RELATIVITY, CAUSALITY, LOCALITY, QUANTIZATION AND DUALITY IN THE $Sp(2M)$ INVARIANT GENERALIZED SPACE-TIME

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We analyze properties of the $Sp(2M)$ conformally invariant field equations in the recently proposed generalized $\frac{1}{2}M(M+1)-$ dimensional space-time $\mathcal{M}_M$ with matrix coordinates. It is shown that classical solutions of these field equations define a causal structure in $\mathcal{M}_M$ and admit a well-defined decomposition into positive and negative frequency solutions that allows consistent quantization in a positive definite Hilbert space. The effect of constraints on the localizability of fields in the generalized space-time is analyzed. Usual $d-$dimensional Minkowski space-time is identified with the subspace of the matrix space $\mathcal{M}_M$ that allows true localization of the dynamical fields. Minkowski coordinates are argued to be associated with some Clifford algebra in the matrix space $\mathcal{M}_M$. The dynamics of a conformal scalar and spinor in $\mathcal{M}_2$ and $\mathcal{M}_4$ is shown to be equivalent, respectively, to the usual conformal field dynamics of a scalar and spinor in the $3d$ Minkowski space-time and the dynamics of massless fields of all spins in the $4d$ Minkowski space-time. An extension of the electro-magnetic duality transformations to all spins is identified with a particular generalized Lorentz transformation in $\mathcal{M}_4$. The $M = 8$ case is shown to correspond to a $6d$ chiral higher spin theory. The cases of $M = 16$ (d=10) and $M = 32$ (d=11) are discussed briefly.

Contents

To the memory of Michael Marinov

During several years I had the opportunity to appreciate the attractiveness and power of Misha Marinov's personality. Participating in the same seminars at Lebedev Physical Institute and Moscow State University we discussed physics many times. I was particularly happy to be able to discuss with Misha some questions related to the theory of higher spin gauge fields, such as star product algebras, symbols of operators and others. These discussions gave me a lot. Misha's attitude to a scientific question always depended on the extent he believed the topic belonged to a true physics area. Of course, every physicist has his own feeling of "true physics". An "invariant" principal part Misha insisted on was that there always must be a particular physical problem behind any formal manipulations. That was certainly a good school. I hope that this contribution fits Misha's high standard.

After Misha moved to the Land of his Forefathers one obvious and not at all surprising change I noticed during our short but full of discussions meeting at a conference in 1996 was how much he appreciated and enjoyed to be an independent Man, the feeling he was not allowed to have while living in the Soviet Union. It is too unfair that Misha was not given more time to live a full life and too unfortunate that no one can any longer discuss physics to Michael Marinov.

1 Introduction

In the recent paper[1] the system of conformally invariant equations of motion for $4d$ massless fields of all spins (for more details on the theory of $4d$ higher spin gauge fields we refer the reader to Refs. 2) was shown to exhibit generalized conformal symmetry[a] $Sp(8)$ and was argued to be equivalent to a system of equations for scalar and spinor in the generalized ten-dimensional space-time. An extension of these equations to the generalized $\frac{1}{2}M(M+1)$-dimensional space-time with arbitrary even[b] M was proposed in the same Ref. 1. Based on the idea of Bogolyubov transform duality[3] between classical and quantum pictures, the proposed equations in $\mathcal{M}_M$ were argued[1] to admit a consistent quantization in a positive definite Hilbert space. The aim of this paper is to reconsider the $Sp(2M)$ invariant equations in $\mathcal{M}_M$ from the perspective of the standard field-theoretical approach. We will show that these equations define a causal structure of the generalized space-time and admit consistent

[a] We use notation $Sp(2M)$ for the noncompact real group $Sp(2M, \mathbf{R})$ constituted by $2M \times 2M$ real matrices that leave invariant a non-degenerate real $2M \times 2M$ antisymmetric bilinear form.

[b] Note that M is required to be even by the AdS version of the model discussed in Ref. 1.

quantization with positive and negative frequency solutions giving rise to the creation and annihilation operators in a positive definite Hilbert space. Usual Minkowski space-time will be identified with some $d \leq M+1-$dimensional submanifold of the generalized space-time that admits localization of fields. From this perspective, the usual space-time can be thought of as a visualization of the generalized space-time. Remarkably, the space-time submanifolds that admit localization turn out to be related to certain Clifford algebras which, in turn, give rise to the usual Minkowski geometry.

The formulations of $Sp\,(2M)$ invariant systems in terms of the generalized space-time $\mathcal{M}_M$ and usual space-time are equivalent and complementary. The description in terms of $\mathcal{M}_M$ provides clear geometric origin for the $Sp\,(2M)$ generalized conformal symmetry. In particular it provides a geometric interpretation of the electromagnetic duality transformations as particular generalized Lorentz transformations. However, to define true local fields, one has to resolve some constraints. The description in terms of the Minkowski space-time, that solves the latter problem, makes some of the symmetries not manifest.

The study of the dynamical equations in $\mathcal{M}_M$ is likely to be of key importance for the analysis of dynamical systems that exhibit $Sp\,(2M)$ symmetries. In particular, the formulation in terms of generalized space-time $\mathcal{M}_M$ is expected to be useful for the investigation of the M-theory through the algebras $sp(32)$, $sp(64)$ and their superextensions.[4,5] Various ideas on a possible structure of alternative to Minkowski space-times have appeared both in the field-theoretical[6−8] and world particle dynamics[9] contexts. In particular, relevance of a $Sp(8)$ invariant 10-dimensional space-time for the description of the massless fields of all spins in four dimensions was emphasized by Fronsdal in Ref. 7 where also a realization of $Sp\,(2M)$ in the $\frac{1}{2}M(M+1)-$ dimensional manifold formed by isotropic $M-$forms was given, which construction is closely related to the one discussed in this paper. To the best of our knowledge, the dynamical equations we study in this paper, that allow for physical interpretation of the generalized space-time, were first suggested in Ref.1.

The generalized flat $\frac{1}{2}M(M+1)-$dimensional space-time $\mathcal{M}_M$ is described by the matrix coordinates $X^{\alpha\beta} = X^{\beta\alpha}$ $(\alpha, \beta = 1 \dots M)$. The $Sp\,(2M)$ generalized conformal symmetry transformations in $\mathcal{M}_M$ are realized by the vector fields[1]

$$P_{\alpha\beta} = -i\frac{\partial}{\partial X^{\alpha\beta}}\,, \tag{1}$$

$$L_\alpha{}^\beta = 2iX^{\beta\gamma}\frac{\partial}{\partial X^{\alpha\gamma}}\,, \tag{2}$$

$$K^{\alpha\beta} = -iX^{\alpha\gamma}X^{\beta\eta}\frac{\partial}{\partial X^{\gamma\eta}}\,. \tag{3}$$

The (nonzero) $sp(2M)$ commutation relations are

$$[L_\alpha{}^\beta\,, L_\gamma{}^\delta] = i\left(\delta_\alpha^\delta L_\gamma{}^\beta - \delta_\gamma^\beta L_\alpha{}^\delta\right)\,, \tag{4}$$

$$[L_\alpha{}^\beta\,, P_{\gamma\delta}] = -i\left(\delta_\gamma^\beta P_{\alpha\delta} + \delta_\delta^\beta P_{\alpha\gamma}\right)\,, \qquad [L_\alpha{}^\beta\,, K^{\gamma\delta}] = i\left(\delta_\alpha^\gamma K^{\beta\delta} + \delta_\alpha^\delta K^{\beta\gamma}\right)\,, \tag{5}$$

$$[P_{\alpha\beta}\,, K^{\gamma\delta}] = \frac{i}{4}\left(\delta_\beta^\gamma L_\alpha{}^\delta + \delta_\alpha^\gamma L_\beta{}^\delta + \delta_\alpha^\delta L_\beta{}^\gamma + \delta_\beta^\delta L_\alpha{}^\gamma\right). \tag{6}$$

Here $P_{\alpha\beta}$ and $K^{\alpha\beta}$ are generators of the generalized translations and special conformal transformations. The gl_M algebra spanned by $L_\alpha{}^\beta$ decomposes into the central subalgebra associated with the generalized dilatation generator

$$D = L_\alpha{}^\alpha \tag{7}$$

and the sl_M generalized Lorentz generators

$$l_\alpha{}^\beta = L_\alpha{}^\beta - \frac{1}{M}\delta_\alpha^\beta D\,. \tag{8}$$

