lambda_{\chi^a} \chi^{(1|a)},\ \Lambda_u = \lambda_{\varphi^u} - \bar{\lambda}_u\,,\\
&H_l^{(1)} = H_l + \lambda_{\phi^l} \phi^{(1|l)},\ H_l = H_{l-1} + \bar{\lambda}_l \varphi^{(1|l)} = H + \sum_{u=1}^{l} \bar{\lambda}_u \varphi^{(1|u)}\,,\\
&\bar{\lambda}_u = -\left[M^{(u)}\right]^{-1} \left\{\varphi^{(u|u)}, H_{u-1} + \varepsilon\right\}\,,\ M^{(u)} = \left\{\varphi^{(u|u)}, \varphi^{(1|u)}\right\},\\
&\det M^{(u)} \neq 0\,,\ \ (H_0 \equiv H)\ .
\end{aligned} \tag{26}$$

All the constraints $\Phi^{(1,\ldots,l)}$ are independent. In passing from l-th stage ($l \leq p-1$) to $(l+1)$-th stage, the only constraints to be reorganized are $\phi^{(i|l)}$.

The constraints obey the relations

$$u < v:\ \left\{\varphi^{(i|u)}, \varphi^{(j|v)}\right\} = \left\{\begin{array}{l} O\left(\Phi^{(1,\ldots,i+j-1)}\right),\ i+j \leq u+1 \\ O\left(\Phi^{(1,\ldots,u)}\right),\ i+j > u+1,\ j \leq u \\ O\left(\Phi^{(1,\ldots,j)}\right),\ j > u\,, \end{array}\right.$$

$$\left\{\varphi^{(i|u)}, \varphi^{(j|u)}\right\} = \left\{\begin{array}{l} O\left(\Phi^{(1,\ldots,i+j-1)}\right),\ i+j \leq u+1 \\ (-1)^{u-i} M^{(u)} + O\left(\Phi^{(1,\ldots,u)}\right),\ i+j = u+1 \\ O\left(\Phi^{(1,\ldots,j)}\right),\ i+j > u+1\,, \end{array}\right. \tag{27}$$

i.e., $\varphi^{(i|u)}$ are of the second class,

$$
\begin{aligned}
&\left\{T_l^{(i)}, T_l^{(j)}\right\} = O\left(\Phi^{(1,\dots,i+j-1)}\right),\ i+j \le l+1\,,\\
&T_l^{(i)} \equiv \left(\phi^{(i|l)},\ \chi^{(i|a)},\ i \le a \le l-1\right);\\
&\left\{\varphi^{(i|u)}, T_l^{(j)}\right\} = \left\{\begin{array}{l} O\left(\Phi^{(1,\dots,i+j-1)}\right),\ i+j \le u+1 \\ O\left(\Phi^{(1,\dots,u)}\right),\ i+j > u+1,\ j \le u \\ O\left(\Phi^{(1,\dots,j)}\right),\ j > u, \end{array}\right.
\end{aligned} \tag{28}
$$

and

$$
\begin{aligned}
&\left\{\varphi^{(i|u)}, H_j + \varepsilon\right\} = \varphi^{(i+1|u)} + O\left(\Phi^{(1,\dots,i)}\right),\ j \ge i,\ u > i\,,\\
&\left\{\varphi^{(i|i)}, H_j + \varepsilon\right\} = O\left(\Phi^{(1,\dots,i)}\right),\ j \ge i\,,\\
&\left\{\phi^{(i|l)}, H_j + \varepsilon\right\} = \phi^{(i+1|l)} + O\left(\Phi^{(1,\dots,i)}\right),\ i \le j,\ i \le l-1\,,\\
&\left\{\chi^{(i|a)}, H_j + \varepsilon\right\} = \chi^{(i+1|a)} + O\left(\Phi^{(1,\dots,i)}\right),\ i \le j,\ i \le l-1,\ a \ge i+1\,,\\
&\left\{\chi^{(i|i)}, H_j + \varepsilon\right\} = O\left(\Phi^{(1,\dots,i)}, \varphi^{(i+1|i+1)}\right),\ i \le j,\ i \le l-1\,,\\
&\left\{\phi^{(l|l)}, H^{(1)} + \varepsilon\right\} = \left\{\phi^{(l|l)}, H_l + \varepsilon\right\} + O\left(\Phi^{(1,\dots,l)}\right).
\end{aligned}
$$

The second-stage constraints satisfy this hypothesis.