The infinitesimal transformations generated by the vector fields (1)-(3) can be integrated to the finite group transformations

$$X^{\alpha\beta} \to \tilde{X}^{\alpha\beta} = X^{\alpha\beta} + a^{\alpha\beta} \tag{9}$$

for generalized translations,

$$X^{\alpha\beta} \to \tilde{X}^{\alpha\beta} = a^\alpha{}_\gamma a^\beta{}_\delta X^{\gamma\delta} \tag{10}$$

for generalized Lorentz SL_M transformations ($det|a^\alpha{}_\gamma| = 1$) or dilatations ($a^\alpha{}_\gamma = \kappa\delta^\alpha{}_\gamma$, $\kappa \neq 0$), and

$$X^{\alpha\beta} \to \tilde{X}^{\alpha\beta} = (X_{\alpha\beta} + a_{\alpha\beta})^{-1} \tag{11}$$

for generalized special conformal transformations where $X_{\alpha\beta}$ is the inverse to $X^{\alpha\beta}$:

$$X_{\alpha\gamma}X^{\gamma\beta} = \delta_\alpha^\beta\,. \tag{12}$$

Like in the standard conformal transformations the full generalized conformal group contains inversion R

$$R(X^{\alpha\beta}) = X_{\alpha\beta}\,, \tag{13}$$

which is involutive

$$R \circ R = Id\,. \tag{14}$$

A generalized special conformal transformation $S(a_{\alpha\beta})$ can be represented as a combination of two inversions and some translation

$$S(a_{\alpha\beta}) = R \circ T(a_{\alpha\beta}) \circ R\,. \tag{15}$$

It is possible to define the action of the $Sp\,(2M)$ transformations on a generalized tensor field $\phi^{\alpha_1\ldots\alpha_n}{}_{\beta_1\ldots\beta_m}(X)$ as follows. A finite translation $T(a)$ with the parameter $a^{\alpha\beta}$ is defined as usual

$$T(a)\Big(\phi^{\alpha_1\ldots\alpha_n}{}_{\beta_1\ldots\beta_m}(X)\Big) = \phi^{\alpha_1\ldots\alpha_n}{}_{\beta_1\ldots\beta_m}(X+a)\,. \tag{16}$$

A finite GL_M transformation with the parameter $a^\alpha{}_\beta$ that contains generalized Lorentz transformations ($det\,|a| = 1$) and dilatations ($a^\alpha{}_\beta \sim \delta^\alpha_\beta$) is

$$\begin{aligned} &G(a)\Big(\phi^{\alpha_1\ldots\alpha_n}{}_{\beta_1\ldots\beta_m}(X^{\nu\mu})\Big) \\ &= \Big(det\,|a|\Big)^{\Delta} a^{-1\,\alpha_1}{}_{\gamma_1}\ldots a^{-1\alpha_n}{}_{\gamma_n} a^{\delta_1}{}_{\beta_1}\ldots a^{\delta_m}{}_{\beta_m}\phi^{\gamma_1\ldots\gamma_n}{}_{\delta_1\ldots\delta_m}(a^\nu{}_\eta a^\mu{}_\sigma X^{\eta\sigma})\,, \end{aligned} \tag{17}$$

where the parameter Δ is the generalized conformal weight of the tensor field ϕ. A finite generalized special conformal transformation with the parameter $a_{\alpha\beta}$ is

$$\begin{aligned} &S(a)\Big(\phi^{\alpha_1\ldots\alpha_n}{}_{\beta_1\ldots\beta_m}(X^{\nu\mu})\Big) \\ &= \Big(det\,|Q|\Big)^{\Delta} Q^{-1\,\alpha_1}{}_{\gamma_1}\ldots Q^{-1\alpha_n}{}_{\gamma_n} Q^{\delta_1}{}_{\beta_1}\ldots Q^{\delta_m}{}_{\beta_m}\phi^{\gamma_1\ldots\gamma_n}{}_{\delta_1\ldots\delta_m}(Q^\nu{}_\eta X^{\eta\mu})\,, \end{aligned} \tag{18}$$

where

$$Q^{-1\,\alpha}{}_\beta = \delta^\alpha_\beta + X^{\alpha\gamma}a_{\gamma\beta}\,. \tag{19}$$

Finally, the inversion (13) interchanges upper and lower indices according to the rule

$$\begin{aligned} &R\Big(\phi^{\alpha_1\ldots\alpha_n}{}_{\beta_1\ldots\beta_m}(X)\Big) = \tilde{\phi}_{\alpha_1\ldots\alpha_n}{}^{\beta_1\ldots\beta_m}(X^{-1})\Big) \\ &= \Big(det\,|X|\Big)^{-\Delta} X^{-1}{}_{\alpha_1\gamma_1}\ldots X^{-1}{}_{\alpha_n\gamma_n} X^{\delta_1\beta_1}\ldots X^{\delta_m\beta_m}\phi^{\gamma_1\ldots\gamma_n}{}_{\delta_1\ldots\delta_m}(X^{-1})\,. \end{aligned} \tag{20}$$

The singularities of special conformal transformations are localized where the matrix $Q^{-1\alpha}{}_\beta$ (19) degenerates. To define a globally defined action of the conformal group the generalized space-time $\mathcal{M}_M$ has to be compactified to $\mathcal{CM}_M$ by adding the "infinity" strata associated with the degenerate matrices $X^{\alpha\beta}$. (The generalized conformal infinity is a stratified manifold because the equations that single out the space of degenerate real matrices of a given rank are singular). Compactified matrix spaces were defined e.g. in Refs. 6 and 7. A simple coset space realization of $\mathcal{CM}_M$ is given in section 9.

The equations of motion proposed in Ref. 1 read

$$\Big(\frac{\partial^2}{\partial X^{\alpha\beta}\partial X^{\gamma\delta}} - \frac{\partial^2}{\partial X^{\alpha\gamma}\partial X^{\beta\delta}}\Big) b(X) = 0 \tag{21}$$

for a scalar field $b(X)$ and

$$\frac{\partial}{\partial X^{\alpha\beta}} f_\gamma(X) - \frac{\partial}{\partial X^{\alpha\gamma}} f_\beta(X) = 0 \tag{22}$$

for a svector field $f_\beta(X)$. (We use the name "svector" (symplectic vector) to distinguish $f_\beta(X)$ from vectors of the usual Lorentz algebra $o(d-1,1)$. Note that svector fields will be shown to obey the Fermi statistics). For $M=2$, because antisymmetrization of any two-component indices α and β is equivalent to their contraction with the 2×2 symplectic form $\epsilon^{\alpha\beta}$, (21) and (22) coincide with the $3d$ massless Klein-Gordon and Dirac equations, respectively. For $M=4$, the equations (21) and (22) in the generalized ten-dimensional space-time $\mathcal{M}_4$ were argued in Ref. 1 to encode the infinite set of the usual $4d$ equations of motion for massless fields of all spins.

The equations (21) and (22) are invariant under the $Sp\,(2M)$ generalized conformal symmetry transformations provided that both $b(X)$ and $f_\alpha(X)$ have conformal weight $\Delta = \frac{1}{2}$. The invariance under generalized translations and Lorentz transformations is obvious. To prove the full invariance it is enough to check that the equations (21) and (22) are invariant under generalized dilatations and inversions. The infinitesimal transformations are[1]

$$\delta b(X) = \bigg(\varepsilon^{\alpha\beta}\frac{\partial}{\partial X^{\alpha\beta}} + \frac{1}{2}\varepsilon^{\alpha}{}_{\alpha} + 2\varepsilon^{\alpha}{}_{\beta}X^{\beta\gamma}\frac{\partial}{\partial X^{\alpha\gamma}} - \varepsilon_{\alpha\beta}\Big[\frac{1}{2}X^{\alpha\beta} + X^{\alpha\gamma}X^{\beta\eta}\frac{\partial}{\partial X^{\gamma\eta}}\Big]\bigg) b(X)\,, \tag{23}$$

$$\delta f_\gamma(X) = \bigg(\varepsilon^{\alpha\beta}\frac{\partial}{\partial X^{\alpha\beta}} + \frac{1}{2}\varepsilon^{\alpha}{}_{\alpha} + 2\varepsilon^{\alpha}{}_{\beta}X^{\beta\eta}\frac{\partial}{\partial X^{\alpha\eta}} - \varepsilon_{\alpha\beta}\Big[\frac{1}{2}X^{\alpha\beta} + X^{\alpha\delta}X^{\beta\eta}\frac{\partial}{\partial X^{\delta\eta}}\Big]\bigg) f_\gamma(X) + \Big(\varepsilon^{\beta}{}_{\gamma} - \varepsilon_{\eta\gamma}X^{\eta\beta}\Big) f_\beta, \tag{24}$$

where $\varepsilon^{\alpha\beta}$, $\varepsilon^{\alpha}{}_{\beta}$ and $\varepsilon_{\alpha\beta}$ are, respectively, $X-$independent parameters of generalized translations, Lorentz transformations along with dilatations, and special conformal transformations. These transformations can be extended to

$OSp\,(1, 2M)$ acting on the supermultiplet formed by scalar $b(X)$ and svector $f_\alpha(X)$ and to extended conformal supersymmetries $OSp\,(L, 2M)$ acting on appropriate sets of scalars and svectors[1] (see also section 8).

Note that we refer to the $Sp\,(2M)$ as to generalized conformal symmetry not only because of similarity with the usual conformal symmetry but also because, as shown in section 7, it extends the usual conformal symmetry $SO(d, 2)$ acting in the theory. (For example, for the case of $d = 4$ ($M = 4$) this is in accordance with the well-known fact that $o(4, 2) \sim su(2, 2) \subset sp(8)$.) An important feature of the equations (21) and (22) is that they do not contain any metric tensor. As a result, the interpretation of $Sp\,(2M)$ as a conformal symmetry associated with some metric re-scaling may not necessarily be relevant for the full theory in $\mathcal{M}_M$. In particular, the inversion R (13) is defined without any metric tensor, as opposed to the usual inversion $x^i \to \frac{x^i}{x^j x^k \eta_{jk}}$ where η_{jk} is the Minkowski metric tensor.

The rest of the paper is organized as follows. In section 2 we solve the equations (21) and (22) by means of Fourier transform and analyze some particular solutions associated with Green functions. The causal structure of the generalized space-time $\mathcal{M}_M$ is investigated in section 3 where the concepts of global Cauchy surface and time are defined. The generalized Lorentz transformations are discussed in section 4. The problem of localizability of fields in $\mathcal{M}_M$ is studied in section 5 where the concept of local Cauchy bundle with the usual space as base manifold σ is introduced. The particular cases of $M = 2$ and $M = 4$ are considered in detail in subsections 5.1 and 5.2. It is also shown in 5.2 that the $4d$ electro-magnetic duality transformations along with their extension to higher spins identify with certain generalized Lorentz transformations. Quantization of the free fields in the generalized space-time is performed in section 6 where the positive-definite Hilbert space of one-particle states is built, $\mathcal{D}$ functions and Green functions are found and the microcausality is analyzed including the analysis of the spin-statistics relationship. A structure of the generalized space-time for higher M is discussed in section 7 with the emphasize on the key role of Clifford algebras in the definition of the space σ for generic M. Generalized electro-magnetic duality is identified with the subgroup of the generalized Lorentz symmetry that leaves invariant a local Cauchy surface σ. Particular attention is paid to the $M = 8$ case that corresponds to a $6d$ chiral higher spin theory. The cases of $M = 16$ and $M = 32$ corresponding to some $10d$ and $11d$ theories are also discussed in section 7. A $osp(2L, 2M)$ invariant superspace extension of the proposed equations is formulated in section 8. Coset constructions for the compactified generalized (super)spaces are given in section 9. A summary of the obtained results and outlook is the content of section 10.

2 Classical solutions

The equation (21) admits a solution of the form

$$b(X) = \phi(\xi_\alpha \xi_\beta X^{\alpha\beta}) \,, \tag{25}$$

with an arbitrary function of one variable $\phi(z)$ and constant parameters ξ_α. Such solutions are analogous to the plane wave light-like solutions in the usual Minkowski space-time with coordinates x^n $(n = 0 \ldots d-1)$

$$b(x) = \phi(x^n k_n) \,, \qquad k_n k^n = 0 \,. \tag{26}$$

For $M = 2$ and $d = 3$ the two formulas are equivalent because ξ_α defines a light-like direction. For $d = 4$, light-like wave vectors admit the analogous twistor representation $k_{a\dot{b}} = \xi_a \bar{\xi}_{\dot{b}}$ $(a, b \ldots = 1, 2; \dot{a}, \dot{b} \ldots = 1, 2)$.

To show that the set of solutions (25) is complete consider Fourier transform. For a particular harmonic

$$b(X) = b_0 \exp i k_{\alpha\beta} X^{\alpha\beta} \,, \tag{27}$$

(21) requires

$$k_{\alpha\beta} k_{\gamma\delta} = k_{\alpha\gamma} k_{\beta\delta} \,. \tag{28}$$

This is solved by the twistor ansatz

$$k_{\alpha\beta} = k \xi_\alpha \xi_\beta \,, \tag{29}$$

with an arbitrary commuting real svector ξ_α and a factor k. The equivalent statement is that any non-zero matrix $k_{\alpha\beta}$ satisfying (28) has rank 1. For the proof it is enough to diagonalize the symmetric matrix $k_{\alpha\beta}$ by a sl_M transformation to see that the product of any two different eigenvalues is zero by (28) at $\alpha \neq \beta$. Modulo rescalings of k and ξ_α, there are two essentially different options in (29) with $k = 1$ or $k = -1$. These correspond to the positive and negative frequency solutions, respectively.

The situation with the Dirac-like equation (22) is analogous because it follows that $f_\alpha(X)$ satisfies (21)

$$\left(\frac{\partial^2}{\partial X^{\alpha\beta} \partial X^{\gamma\delta}} - \frac{\partial^2}{\partial X^{\alpha\gamma} \partial X^{\beta\delta}} \right) f_\sigma(X) = \frac{\partial}{\partial X^{\sigma\delta}} \left(\frac{\partial}{\partial X^{\alpha\beta}} f_\gamma(X) - \frac{\partial}{\partial X^{\alpha\gamma}} f_\beta(X) \right) = 0 \,. \tag{30}$$

A plane wave solution for the svector field $f_\alpha(X)$ has a form which is analogous to (25)

$$f_\alpha(X) = \xi_\alpha \phi(\xi_\gamma \xi_\beta X^{\gamma\beta})\,. \tag{31}$$

The Dirac-like equation (22) requires f_α to be proportional to ξ_α. As a result, the harmonic svector plane wave solution has the form

$$f_\alpha(X) = f_0 \xi_\alpha \exp ik\xi_\gamma \xi_\beta X^{\gamma\beta}\,. \tag{32}$$

Scalar and svector therefore have equal numbers of on-mass-shell degrees of freedom.

An important particular solution of the equation (21) has the form

$$b(X) = det^{-\frac{1}{2}}|X - X_0|\,, \tag{33}$$

where X_0 is any fixed point of $\mathcal{M}_M$. Setting $X_0 = 0$, one gets

$$\frac{\partial}{\partial X^{\alpha\beta}} det^{-\frac{1}{2}}|X| = -\frac{1}{2} X_{\alpha\beta} det^{-\frac{1}{2}}|X|\,. \tag{34}$$

Taking into account

$$\frac{\partial}{\partial X^{\alpha\beta}} X_{\gamma\delta} = -\frac{1}{2}\Big(X_{\alpha\delta}X_{\beta\gamma} + X_{\alpha\gamma}X_{\beta\delta}\Big)\,, \tag{35}$$

one obtains

$$\frac{\partial^2}{\partial X^{\alpha\beta}\partial X^{\gamma\delta}} det^{-\frac{1}{2}}|X| = \frac{1}{4}\Big(X_{\alpha\beta}X_{\gamma\delta} + X_{\alpha\gamma}X_{\beta\delta} + X_{\alpha\delta}X_{\beta\gamma}\Big) det^{-\frac{1}{2}}|X|\,. \tag{36}$$

This expression is totally symmetric with respect to $\alpha, \beta, \gamma, \delta$. As a result, the antisymmetric part of the second derivative of $det^{-\frac{1}{2}}|X|$ corresponding to the left hand side of (21) vanishes.

Analogously, the svector equation (22) admits a solution

$$f_\alpha(X) = (X - X_0)_{\alpha\beta}\eta^\beta det^{-1/2}|X - X_0|\,, \tag{37}$$

with an arbitrary constant polarization svector η^β.

Formulas (33) and (37) solve (21) and (22) at least in the regions where the matrix $X^{\alpha\beta} - X_0^{\alpha\beta}$ is nondegenerate. In section 6 we show that these solutions are related to the Green functions of the scalar and svector in the generalized space-time and give more precise definition of their singularities.

3 Causal structure and time

The equations (21) and (22) imply propagation along the generalized light-like directions

$$\Delta X^{\alpha\beta} = \eta^\alpha \eta^\beta \,, \tag{38}$$

where η^α is a twistor dual to ξ_β. The sign choice on the right hand side of (38) fixes a choice of the time arrow.

One way to reach this conclusion is to analyze the characteristic equation for the front of discontinuity of a field amplitude. Let $n_{\alpha\beta}$ be proportional to the infinite part of the derivative $\frac{\partial}{\partial X^{\alpha\beta}} b(X)$. For the front discontinuity of co-dimension one to be compatible with the field equations the normal vector $n_{\alpha\beta}$ has to satisfy the equation analogous to (28) and therefore has to be of the form

$$n_{\alpha\beta}(z) = \pm \xi_\alpha(z)\xi_\beta(z) \,, \tag{39}$$

where the coordinates z parametrize the wave front. For example, this formula is true for the solution (25) with the step-function $\phi(z)$. The discontinuity fronts of the solutions (33) and (37) are described by the surfaces of degenerate matrices

$$det|X - X_0| = 0 \,. \tag{40}$$

For a front of co-dimension one its normal vector is described by a rank 1 matrix $n_{\alpha\beta}$ satisfying

$$(X^{\alpha\beta} - X_0^{\alpha\beta}) n_{\beta\gamma} = 0 \,. \tag{41}$$

The svector ξ_α in the formula (39) identifies with the null-vector of the matrix $X^{\alpha\beta} - X_0^{\alpha\beta}$.