2.4 $(p+1)$-th stage

Let us consider the $(p+1)$-th stage of the refined DP. The consistency conditions for the constraints $\phi^{(p|p)}$ result in the $(p+1)$-th stage constraints,

$$
\left\{\phi^{(p|p)}, H_p + \varepsilon\right\} \equiv \phi'^{(p+1|p)} = 0\,. \tag{29}
$$

The $(p+1)$-th stage constraints $\phi'^{(p+1|p)} = \phi'^{(p+1|p)}_{\alpha_p}$ together with constraints of the previous stages may form a dependent set of constraints. We suppose that the matrix $\partial\big(\Phi^{(1,\dots,p)}, \phi'^{(p+1|p)}\big)/\partial\eta$ has a constant rank,

$$
\mathrm{rank}\,\frac{\partial\left(\Phi^{(1,\dots,\mathrm{p})}, \phi'^{(\mathrm{p}+1|\mathrm{p})}\right)}{\partial\eta} = \sum_{\mathrm{i}=1}^{\mathrm{p}} \mathrm{m_i} + \mathrm{m_{p+1}} \le \left[\Phi^{(1,\dots,\mathrm{p})}\right] + \left[\phi'^{(\mathrm{p}+1|\mathrm{p})}\right]. \tag{30}
$$

We first reorganize the constraints $\phi'^{(p+1|p)}$ to $\phi^{(p+1|p)}$ as follows:

$$\phi'^{(p+1|p)}_{\alpha_p} \rightarrow \phi^{(p+1|p)}_{\alpha_p} = \phi'^{(p+1|p)}_{\alpha_p}$$
$$- \sum_{u=2}^{p} \sum_{i=1}^{u-1} \varphi^{(i|u)} \left[M^{(u)}\right]^{-1} \left\{\varphi^{(u+1-i|u)}, \phi'^{(p+1|p)}_{\alpha_p}\right\}.$$

The new constraints $\phi^{(p+1|p)}$ obey the relations

$$\phi^{(p+1|p)} = \left\{\phi^{(p|p)}, H_p + \varepsilon\right\} + O\left(\Phi^{(1,\ldots,p)}\right) ,$$
$$\left\{\varphi^{(i|u)}, \phi^{(p+1|p)}\right\} = O\left(\Phi^{(1,\ldots,p)}\right),\ 2 \le i \le p,\ i \le u \le p\,,$$
$$\left\{\varphi^{(1|u)}, \phi^{(p+1|p)}\right\} = \left\{\varphi^{(1|u)}, \left\{\phi^{(p|p)}, H_p + \varepsilon\right\}\right\} + O\left(\Phi^{(1,\ldots,p)}\right)$$
$$= \left\{\left\{\varphi^{(1|u)}, \phi^{(p|p)}\right\}, H_p + \varepsilon\right\} + \left\{\phi^{(p|p)}, \left\{\varphi^{(1|u)}, H_p + \varepsilon\right\}\right\}$$
$$+ O\left(\Phi^{(1,\ldots,p)}\right) = O\left(\Phi^{(1,\ldots,p)}, \phi^{(p+1|p)}\right),\ u = 1, \ldots, p,$$
$$\left\{\chi^{(1|a)}, \phi^{(p+1|p)}\right\} = O\left(\Phi^{(1,\ldots,p)}, \phi^{(p+1|p)}\right),\ a = 1, \ldots, p-1\,,$$
$$\left\{\phi^{(p+1|p)}, H^{(1)} + \varepsilon\right\} = \left\{\phi^{(p+1|p)}, H^{(1)}_p + \varepsilon\right\}$$
$$+ O\left(\Phi^{(1,\ldots,p)}, \phi^{(p+1|p)}\right). \tag{31}$$