Suppose that a light-like signal emitted from some point $X_0^{\alpha\beta}$ of the generalized space-time reaches some other point $X_1^{\alpha\beta} = X_0^{\alpha\beta} + \eta^\alpha\eta^\beta$ switching on a new process that emits a signal in a different light-like direction.[c] Provided this happens several times, any point

$$\Delta X^{\alpha\beta} = \sum_{i=0}^{M} \eta_i^\alpha \eta_i^\beta \,, \tag{42}$$

can be reached where η_i^α is a complete set of contravariant svectors dual to the complete set of covariant svectors ξ^i_α

$$\xi^i_\alpha \eta_i^\beta = \delta_\alpha^\beta \,, \qquad \xi^i_\alpha \eta_j^\alpha = \delta^i_j \,. \qquad i,j = 1 \dots M \,. \tag{43}$$

[c] Let us note that the assumption that a process can be switched on locally may or may not be true for particular dynamical equations. In fact, as discussed in more detail in section 5, the equations (21), (22) do not admit true localization in $\mathcal{M}_M$. We do not expect this to affect our conclusions on the causal structure of the generalized space-time, however.

Formula (42) describes a general positive semi-definite symmetric matrix $\Delta X^{\alpha\beta}$. Let us note that analogous representation of positive semi-definite matrices in the context of analysis of BPS states was used recently in Ref. 10. The authors called the elementary twistors η_i^α preons.

We see that the relativistic geometry that follows from the equation (21) identifies the future cone $\mathcal{C}^+_{X_0}$ of a point X_0 with the set of matrices $X^{\alpha\beta}$ such that $\Delta X^{\alpha\beta} = X^{\alpha\beta} - X_0^{\alpha\beta}$ is positive semi-definite. Time-like vectors are described by positive definite matrices

$$\Delta X^{\alpha\beta}\xi_\alpha\xi_\beta > 0\,, \qquad \forall\xi_\alpha\,. \tag{44}$$

Light-like vectors identify with degenerate positive semi-definite matrices

$$det|\Delta X| = 0\,, \qquad \Delta X^{\alpha\beta}\xi_\alpha\xi_\beta \geq 0\,, \qquad \forall\xi_\alpha\,. \tag{45}$$

We will distinguish between rank - k light-like directions described by matrices of rank k. The concepts of time-like and rank - k light-like vectors are invariant under the generalized Lorentz group SL_M.

The equation (21) describes propagation of signals along the most degenerate light-like directions of rank 1. Using the technique developed in Ref. 1 one can work out a form of the equations that describe propagation along less degenerate light-like directions. We will come back to this question elsewhere.

The past cone $\mathcal{C}^-_{X_0}$ is defined analogously as the set of negative semi-definite matrices

$$(X^{\alpha\beta} - X_0^{\alpha\beta})\xi_\alpha\xi_\beta \leq 0\,, \qquad \forall\xi_\alpha\,. \tag{46}$$

If $Y \in \mathcal{C}^+_X$ then $X \in \mathcal{C}^-_Y$ and $2X - Y \in \mathcal{C}^-_X$. Note that $\mathcal{C}^+_X$ is the convex cone: $\forall X_1, X_2 \in \mathcal{C}^+_X, \lambda, \mu \in R^+\,, \lambda X_1 + \mu X_2 \in \mathcal{C}^+_X$.

To define the concept of time let us first introduce a concept of space-like global Cauchy surface as such a submanifold Σ of some (generalized) space-time manifold $\mathcal{M}$ that

(i) $\forall X_1, X_2 \in \Sigma$, $X_1 \notin \mathcal{C}^\pm_{X_2}$ and $X_2 \notin \mathcal{C}^\pm_{X_1}$ for $X_1 \neq X_2$.

(ii) any point of $\mathcal{M}$ belongs to either future or past cone of some point of Σ. No point $Y \in \mathcal{M}$ can belong to the future cone of some point $X_1 \in \Sigma$ and past cone of some other point $X_2 \in \Sigma$.

The meaning of the definition of global Cauchy surface is obvious: no pair of observers on Σ are allowed to exchange causal signals, i.e. a global Cauchy surface is space-like; emitting causal signals from a Cauchy surface, one can reach any point in the future, and the space-like global Cauchy surface can be reached by a signal from any space-time point in the past.[d] Note that for

[d] Note that this definition can be adjusted to a particular type of signals by replacing in the

the particular case under consideration with $\mathcal{M} = \mathcal{M}_M$ being a linear space $R^{\frac{1}{2}M(M+1)}$ and convex future (past) cones the second part of the requirement (ii) is a consequence of (i).

Provided that $\mathcal{M}$ admits a fibration into a set of space-like global Cauchy surfaces Σ_t parametrized by some parameter(s) t, this defines the concept of time(s).

Let $T^{\alpha\beta}$ be some positive definite matrix. The axioms (i) and (ii) are satisfied with the space-like global Cauchy surfaces Σ_t parametrized as

$$X^{\alpha\beta} \in \Sigma_t : \qquad X^{\alpha\beta} = x^{\alpha\beta} + tT^{\alpha\beta}\,, \tag{47}$$

where the space coordinates $x^{\alpha\beta}$ are arbitrary $T-$ traceless matrices

$$x^{\alpha\beta}T_{\alpha\beta} = 0\,, \qquad T_{\alpha\beta}T^{\beta\gamma} = \delta_\alpha^\gamma\,. \tag{48}$$

Indeed, the difference of any two matrices of the form (47) at fixed t is traceless and therefore it is neither positive definite nor negative definite. As a result, any two points of Σ_t at some fixed t are separated by a space-like interval. The rest of the axioms is a consequence of the trivial decomposition (47) of a matrix into the sum of its trace and traceless parts.

An important output of this analysis is that the generalized space-time $\mathcal{M}_M$ has just one evolution parameter

$$t = \frac{1}{M}X^{\alpha\beta}T_{\alpha\beta}\,. \tag{49}$$

The ambiguity in the choice of a positive definite matrix $T^{\alpha\beta}$ parametrizes the ambiguity in the choice of a particular coordinate frame like in Einstein special relativity: any two positive definite matrices $T^{\alpha\beta}_{1,2}$ with equal determinants are related by some generalized SL_M Lorentz transformation. The dilatation allows one to fix a scale of time in an arbitrary way.

Note that rank - k light-cone time parameters can be defined analogously with positive semi-definite matrices $T^{\alpha\beta}$ in (47).

4 Generalized Lorentz transformations

Having defined the concept of time, we are now in a position to analyze the generalized Lorentz transformations

$$X'^{\alpha\beta} = a^\alpha{}_\gamma a^\beta{}_\delta X^{\gamma\delta}\,. \tag{50}$$

condition (ii) the future and past cones by their boundary of a particular type (say, rank 1 for the case under consideration). Such a specification does not make difference at least for the particular dynamical system we study.

Here $a^{\alpha}{}_{\gamma}$ is an SL_M matrix

$$det\,|a^{\alpha}{}_{\gamma}| = 1\,. \tag{51}$$

The space symmetry subalgebra of the Lorentz-like group SL_M consists of the elements that leave invariant the positive definite symmetric matrix $T^{\alpha\beta}$ associated with time

$$T^{\alpha\beta} = a_o{}^{\alpha}{}_{\gamma} a_o{}^{\beta}{}_{\delta} T^{\gamma\delta}\,. \tag{52}$$

It is isomorphic to the compact group $SO(M)$ being the maximal compact subgroup of SL_M. $SO(M)$ is the analog of the usual space rotations $SO(d-1)$ of the $SO(d-1,1)$ Lorentz invariant space-time. The dimension of the coset space $SL_M/SO(M)$ equals to the number of the space (traceless) matrix coordinates $\frac{1}{2}M(M+1)-1$. As a result, very much as for the usual Lorentz transformations, the parameters of the generalized Lorentz group turn out to be associated with the space symmetry and the generalized velocities.

Let us introduce the following quantities

$$u^{\alpha\beta}(a) = a^{\alpha}{}_{\gamma} a^{\beta}{}_{\delta} T^{\gamma\delta} - T^{\alpha\beta}\gamma(a)\,, \tag{53}$$

$$r_{\gamma\delta}(a) = a^{\alpha}{}_{\gamma} a^{\beta}{}_{\delta} T_{\alpha\beta} - T_{\gamma\delta}\gamma(a)\,, \tag{54}$$

$$\gamma(a) = \frac{1}{M} a^{\alpha}{}_{\gamma} a^{\beta}{}_{\delta} T^{\gamma\delta} T_{\alpha\beta}\,, \tag{55}$$

defined in such a way that

$$u^{\alpha\beta} T_{\alpha\beta} = 0\,, \qquad r_{\alpha\beta} T^{\alpha\beta} = 0\,. \tag{56}$$

According to these definitions, $u^{\alpha\beta}(a)$ and $r_{\alpha\beta}(a)$ parametrize the right and left coset spaces $SL_M/SO(M)$, respectively,

$$u^{\alpha\beta}(aa_o) = u^{\alpha\beta}(a)\,, \qquad r_{\alpha\beta}(a_o a) = r_{\alpha\beta}(a)\,, \tag{57}$$

where $a_o \in SO(M)$ is any generalized space rotation satisfying (52). The parameter $\gamma(a)$ is left and right invariant

$$\gamma(aa_o) = \gamma(a_o a) = \gamma(a)\,. \tag{58}$$

It is not independent, but can be uniquely expressed in terms of either $u^{\alpha\beta}$ or $r_{\alpha\beta}$. To see this, consider the positive definite symmetric matrices

$$U^{\alpha\beta}(a) = a^{\alpha}{}_{\gamma} a^{\beta}{}_{\delta} T^{\gamma\delta}\,, \qquad R_{\gamma\delta}(a) = a^{\alpha}{}_{\gamma} a^{\beta}{}_{\delta} T_{\alpha\beta}\,, \tag{59}$$

and their characteristic equations

$$det\Big|U^{\alpha\beta}(a)-\lambda T^{\alpha\beta}\Big|=0\,,\qquad det\Big|R_{\alpha\beta}(a)-\lambda T_{\alpha\beta}\Big|=0\,. \tag{60}$$

It is elementary to see that, the equations on U and R in (60) are equivalent and, therefore, have the same sets of eigenvalues λ_i. Since $U^{\alpha\beta}, R_{\alpha\beta}$ and $T_{\alpha\beta}$ are positive definite, all eigenvalues are strictly positive [e]

$$\lambda_i > 0\,. \tag{61}$$

From (51) it follows that

$$\prod_{i=1}^{M}\lambda_i = 1\,. \tag{62}$$

The eigenvalues of $u^{\alpha\beta}$ and $r_{\alpha\beta}$ are

$$u_i = \lambda_i - \frac{1}{M}\sum_{j=1}^{M}\lambda_j\,. \tag{63}$$

Taking into account that

$$\gamma(a) = \frac{1}{M}\sum_{j=1}^{M}\lambda_j\,, \tag{64}$$

(62) acquires the form

$$\prod_{i=1}^{M}(u_i+\gamma) = 1\,. \tag{65}$$

This equation allows one to express γ in terms of u_i uniquely at the condition that all factors $(u_i+\gamma)$ are strictly positive (the function $\prod_{i=1}^{M}(u_i+\gamma)$ is monotonic in γ in the region where the factors $(u_i+\gamma)$ are positive). This proves that γ expresses in terms of both $r_{\alpha\beta}$ and $u^{\alpha\beta}$. As a consequence of the well-known inequality between the arithmetic and geometric averages, from (62) and (64) it follows that

$$\gamma(u) \geq 1\,. \tag{66}$$

[e] Let us note that the construction of the representatives of the left and right coset spaces $U^{\alpha\beta}$ and $R_{\alpha\beta}$ is analogous to the construction of the metric tensor in the frame formulation of gravity ($a^{\alpha}{}_{\beta}$ and $T^{\alpha\beta}$ are analogues of the frame field and flat metric, respectively). The eigenvalues (63) parametrize the double coset space $SO(M)\setminus SL_M/SO(M)$.

Now we rewrite the generalized Lorentz transformations in terms of the decomposition (47)

$$x'^{\alpha\beta} = a^{\alpha}{}_{\gamma}a^{\beta}{}_{\delta}x^{\gamma\delta} - \frac{1}{M}T^{\alpha\beta}r_{\gamma\delta}x^{\gamma\delta} + tu^{\alpha\beta}\,. \tag{67}$$

$$t' = \frac{1}{M}r_{\alpha\beta}x^{\alpha\beta} + \gamma t\,. \tag{68}$$

In this analysis we assume that the space-time decomposition (47) is defined with respect to the same matrix $T^{\alpha\beta}$ in the both frames. If one would rotate the matrix $T^{\alpha\beta}$ by the same Lorentz transformation the description in the two systems would be identical. For example the $3d$ coordinates in the decomposition $X^{\alpha\beta} = tI^{\alpha\beta} + x^1\sigma_1^{\alpha\beta} + x^2\sigma_3^{\alpha\beta}$ are defined for the fixed basis matrices $T^{\alpha\beta} = I^{\alpha\beta}$ and $\sigma_i^{\alpha\beta}$.

From (68) one observes that $\gamma(u)$ is a generalization of the relativistic Lorentz factor that relates the time scales in the two systems. The velocity of the origin of coordinates $x^{\alpha\beta}$ of the unprimed generalized inertial coordinate frame with respect to the primed one is

$$v^{\alpha\beta} = \frac{u^{\alpha\beta}}{\gamma(u)}\,. \tag{69}$$

The eigenvalues v_i of the traceless matrix $v^{\alpha\beta}$ are

$$v_i = \frac{M\lambda_i}{\sum_{j=1}^{M}\lambda_j} - 1\,. \tag{70}$$

From this formula it follows that the eigenvalues satisfy the restrictions

$$M - 1 \geq v_i \geq -1\,. \tag{71}$$

If one of the eigenvalues $\lambda_i \to \infty$, v_i saturates the upper bound while v_j at $j \neq i$ saturate the lower bound. For these limiting cases the relativistic factor $\gamma(v) \to \infty$. It is elementary to see using (51) that this is a general phenomenon: if some $v_i \to M-1$ then $v_j \to -1$ $j \neq i$ and $\gamma(v) \to \infty$. Moreover, $\gamma(v) \to \infty$ whenever at least one of the eigenvalues $v_j \to -1$ (while the upper bound may not be saturated).

The condition (71) is a generalization of the Lorentz geometry restriction that one system cannot move with respect to another with the speed exceeding the speed of light. Let us note that there is no symmetry $v_i \to -v_i$ in the generalized geometry because eigenvalues v_i are invariants of the space symmetry group $SO(M)$. In the usual Lorentzian geometry a sign of the velocity vector can be changed by a space rotation. A generalization of this symmetry to the generalized geometry is the symmetry under permutations of eigenvalues $v_i \leftrightarrow v_j$. Note that, according to the realization of Lorentz boosts (160) of section 7, usual Lorentz transformations identify with such generalized Lorentz transformations that there are only two different eigenvalues among v_i which, therefore, can differ only by sign, thus belonging to the interval $1 \geq v_i \geq -1$. In particular, this is obviously the case for $M = 2$.

The transform from the primed frame to the unprimed one is described by the inverse SL_M transform and, therefore is characterized by the replacement $\lambda_i \to \lambda_i^{-1}$. Note that the relativistic factors $\gamma(v)$ of the direct and inverse transforms are not necessarily equal to each other. However, because of (51) they tend to infinity simultaneously, i.e. if a system A is ultrarelativistic with respect to the system B, then the system B is ultrarelativistic with respect to A. A system at rest is characterized by $v_i = 0$.

5 Field localizability

The definition of a global Cauchy surface suggests that it is enough to know values of the fields along with some their time derivatives on a global Cauchy surface to fix a particular solution in the whole generalized space-time $\mathcal{M}_M$. This is certainly true. The question is what is a set of initial data that can be fixed arbitrarily to determine the time evolution of fields. Because a single field $b(X)$ satisfies the system of equations (21), some of these equations play a role of constraints on the global Cauchy surface thus restricting a possible choice of the initial data. This is analogous to the usual constraint dynamics. For example the Gauss low constraint $\partial_i E^i = 0$ in the pure electrodynamics restricts initial data for the electric field E^i.