Consider the consistency conditions for $\phi^{(p+1|p)}$ (we can use $H^{(1)}_p$ instead of $H^{(1)}$ in DP)

$$\{\phi^{(p+1|p)}, H_p + \epsilon\} + C^{(p+1)}\lambda_p = 0,\ C^{(p+1)} = C^{(p+1)}_{\alpha_p\beta_p} = \left\{\phi^{(p+1|p)}_{\alpha_p}, \phi^{(1|p)}_{\beta_p}\right\}.$$

We can see that the matrix $C^{(p+1)}$ and all the matrices $\left\{\phi^{(p+2-i|p)}_{\alpha_p}, \phi^{(i|p)}_{\beta_p}\right\}$, $i = 2, \ldots, p$ coincide up to a sign and are (anti)symmetric on the constraint surface $\Phi^{(1,\ldots,p)} = \phi^{(p+1|p)} = 0$:

$$C^{(p+1)}_{\alpha_p\beta_p} = \left\{\left\{\phi^{(p|p)}_{\alpha_p}, H_p + \epsilon\right\}, \phi^{(1|p)}_{\beta_p}\right\} + O\left(\Phi^{(1,\ldots,p)}\right) = -\left\{\phi^{(p|p)}_{\alpha_p}, \phi^{(2|p)}_{\beta_p}\right\}$$
$$+ O\left(\Phi^{(1,\ldots,p)}, \phi^{(p+1|p)}\right) = \cdots = (-1)^i \left\{\phi^{(p+1-i|p)}_{\alpha_p}, \phi^{(i+1|p)}_{\beta_p}\right\}$$
$$+ O\left(\Phi^{(1,\ldots,p)}, \phi^{(p+1|p)}\right) = \cdots = (-1)^p \left\{\phi^{(1|p)}_{\alpha_p}, \phi^{(p+1|p)}_{\beta_p}\right\}$$
$$+ O\left(\Phi^{(1,\ldots,p)}, \phi^{(p+1|p)}\right) = (-1)^{p+1}\, C^{(p+1)}_{\beta_p\alpha_p} + O\left(\Phi^{(1,\ldots,p)}, \phi^{(p+1|p)}\right) .$$

Here, we have used the Jacobi identity and Eqs. (26, 27, 28, 31).

We suppose that the matrix $C^{(p+1)}$ has a constant rank, $\operatorname{rank} C^{(p+1)} = r_{p+1}$. We then perform the $Z-$reorganization. Namely, we consider the equation

$$C^{(p+1)}Z^{(p+1)} = O\left(\Phi^{(1,\ldots,p)}, \phi^{(p+1|p)}\right),$$

which has $m''_{p+2} = m'_{p+1} - r_{p+1}$ linearly independent solutions $Z^{(p+1)}_{\sigma_p} = Z^{(p+1)\alpha_p}_{\sigma_p}$, $[\sigma_p] = m''_{p+2}$, such that $\det Z^{(p+1)\sigma'_p}_{\sigma_p} \neq 0$ $(\alpha_p = (\mu_{p+1}, \sigma_p)$, $[\mu_{p+1}] = r_{p+1})$. These solutions together with the vectors $Z^{(p+1)}_{\mu_{p+1}} = Z^{(p+1)\alpha_p}_{\mu_{p+1}} = \delta^{\alpha_p}_{\mu_{p+1}}$ form the set of m'_{p+1} linearly independent vectors. We reorganize the constraints $\phi^{(i|p)}, i = 1, ..., p+1$ using a nonsingular matrix Z_{p+1}:

$$\phi^{(i|p)} \to Z_{p+1}\phi^{(i|p)} = \begin{pmatrix} \varphi'^{(i|p+1)}_{\mu_{p+1}} = \phi^{(i|p)}_{\mu_{p+1}} \\ \Psi'^{(i|p)}_{\sigma_p} = Z^{(p+1)\alpha_p}_{\sigma_p}\phi^{(i|p)}_{\alpha_p} \end{pmatrix}, \quad \alpha_p = (\mu_{p+1}, \sigma_p)\ ,$$

$$Z_{p+1} = Z^{\beta_p}_{p+1\alpha_p} = \begin{pmatrix} \delta^{\nu_{p+1}}_{\mu_{p+1}} & 0 \\ Z^{(p+1)\nu_{p+1}}_{\sigma_p} & Z^{(p+1)\sigma'_p}_{\sigma_p} \end{pmatrix}, \quad \beta_p = \left(\nu_{p+1}, \sigma'_p\right)\ .$$