Indeed, using the decomposition (47) we have

$$\frac{\partial}{\partial X^{\alpha\beta}} = \frac{1}{M} T_{\alpha\beta} \frac{\partial}{\partial t} + \frac{\partial}{\partial x^{\alpha\beta}}\,, \tag{72}$$

where the space coordinates $x^{\alpha\beta}$ are traceless in the sense of (48). From (21) one derives the wave equation

$$\Big(\frac{\partial^2}{\partial t^2} - \frac{M}{M-1} T^{\alpha\gamma} T^{\beta\delta} \frac{\partial^2}{\partial x^{\alpha\beta}\partial x^{\gamma\delta}}\Big) b(X) = 0 \tag{73}$$

and constraints

$$\begin{aligned}&\Big(T_{\alpha\beta}\frac{\partial}{\partial x^{\gamma\delta}} + T_{\gamma\delta}\frac{\partial}{\partial x^{\alpha\beta}}\Big) p(X) \\ &+ \Big(M \frac{\partial^2}{\partial x^{\alpha\beta}\partial x^{\gamma\delta}} + \frac{1}{M-1} T_{\alpha\beta} T_{\gamma\delta} T^{\nu\mu} T^{\rho\sigma} \frac{\partial^2}{\partial x^{\nu\rho}\partial x^{\mu\sigma}}\Big) b(X) - (\beta \leftrightarrow \gamma) = 0\,,\end{aligned} \tag{74}$$

where $p(X) = \frac{\partial}{\partial t} b(X)$. Note that the totally $T-$transversal part of the constraints (74) is independent of the momentum p thus imposing constraints on the space derivatives of $b(X)$.

The constraints restrict initial data for the field $b(x, t_0)$ and its first derivative $p(x, t_0)$ on the global Cauchy surface. In particular, it is not possible to choose initial data localized at some point x_0 with $b(x, t_0) \sim \delta(x - x_0)$, $p(x, t_0) \sim \delta(x - x_0)$. This is why we say that the fields $b(X)$ and $f_\alpha(X)$ are not localizable in the generalized space-time $\mathcal{M}_M$.

This conclusion is not surprising because, as shown in section 2, the generic solutions of the equations (21) and (22) have the form

$$b(X) = \frac{1}{\pi^{\frac{M}{2}}} \int d^M \xi \left(b^+(\xi) \exp i\xi_\alpha \xi_\beta X^{\alpha\beta} + b^-(\xi) \exp -i\xi_\alpha \xi_\beta X^{\alpha\beta} \right), \quad (75)$$

$$f_\gamma(X) = \frac{1}{\pi^{\frac{M}{2}}} \int d^M \xi \, \xi_\gamma \Big(f^+(\xi) \exp i\xi_\alpha \xi_\beta X^{\alpha\beta} + f^-(\xi) \exp -i\xi_\alpha \xi_\beta X^{\alpha\beta} \Big). \quad (76)$$

Both for the scalar $b(X)$ and svector $f_\alpha(X)$, the space of solutions is parametrized by two functions of M variables ξ_α. Because odd functions $b^\pm(\xi)$ and even functions $f^\pm(\xi)$ do not contribute to (75) and (76), respectively, we require

$$b^\pm(\xi) = b^\pm(-\xi), \qquad f^\pm(\xi) = -f^\pm(-\xi). \quad (77)$$

The integration in (75) and (76) is thus carried out over R^M/Z_2. The origin of coordinates $\xi_\alpha = 0$ is invariant under the Z_2 reflection $\xi_\alpha \to -\xi_\alpha$ and therefore is a singular point of the conical orbifold R^M/Z_2.

Using the ambiguity in $b^\pm(\xi)$ and $f^\pm(\xi)$ it may be possible to achieve localization in at most M coordinates that, generically, is much less than the dimension of the global Cauchy surface $dim(\Sigma) = \frac{1}{2}M(M+1) - 1$.

Let us now introduce the concept of local Cauchy bundle. Naively, one might try to identify it with some submanifold σ of the global Cauchy surface Σ such that, fixing a certain number of arbitrary functions on σ, the constraints (74) reconstruct the initial data on Σ. This would imply that the fields would allow a true localization on σ rather than on Σ. Since local observers can only distinguish between local events such a picture would mean that σ is a visualization of the global Cauchy surface by means of a particular field dynamics under consideration.

This idea is basically true with the correction that a number of space coordinates $d - 1$ that allow true localization may be even less than M. A relevant object called local Cauchy bundle E is a $M-$dimensional fiber bundle over an appropriate $d-1-$dimensional base manifold $\sigma \in \Sigma$ called local Cauchy surface and treated as the space manifold. The local Cauchy surface σ is a submanifold of $\mathcal{M}_M$. The space-time manifold is $R\times\sigma\subset\mathcal{M}_M$ where R is the time axis. Note that the local Cauchy bundle E is not necessarily a submanifold of $\mathcal{M}_M$. Since the dynamics in the generalized space-time was argued in section 3 to be compatible with the causal structure of the generalized space-time, the projection of this dynamics to the space-time $R\times\sigma$ that admits localization of the fields is expected to be compatible with the microcausality principle, i.e. the restriction of the Green functions to some local Cauchy surface σ is expected to vanish for the space-like separated regions. We will come back to this point in section 6.

Let us stress that different types of fields that may live in the same generalized space-time may require local Cauchy bundles of different dimensions thus providing different visualizations of the same generalized space-time. This phenomenon is analogous to the fact known after Dirac[11] that the singleton field lives on the boundary of AdS_4 while all other AdS_4 fields live in the bulk. The parallels with the ideas of holography[12,13] and brane dynamics are self-suggestive either. Note that even for the same dynamical system a choice of a particular local Cauchy bundle E may *a priori* be not unique. Different choices of E can lead to different descriptions of the same dynamical system. Although being equivalent, the descriptions in terms of different local Cauchy bundles may look differently and, in fact, describe dual versions of the same model that has a uniform description in the full generalized space-time.

Let us consider some examples.

5.1 M=2

For $M=2$, the generalized space-time reduces to the usual $3d$ space-time geometry while the equations (21) and (22) are equivalent to the usual massless equations for $3d$ scalar and spinor. There are no constraints (74) for $M=2$. Let us show that, for this case, the representation (75) allows for the usual field localizability.

We set

$$T^{\alpha\beta}=\delta^{\alpha\beta}\,,$$
$$X^{\alpha\beta}=t\delta^{\alpha\beta}+x^1\sigma_1^{\alpha\beta}+x^2\sigma_3^{\alpha\beta}\,,$$

where $\sigma_{1,3}^{\alpha\beta}$ are the two traceless symmetric Pauli matrices having unit square. Restriction of the solution (75) to the global Cauchy surface $t=0$ gives

$$b(x,0)=\frac{1}{\pi}\int d^2\xi\Big(b^+(\xi)\exp i(k_1x^1+k_2x^2)+b^-(\xi)\exp -i(k_1x^1+k_2x^2)\Big), \quad (78)$$

where

$$k_1 = \xi_1^2 - \xi_2^2 \,, \qquad k_2 = 2\,\xi_1\xi_2 \,. \tag{79}$$

The combinations of the integration variables (79) map R^2/Z_2 on R^2. This map is bijective with the expected singularity at $\xi_\alpha = 0$

$$dk_1 \wedge dk_2 = 4\Big(\xi_1^2 + \xi_2^2\Big) d\xi_1 \wedge d\xi_2 \,. \tag{80}$$

Note that

$$\xi_1^2 + \xi_2^2 = \sqrt{k_1^2 + k_2^2} \,. \tag{81}$$

Setting

$$b^{\pm}(\xi) = \frac{1}{2\pi}\Big(\xi_1^2 + \xi_2^2\Big) \exp \mp i k_i x_0^i \,, \tag{82}$$

one obtains

$$b(x,0) = \delta(x^i - x_0^i) \,, \qquad \frac{\partial}{\partial t} b(x,t)\Big|_{t=0} = 0 \,. \tag{83}$$

Analogously, setting

$$b^{\pm}(\xi) = \pm\frac{1}{2\pi i} \exp \mp i k_i x_0^i \,, \tag{84}$$

we obtain

$$b(x,0) = 0 \,, \qquad \frac{\partial}{\partial t} b(x,t)\Big|_{t=0} = \delta(x^i - x_0^i) \,. \tag{85}$$

Thus, the twistor parametrization (75) of the solutions of the $3d$ field equations is equivalent to the standard Minkowski parametrization with the integration measure $d^3k\delta(k^2)$. As expected, the initial data for $b(X)$ and its first time derivative can be fixed in an arbitrary way on the global Cauchy surface Σ. The analysis of the fermionic solutions (76) is analogous.

5.2 M=4

The case of $M = 4$ was argued in Ref. 1 to be equivalent to the free field equations of $4d$ conformal fields of all spins. The situation here is more interesting because the generalized space-time is ten-dimensional while the physical space-time is four-dimensional. Having four twistor integration parameters in (75), a submanifold of the global Cauchy surface that admits localization of fields can be at most four-dimensional. Let us show that the local Cauchy bundle is $R^3 \times S^1$ where $\sigma = R^3$ is the usual Cauchy surface of the $4d$ Minkowski space-time while the S^1 harmonics distinguish between spins of $4d$ conformal fields.