Next, a set of additional reorganizations must be carried out (by adding some previous-stage constraints). We first reorganize the constraints $\varphi'^{(i|p+1)}$:

$$\begin{aligned}
&\varphi'^{(1|p+1)}_{\mu_{p+1}} \to \varphi''^{(1|p+1)}_{\mu_{p+1}} = \varphi'^{(1|p+1)}_{\mu_{p+1}}\,, \\
&\varphi'^{(i|p+1)}_{\mu_{p+1}} \to \varphi''^{(i|p+1)}_{\mu_{p+1}} = \varphi'^{(i|p+1)}_{\mu_{p+1}} - \varphi'^{(1|p+1)}\left(\left\{\varphi'^{(p+1|p+1)}, \varphi'^{(1|p+1)}\right\}\right)^{-1} \\
&\times\left\{\varphi'^{(p+1|p+1)}, \varphi'^{(i|p+1)}_{\mu_{p+1}}\right\},\ i = 2, ..., p\,; \\
&\varphi'^{(p+1|p+1)}_{\mu_{p+1}} \to \varphi''^{(p+1|p+1)}_{\mu_{p+1}} = \varphi'^{(p+1|p+1)}_{\mu_{p+1}} \\
&-\frac{1}{2}\varphi'^{(1|p+1)}\left(\left\{\varphi'^{(p+1|p+1)}, \varphi'^{(1|p+1)}\right\}\right)^{-1}\left\{\varphi'^{(p+1|p+1)}, \varphi'^{(p+1|p+1)}_{\mu_{p+1}}\right\};
\end{aligned}$$

then, we reorganize the constraints $\varphi''^{(i|p+1)}$:

$$\varphi''^{(i|p+1)}_{\mu_{p+1}} \to \varphi'''^{(i|p+1)}_{\mu_{p+1}} = \varphi''^{(i|p+1)}_{\mu_{p+1}}, \quad i = 1, 2, p+1\,,$$

$$\varphi''^{(i|p+1)}_{\mu_{p+1}} \to \varphi'''^{(i|p+1)}_{\mu_{p+1}} = \varphi''^{(i|p+1)}_{\mu_{p+1}} - \varphi''^{(2|p+1)} \left(\left\{\varphi''^{(p|p+1)}, \varphi''^{(2|p+1)}\right\}\right)^{-1}$$

$$\times \left\{\varphi''^{(p|p+1)}, \varphi''^{(i|p+1)}_{\mu_{p+1}}\right\}, \quad i = 3, ..., p-1\,;$$

$$\varphi''^{(p|p+1)}_{\mu_{p+1}} \to \varphi'''^{(p|p+1)}_{\mu_{p+1}} = \varphi''^{(p|p+1)}_{\mu_{p+1}}$$

$$- \frac{1}{2}\varphi''^{(2|p+1)} \left(\left\{\varphi''^{(p|p+1)}, \varphi''^{(2|p+1)}\right\}\right)^{-1} \left\{\varphi''^{(p|p+1)}, \varphi''^{(p|p+1)}_{\mu_{p+1}}\right\};$$

etc.. This set of reorganizations ends up with producing the φ's with d primes, where d is equal to the integer part of $(p+1)/2$ $(d = [(p+1)/2])$,

$$\varphi''^{\ldots(i|p+1)}_{\mu_{p+1}} \to \varphi''^{\ldots\prime(i|p+1)}_{\mu_{p+1}} = \varphi''^{\ldots(i|p+1)}_{\mu_{p+1}}, \quad i = 1, ..., d, p+3-d, ..., p+1;$$

$$\varphi''^{\ldots(i|p+1)}_{\mu_{p+1}} \to \varphi''^{\ldots\prime(i|p+1)}_{\mu_{p+1}} = \varphi''^{\ldots(i|p+1)}_{\mu_{p+1}} - \varphi''^{\ldots(l|p+1)}$$

$$\times \left(\left\{\varphi''^{\ldots(p+2-d|p+1)}, \varphi''^{\ldots(d|p+1)}\right\}\right)^{-1} \left\{\varphi''^{\ldots(p+2-d|p+1)}, \varphi''^{\ldots(i|p+1)}_{\mu_{p+1}}\right\},$$

$$i = d+1, p+1-d;$$

$$\varphi''^{\ldots(p+2-d|p+1)}_{\mu_{p+1}} \to \varphi''^{\ldots\prime(p+2-d|p+1)}_{\mu_{p+1}} = \varphi''^{\ldots(p+2-d|p+1)}_{\mu_{p+1}} - \frac{1}{2}\varphi''^{\ldots(d|p+1)}$$

$$\times \left(\left\{\varphi''^{\ldots(p+2-d|p+1)}, \varphi''^{\ldots(d|p+1)}\right\}\right)^{-1} \left\{\varphi''^{\ldots(p|p+1)}, \varphi''^{\ldots(p+2-d|p+1)}_{\mu_{p+1}}\right\}.$$

In what follows, we omit all the primes such that $\varphi^{(i|p+1)}$ are the final reorganized constraints. These constraints satisfy the relations (26)-(28) with $p \to p+1$.

We now reorganize the constraints $\Psi'^{(i|p)}$:

$$\Psi'^{(1|p)}_{\sigma_p} \to \Psi^{(1|p)}_{\sigma_p} = \Psi'^{(1|p)}_{\sigma_p}; \;\; \Psi'^{(i|p)}_{\sigma_p} \to \Psi^{(i|p)}_{\sigma_p} = \Psi'^{(i|p)}_{\sigma_p} - \sum_{j=1}^{i-1} \varphi^{(j|p+1)}$$

$$\times \left(\left\{\varphi^{(p+2-j|p+1)}, \varphi^{(j|p+1)}\right\}\right)^{-1} \left\{\varphi^{(p+2-j|p+1)}, \Psi'^{(i|p)}_{\sigma_p}\right\}\,,\; i = 2, ..., p+1\,.$$

The constraints $\Psi^{(i|p)}$ have the properties

$$\left\{\varphi^{(i|u)}, \Psi^{(j|p)}\right\} = \begin{cases} O\left(\Phi^{(1,\dots,i+j-1)}\right), \ i+j \leq u+1 \\ O\left(\Phi^{(1,\dots,u)}\right), \ i+j > u+1, \ j \leq u \\ O\left(\Phi^{(1,\dots,j)}\right), \ j > u, \end{cases}$$

$$\left\{\Psi^{(i|p)}, \Psi^{(j|p)}\right\} = O\left(\Phi^{(1,\dots,i+j-1)}\right), \ i+j \leq (p+1)+1\,.$$

We can see that the constraints $\left(\Phi^{(1,\dots,p)}, \varphi^{(p+1|p+1)}\right)$ are independent. Taking this and Eq. (30) into account, we reorganize the constraints $\Psi^{(p+1|p)}$: $\Psi^{(p+1|p)}_{\sigma_p} \rightarrow \left(\phi^{(p+1|p+1)}_{\alpha_{p+1}} = \Psi^{(p+1|p+1)}_{\alpha_{p+1}}, \chi^{(p+1|p)}_{\rho_p} = U^{(p+1)\sigma_p}_{\rho_p}\Psi^{(p+1|p)}_{\sigma_p}\right)$, $\sigma_p = (\alpha_{p+1}, \rho_p)$, $\left[\phi^{(p+1|p+1)}\right] = m_{p+1} - r_{p+1} \equiv m'_{p+2}$, $U^{(p+1)}_{\rho_p}$ is a set of independent vectors, $\left[U^{(p+1)}\right] = \left[\chi^{(p+1|p)}\right] = m''_{p+2} - m'_{p+2} \equiv s_p$, such that the constraints $\Phi^{(1,\dots,p+1)}$, where $\Phi^{(p+1)} = \left(\varphi^{(p+1|p+1)}, \phi^{(p+1|p+1)}\right)$, $\left[\Phi^{(p+1)}\right] = m_{p+1}$, are independent and $\chi^{(p+1|p)} = O\left(\Phi^{(1,\dots,p)}, \varphi^{(p+1|1,\dots,p+1)}\right)$. Then, we reorganize the constraints $\Psi^{(i|p)}$, $i = 1, ..., p$:

$$\Psi^{(i|p)}_{\sigma_p} \rightarrow \left(\phi^{(i|p+1)}_{\alpha_{p+1}} = \Psi^{(i|p+1)}_{\alpha_{p+1}}, \chi^{(i|p)}_{\rho_p} = U^{(p+1)\sigma_p}_{\rho_p}\Psi^{(i|p)}_{\sigma_p}\right)\,,$$

$$\left[\chi^{(i|p)}\right] = s_p\,, \quad m_i = \sum_{j=i}^{p+1} r_j + \sum_{a=i}^{p} s_a + m'_{p+2}\,.$$

The consistency conditions for the constraints $\chi^{(p+1|p)}$ do not produce any new constraints.

Introducing the Hamiltonian $H^{(1)}_{p+1}$,

$$H^{(1)} = H^{(1)}_{p+1} + \sum_{u=1}^{p+1} \Lambda_u \varphi^{(1|u)} + \sum_{a=1}^{p} \lambda_{\chi^a}\chi^{(1|a)}, \ \Lambda_u = \lambda_{\varphi^u} - \bar{\lambda}_u\,,$$

$$H^{(1)}_{p+1} = H_{p+1} + \lambda_{\phi^{p+1}}\phi^{(1|p+1)}, \ H_{p+1} = H_p + \bar{\lambda}_{p+1}\varphi^{(1|p+1)}\,,$$

$$\bar{\lambda}_{p+1} = -\left[M^{(p+1)}\right]^{-1}\left\{\varphi^{(p+1|p+1)}, H_p + \varepsilon\right\}\,,$$

$$M^{(p+1)} = M^{(p+1)}_{\mu_{p+1}\nu_{p+1}} = C^{(p+1)}_{\mu_{p+1}\nu_{p+1}}\,,$$

we can straightforwardly verify that all the constraints reorganized up to the $(p+1)$-th stage satisfy the induction hypothesis.

2.5 Final stage

DP ends up at the k-th stage if $\left[\phi^{(k|k)}\right] = m'_{k+1} = 0$. The latter means that $\left[\phi^{(i|k)}\right] = 0,\ \ i = 1, ..., k-1$, and $T_k^{(i)} = \left(\chi^{(i|a)}\right)$, see (27) . The constraints $\chi^{(i|a)}$ therefore commute as follows :

$$\left\{\chi^{(i|a)}, \chi^{(j|b)}\right\} = O\left(\Phi^{(1,\ldots,i+j-1)}\right),\ i+j \leq k+1\,.$$

Considering the relations

$$\begin{aligned} &O\left(\Phi^{(1,\ldots,k)}\right) = \left\{\chi^{(1|a)}, \left\{\chi^{(k-1|k-1)}, H_k + \epsilon\right\}\right\} \\ &= \left\{\chi^{(k-1|k-1)}, \chi^{(2|a)}\right\} + O\left(\Phi^{(1,\ldots,k)}\right)\ , \end{aligned}$$

we obtain

$$\left\{\chi^{(i|a)}, \chi^{(j|b)}\right\} = O\left(\Phi^{(1,\ldots,k)}\right),\ i+j = k+2\,.$$

Then, considering the double commutator $\left\{\chi^{(2|a)}, \left\{\chi^{(k-1|k-1)}, H_k + \epsilon\right\}\right\}$, we obtain

$$\left\{\chi^{(i|a)}, \chi^{(j|b)}\right\} = O\left(\Phi^{(1,\ldots,k)}\right),\ i+j = k+3\,,$$

and so on. We finally have

$$\left\{\chi^{(i|a)}, \chi^{(j|b)}\right\} = \begin{cases} O\left(\Phi^{(1,\ldots,i+j-1)}\right),\ i+j < k+1 \\ O\left(\Phi^{(1,\ldots,k)}\right),\ i+j \geq k+1 \end{cases}\,. \tag{32}$$

In particular, we can conclude (only at the last stage!) that all the constraints $\chi^{(i|a)}$ are FCC.