Using the language of two-component complex spinors we set

$$X^{\alpha\beta} = \left(x^{ab}, \mathcal{X}^{a\dot{a}}, \overline{x}^{\dot{a}\dot{b}}\right), \tag{86}$$

with the convention that the complex conjugation transforms dotted indices $a, b = 1, 2$ to the undotted ones $\dot{a}, \dot{b} = 1, 2$ and vice versa so that $\overline{x}^{\dot{a}\dot{b}}$ is complex conjugated to x^{ab} while $\mathcal{X}^{a\dot{a}}$ is hermitian. We choose the global Cauchy fibration as follows

$$X^{a\dot{a}} = \mathcal{X}^{a\dot{a}} = t\delta^{a\dot{b}} + x^i \sigma_i^{a\dot{b}}\,, \qquad X^{ab} = x^{ab}\,, \qquad \overline{X}^{\dot{a}\dot{b}} = \overline{x}^{\dot{a}\dot{b}}\,, \tag{87}$$

where $\sigma_i^{a\dot{b}}$ are hermitian traceless Pauli matrices normalized to have unit square. Having four integration variables ξ_α we can try to localize some four coordinates from among those nine that parametrize the global Cauchy surface.

The coordinates of the local Cauchy surface σ can be identified with the three space coordinates x^i of the usual $4d$ space-time. Let us use the complex notation for the space coordinates in the $1:2$ plane

$$x = x^1 + ix^2\,, \qquad \bar{x} = x^1 - ix^2\,. \tag{88}$$

The combinations of svectors ξ_α dual to the coordinates x^i

$$k_3 = \xi_1\bar{\xi}_{\dot{1}} - \xi_2\bar{\xi}_{\dot{2}}\,, \qquad \overline{k} = 2\xi_1\bar{\xi}_{\dot{2}}\,, \qquad k = 2\bar{\xi}_{\dot{1}}\xi_2\,, \tag{89}$$

map R_4/Z_2 on R_3, i.e. k_3, k and $\overline{k}$ can take arbitrary values. The leftover ambiguity in the integration variables ξ_α for fixed k_i is the overall phase factor $\xi_\alpha \to \exp \frac{1}{2} i\varphi\, \xi_\alpha$, $\varphi \in [0, 2\pi]$. (Recall that ξ_α is identified with $-\xi_\alpha$.) We set

$$\exp i\phi = 2\frac{\xi_1\xi_2}{k}\,, \qquad \exp -i\phi = 2\frac{\bar{\xi}_{\dot{1}}\bar{\xi}_{\dot{2}}}{\overline{k}}\,. \tag{90}$$

For the integration measure we obtain

$$\begin{aligned} d(\xi_1\bar{\xi}_{\dot{1}} - \xi_2\bar{\xi}_{\dot{2}}) \wedge d(\xi_1\bar{\xi}_{\dot{2}}) \wedge d(\bar{\xi}_{\dot{1}}\xi_2) \wedge d(\xi_1\xi_2) \\ = 2\xi_1\xi_2(\xi_1\bar{\xi}_{\dot{1}} + \xi_2\bar{\xi}_{\dot{2}}) d\xi_1 \wedge d\bar{\xi}_{\dot{1}} \wedge d\xi_2 \wedge d\bar{\xi}_{\dot{2}}\,. \end{aligned} \tag{91}$$

This is equivalent to

$$dk_3 \wedge dk \wedge d\bar{k} \wedge d\phi = -8i(\xi_1\bar{\xi}_{\dot{1}} + \xi_2\bar{\xi}_{\dot{2}}) d\xi_1 \wedge d\bar{\xi}_{\dot{1}} \wedge d\xi_2 \wedge d\bar{\xi}_{\dot{2}}\,. \tag{92}$$

The map (89), (90) from R^4/Z_2 associated with the integration variables ξ_α to $R^3 \times S^1$ described by the variables k_i, ϕ is non-degenerate except the expected singularity at $\xi_\alpha = 0$. Note that

$$\xi_1\bar{\xi}_{\dot{1}} + \xi_2\bar{\xi}_{\dot{2}} = \sqrt{k\bar{k} + k_3^2} = \sqrt{k_1^2 + k_2^2 + k_3^2}\,. \tag{93}$$

The integration over three noncompact momentum variables is expected to localize three space coordinates associated with the local Cauchy surface $\sigma = R^3$. Using the ambiguity in the cyclic momentum variable ϕ one can distinguish between different angular dependencies on the complex coordinate

$$z = x^{12}\,, \quad \bar{z} = \overline{x}^{\dot{1}\dot{2}}\,. \tag{94}$$

Fixing $|z| = r$, this is equivalent to considering functions on S^1.

Setting all other coordinates to zero, consider the following restriction of the solution of the field equations

$$\begin{aligned} b(x^3, x, \bar{x}, z, \bar{z}) = & \frac{1}{\pi^2} \int d^4\xi \\ & \times \Big(b^+(\xi) \exp i(x^3 k_3 + x\bar{k} + \bar{x}k + zk \exp i\phi + \bar{z}\bar{k} \exp -i\phi) \\ & + b^-(\xi) \exp -i(x^3 k_3 + x\bar{k} + \bar{x}k + zk \exp i\phi + \bar{z}\bar{k} \exp -i\phi) \Big)\,. \end{aligned} \tag{95}$$

From this expression with the coefficients $b^\pm(\xi)$ of the form

$$b^\pm(\xi) = \frac{1}{8\pi^2}(\xi_1\bar{\xi}_{\dot{1}} + \xi_2\bar{\xi}_{\dot{2}}) f(\exp -i\phi) \exp \mp i(x_0^3 k_3 + x_0\bar{k} + \bar{x}_0 k)\,, \tag{96}$$

or

$$b^\pm(\xi) = \pm\frac{1}{8\pi^2 i} f(\exp -i\phi) \exp \mp i(x_0^3 k_3 + x_0\bar{k} + \bar{x}_0 k)\,, \tag{97}$$

where $f(w^{-1})$ is some Laurant polynomial, it is clear that, for $r \neq 0$, the solution (95) is not localized at $x^i = x_0^i$ because it contains an infinite power series in the derivatives of $\delta(x^i - x_0^i)$ with higher powers of $r^2 = z\bar{z}$ in front of the higher derivatives of the delta-functions. It is therefore impossible to achieve further localization on $R^3 \times S^1 \subset \mathcal{M}_M$ using the ambiguity in the cyclic momentum variable ϕ. This problem can be avoided by taking the limit $r \to 0$ and keeping the leading terms of a given phase. This is equivalent to neglecting all terms that contain $z\bar{z}$, i.e., to considering analytic or antianalytic functions in z. The picture with $r = 0$ is most appropriate physically because the whole dynamical information is then localized at some point of the space R^3 equipped with an auxiliary S^1 that does not affect the analysis of locality and causality. This is why the $M = 4$ local Cauchy bundle $E = R^3 \times S^1$ is not a submanifold of the global Cauchy surface.

Indeed, from (96) one obtains that

$$b(x^3, x, \bar{x}, z, 0)\Big|_{t=0} = \frac{1}{2\pi i} \oint \frac{dw}{w} f(w^{-1}) \delta(x^3 - x_0^3)\delta(x - x_0)\delta(\bar{x} - \bar{x}_0 + zw) \tag{98}$$

$$b(x^3,x,\bar{x},0,\bar{z})\Big|_{t=0} = \frac{1}{2\pi i}\oint \frac{dw}{w} f(w^{-1})\delta(x^3-x^3_0)\delta(x-x_0+\bar{z}w^{-1})\delta(\bar{x}-\bar{x}_0)\,, \tag{99}$$

$$\frac{\partial}{\partial t} b(x^3,x,\bar{x},z,\bar{z})\Big|_{t=0} = 0\,. \tag{100}$$

Analogously, from (97) one obtains that

$$b(x^3,x,\bar{x},z,\bar{z})\Big|_{t=0} = 0\,, \tag{101}$$

$$\frac{\partial}{\partial t} b(x^3,x,\bar{x},z,0)\Big|_{t=0} = \frac{1}{2\pi i}\oint \frac{dw}{w} f(w^{-1})\delta(x^3-x^3_0)\delta(x-x_0)\delta(\bar{x}-\bar{x}_0+zw)\,, \tag{102}$$

$$\frac{\partial}{\partial t} b(x^3,x,\bar{x},0,\bar{z})\Big|_{t=0} = \frac{1}{2\pi i}\oint \frac{dw}{w} f(w^{-1})\delta(x^3-x^3_0)\delta(x-x_0+\bar{z}w^{-1})\delta(\bar{x}-\bar{x}_0)\,. \tag{103}$$

A power of a polynomial in z or $\bar{z}$ equals to the spin of the $4d$ field associated with a particular S^1 harmonic. Therefore, the higher spin is the more derivatives of the space delta-function appear in the equation (98). This property manifests the fact that the conformal higher spin fields contained in the generating function $b(X)$ admit interpretation as order-s derivatives of the dynamical potential fields like Maxwell field strength for spin 1, Weyl tensor for spin 2 etc (for more details see Refs. 14 and 2). The fundamental higher spin gauge fields (potentials) are expected to allow $\delta-$functional localization without extra derivatives on the $3d$ local Cauchy surface for any spin. The analysis of the fermionic equation (22) is analogous.