3 Summary

We here summarize the constraint structure consistent with DP. To make things more clear, we repeat some points.

It is possible to reorganize the complete set of constraints obtained by DP to the following form:

$$\begin{aligned} &\Phi = \Phi^{(1,\ldots,k)} = \left(\Phi^{(i)}\right), \quad \Phi^{(i)} = \left(\varphi^{(i|u)},\ \chi^{(i|a)}\right), \\ &\left[\varphi^{(i|u)}\right] = r_u\,, \quad \left[\chi^{(i|a)}\right] = s_a\,, \quad \operatorname{rank}\frac{\partial\Phi}{\partial\eta} = [\Phi]\,, \\ &i \leq u \leq k,\ 1 \leq i \leq k,\ i \leq a \leq k-1\,. \end{aligned} \tag{33}$$

The total Hamiltonian and and the Lagrange multipliers λ have the form

$$
\begin{aligned}
&H^{(1)} = H + \lambda_{\varphi^u}\varphi^{(1|u)} + \lambda_{\chi^a}\chi^{(1|a)} = H_k + \sum_{u=1}^{k}\Lambda_u\varphi^{(1|u)} + \sum_{a=1}^{k-1}\lambda_{\chi^a}\chi^{(1|a)}\,,\\
&H_l = H_{l-1} + \bar{\lambda}_l\varphi^{(1|l)} = H + \sum_{u=1}^{l}\bar{\lambda}_u\varphi^{(1|u)}\,,\ l = 1,...,k\,,\\
&\Lambda_u = \lambda_{\varphi^u} - \bar{\lambda}_u\,,\quad \bar{\lambda}_u = -\left[M^{(u)}\right]^{-1}\left\{\varphi^{(u|u)}, H_{u-1}+\varepsilon\right\},\\
&M^{(u)} = \left\{\varphi^{(u|u)}, \varphi^{(1|u)}\right\},\quad (H_0 \equiv H)\ . \qquad (34)
\end{aligned}
$$

The mutual commutation relations between the constraints are

$$
u < v:\quad \left\{\varphi^{(i|u)}, \varphi^{(j|v)}\right\} = \left\{\begin{array}{l} O\left(\Phi^{(1,\dots,i+j-1)}\right),\quad i+j \le u+1 \\ O\left(\Phi^{(1,\dots,u)}\right),\quad i+j > u+1,\ j \le u \\ O\left(\Phi^{(1,\dots,j)}\right),\quad j > u, \end{array}\right.
$$

$$
\left\{\varphi^{(i|u)}, \varphi^{(j|u)}\right\} = \left\{\begin{array}{l} O\left(\Phi^{(1,\dots,i+j-1)}\right),\quad i+j \le u+1 \\ (-1)^{u-i}M^{(u)} + O\left(\Phi^{(1,\dots,u)}\right),\quad i+j = u+1 \\ O\left(\Phi^{(1,\dots,j)}\right),\quad i+\ j > u+1, \end{array}\right.
$$

and

$$
\begin{aligned}
&\left\{\varphi^{(i|u)}, \chi^{(j|a)}\right\} = \left\{\begin{array}{l} O\left(\Phi^{(1,\dots,i+j-1)}\right),\quad i+j \le u+1 \\ O\left(\Phi^{(1,\dots,u)}\right),\quad i+j > u+1,\ j \le u \\ O\left(\Phi^{(1,\dots,j)}\right),\quad j > u, \end{array}\right.\\
&\left\{\chi^{(i|a)}, \chi^{(j|b)}\right\} = \left\{\begin{array}{l} O\left(\Phi^{(1,\dots,i+j-1)}\right),\quad i+j < k+1 \\ O\left(\Phi^{(1,\dots,k)}\right),\quad i+j \ge k+1, \end{array}\right.\\
&\left\{\varphi^{(i|u)}, H_j+\varepsilon\right\} = \left\{\varphi^{(i|u)}, H^{(1)}+\varepsilon\right\} + O\left(\Phi^{(1,\dots,i)}\right)\\
&= \varphi^{(i+1|u)} + O\left(\Phi^{(1,\dots,i)}\right),\quad j \ge i,\ u > i\,,\\
&\left\{\varphi^{(i|i)}, H_j+\varepsilon\right\} = O\left(\Phi^{(1,\dots,i)}\right),\quad j \ge i\,,\\
&\left\{\chi^{(i|a)}, H_j+\varepsilon\right\} = \left\{\chi^{(i|a)}, H^{(1)}+\varepsilon\right\} + O\left(\Phi^{(1,\dots,i)}\right)\\
&= \chi^{(i+1|a)} + O\left(\Phi^{(1,\dots,i)}\right),\quad i <\ a \le k-2\,,\\
&\left\{\chi^{(i|i)}, H_j+\varepsilon\right\} = \left\{\chi^{(i|i)}, H^{(1)}+\varepsilon\right\} + O\left(\Phi^{(1,\dots,i)}\right)\\
&= O\left(\Phi^{(1,\dots,i)}, \varphi^{(i+1|i+1)}\right),\quad i \le k-1\,. \qquad (35)
\end{aligned}
$$

The Poisson brackets between SCC from different chains vanish on the constraint surface. The Lagrange multipliers λ_χ are not determined by DP (and by the complete set of equations of motion). Whenever FCC (SCC) exist, the corresponding primary FCC (SCC) do exist.

We note that imposing more restrictions on the theories under consideration, we can obtain a more detailed constraint structure. For example, if we suppose the rank constancy for the matrices $\left\{\Phi^{(1,\dots,i)}, \Phi^{(1,\dots,i)}\right\}$, then we can conclude, at the same i-th stage, that the constraints $\chi^{(i-1|a)}$ are of the first class. However, the less restrictive condition of the rank constancy for the matrix $\left\{\Phi^{(1)}, \Phi^{(i)}\right\}$ is already sufficient for DP to be applicable. There exist some models that obey the latter conditions only.

It is important to stress that commutation relations (35) for χ mean that the property of a constraint to be or not to be of the first class can be established only after completing the Dirac procedure. As a consequence, in the general case, it is impossible to find special variables (see Ref. 2) such that the first-class constraints have the canonical form , i.e., it is impossible that FCC be some canonical momenta classified according to the stages of the Dirac procedure. We consider an example of the theory that confirms this statement. The corresponding Lagrange function is

$$L = \frac{1}{2}\left[\left(\dot{y}^i + x^i\right)^2 + \left(\dot{z}^i + y^i + E^i_{jk}x^j y^k\right)^2\right], \ i,j,k = 1,2\,, \tag{36}$$

where E^i_{jk} is a constant matrix, which is antisymmetric with respect to the lower indices. The total Hamiltonian and primary constraints are

$$\begin{aligned} &H^{(1)} = \frac{1}{2}\left[p^2_{y^i} + p^2_{z^i}\right] - x^i p_{y^i} - y^i p_{z^i} - E^i_{jk}x^j y^k p_{z^i} + \lambda^i \Phi^{(1)}_i\,, \\ &\Phi^{(1)}_i = p_{x^i} = 0\,. \end{aligned} \tag{37}$$

We find that the second- and third-stage constraints are

$$\Phi^{(2)}_i = p_{y^i} + E^j_{ik}y^k p_{z^j} = 0\,,\ \Phi^{(3)}_i = p_{z^i} = 0\,.$$

All the constraints $\left(\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}\right)$ are of the first class. In this case, all λ's remain undetermined and new constraints do not arise. The commutator between the second-stage constraints,

$$\{\Phi^{(2)}_i, \Phi^{(2)}_j\} = 2E^k_{ij}\Phi^{(3)}_k, \tag{38}$$

is proportional to the third-stage constraints. We see that the second-stage constraints are of the first class only with respect to the complete set of constraints $\left(\Phi^{(1)}, \Phi^{(2)}, \Phi^{(3)}\right)$ and not of the first class with respect to the constraints $\left(\Phi^{(1)}, \Phi^{(2)}\right)$ of the two first stages. Any constraint reorganization, which respects the decomposition of the constraints according to the stages of the Dirac procedure can not change this situation. The existence of special variables, in which all $\Phi^{(1)}$ and $\Phi^{(2)}$ are canonical momenta, thus contradicts to relation (38).

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