The analysis of this subsection proves the conjecture of Ref. 1 that the system of equations (21) and (22) at $M=4$ is equivalent to the infinite set of $4d$ equations of motion for massless fields of all spins. This fact is not trivial because the consideration of Ref. 1 was essentially local in the extra six coordinates of the generalized space-time. To summarize, what happens is that the independent degrees of freedom of the fields satisfying the equations (21) and (22) in $\mathcal{M}_M$ live on a four-dimensional local Cauchy bundle $E = R^3\times S^1$ with the base manifold R^3 identified with the usual space and the fiber S^1 giving rise to the infinite tower of spins. The dependence on the extra five coordinates of the global Cauchy surface is reconstructed uniquely by the constraints (74). As a result, propagation of the fields $b(X)$ and $f_\alpha(X)$ in the generalized space-time is equivalent to the propagation of local higher spin fields in the $4d$ space-time supplemented with one additional coordinate for spin.

Let us show how one can see this directly from the equations (21) and (22) focusing for definiteness on the bosonic case. According to (86) we write $b = b(\mathcal{X}^{a\dot{b}}, x^{ab}, \bar{x}^{\dot{a}\dot{b}})$. Eq. (21) decomposes into the three types of equations

$$\Big(\frac{\partial^2}{\partial x^{ab}\partial \bar{x}^{\dot{a}\dot{b}}} - \frac{\partial^2}{\partial \mathcal{X}^{a\dot{a}}\partial \mathcal{X}^{b\dot{b}}}\Big) b(\mathcal{X}, x, \bar{x}) = 0\,, \tag{104}$$

$$\Big(\frac{\partial^2}{\partial x^{ab}\partial x^{cd}} - \frac{\partial^2}{\partial x^{ac}\partial x^{bd}}\Big) b(\mathcal{X}, x, \bar{x}) = 0\,, \quad \Big(\frac{\partial^2}{\partial \bar{x}^{\dot{a}\dot{b}}\partial \bar{x}^{\dot{c}\dot{d}}} - \frac{\partial^2}{\partial \bar{x}^{\dot{a}\dot{c}}\partial \bar{x}^{\dot{b}\dot{d}}}\Big) b(\mathcal{X}, x, \bar{x}) = 0\,, \tag{105}$$

$$\Big(\frac{\partial^2}{\partial x^{ab}\partial \mathcal{X}^{c\dot{a}}} - \frac{\partial^2}{\partial x^{ac}\partial \mathcal{X}^{b\dot{a}}}\Big) b(\mathcal{X}, x, \bar{x}) = 0, \quad \Big(\frac{\partial^2}{\partial \bar{x}^{\dot{a}\dot{b}}\partial \mathcal{X}^{c\dot{d}}} - \frac{\partial^2}{\partial \bar{x}^{\dot{a}\dot{d}}\partial \mathcal{X}^{c\dot{b}}}\Big) b(\mathcal{X}, x, \bar{x}) = 0. \tag{106}$$

The equation (104) has two consequences. First, it determines the dependence on the complex coordinates x^{ab} and their conjugates $\bar{x}^{\dot{a}\dot{b}}$ in terms of $\mathcal{X}$-derivatives of the analytic and antianalytic functions $c(\mathcal{X}, x) = b(\mathcal{X}, x, 0)$ and $\bar{c}(\mathcal{X}, \bar{x}) = b(\mathcal{X}, 0, \bar{x})$. Second, its part antisymmetric in a, b (equivalently, $\dot{a}, \dot{b}$) implies the massless Klein-Gordon equation

$$\epsilon^{ab}\epsilon^{\dot{a}\dot{b}} \frac{\partial^2}{\partial \mathcal{X}^{a\dot{a}}\partial \mathcal{X}^{b\dot{b}}} b(\mathcal{X}, x, \bar{x}) = 0\,, \tag{107}$$

where ϵ^{ab} and $\epsilon^{\dot{a}\dot{b}}$ are the 2×2 antisymmetric symbols.

The equations (105) imply that the coefficients of the expansion of $b(\mathcal{X}, x, \bar{x})$ in powers of x and $\bar{x}$

$$b(\mathcal{X}, x, \bar{x}) = \sum b(\mathcal{X})_{a_1b_1, a_2b_2, \ldots; \dot{a}_1\dot{b}_1, \dot{a}_2\dot{b}_2, \ldots} x^{a_1b_1} x^{a_2b_2} \ldots ; \bar{x}^{\dot{a}_1\dot{b}_1} \bar{x}^{\dot{a}_2\dot{b}_2} \ldots \tag{108}$$

are totally symmetric both in undotted and dotted indices (equivalently, these equations are recognized as complexified $3d$ equations (21) to be solved by the $3d$ twistor ansatz). In particular, the holomorphic and antiholomorphic parts

$$\begin{aligned} c(\mathcal{X}, x) &= \sum c(\mathcal{X})_{a_1a_2a_3a_4\ldots} x^{a_1a_2} x^{a_3a_4} \ldots ; \\ \bar{c}(\mathcal{X}, \bar{x}) &= \sum \bar{c}(\mathcal{X})_{\dot{a}_1\dot{a}_2, \dot{a}_3\dot{a}_4\ldots} \bar{x}^{\dot{a}_1\dot{a}_2} \bar{x}^{\dot{a}_3\dot{a}_4} \ldots \end{aligned} \tag{109}$$

expand in powers of x and $\bar{x}$ with totally symmetric coefficients $c(\mathcal{X})_{a_1\ldots a_{2s}}$, $\bar{c}(\mathcal{X})_{\dot{a}_1\ldots\dot{a}_{2s}}$ to be identified with the (anti)selfdual $4d$ components of the higher spin fields. These satisfy the equations (106) equivalent to

$$\epsilon^{ac} \frac{\partial^2}{\partial x^{ab}\partial \mathcal{X}^{c\dot{a}}} c(\mathcal{X}, x) = 0\,, \qquad \epsilon^{\dot{a}\dot{d}} \frac{\partial^2}{\partial \bar{x}^{\dot{a}\dot{b}}\partial \mathcal{X}^{c\dot{d}}} \bar{c}(\mathcal{X}, \bar{x}) = 0\,, \tag{110}$$

which, in turn, reduce to the usual $4d$ massless equations for massless fields of all spins $c(\mathcal{X})_{a_1\dots a_{2s}}$ and $\bar{c}(\mathcal{X})_{\dot{a}_1\dots\dot{a}_{2s}}$. Let us note that for all spins $s\neq 0$ the Klein-Gordon equation is a consequence of (110) by virtue of

$$\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\frac{\partial^2}{\partial\mathcal{X}^{a\dot{a}}\partial\mathcal{X}^{b\dot{b}}}\frac{\partial}{\partial x^{cd}}c(\mathcal{X},x)=0\,,\qquad \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\frac{\partial^2}{\partial\mathcal{X}^{a\dot{a}}\partial\mathcal{X}^{b\dot{b}}}\frac{\partial}{\partial\bar{x}^{\dot{c}\dot{d}}}\bar{c}(\mathcal{X},\bar{x})=0\,. \tag{111}$$

For the $s=0$ field described by $c(\mathcal{X})=c(\mathcal{X},0)=\bar{c}(\mathcal{X},0)$ it cannot be derived this way because $\frac{\partial}{\partial x^{cd}}c(\mathcal{X})=\frac{\partial}{\partial\bar{x}^{\dot{c}\dot{d}}}\bar{c}(\mathcal{X})=0$.

Clearly, a number of coordinates x and $\bar{x}$ in the expansions for $c(\mathcal{X},x)$ and $\bar{c}(\mathcal{X},\bar{x})$ associated with spin can be equivalently described by a cyclic variable ϕ introduced previously. The fermionic case can be analyzed analogously.

The component fields $c(\mathcal{X})_{a_1\dots a_{2s}}$ and $\bar{c}(\mathcal{X})_{\dot{a}_1\dots\dot{a}_{2s}}$ describe, respectively, self-dual and antiself-dual components of the generalized Weyl tensors of massless fields of all spins s (both integer contained in $b(X)$ and half-integer contained in $f_\alpha(X)$). In particular, $c(\mathcal{X})_{a_1a_2}$ and $\bar{c}(\mathcal{X})_{\dot{a}_1\dot{a}_2}$ describe (anti) self-dual Maxwell field strengths, $c(\mathcal{X})_{a_1a_2a_3}$ and $\bar{c}(\mathcal{X})_{\dot{a}_1\dot{a}_2\dot{a}_3}$ describe (anti) self-dual gravitino field strengths, $c(\mathcal{X})_{a_1\dots a_4}$ and $\bar{c}(\mathcal{X})_{\dot{a}_1\dots\dot{a}_4}$ describe (anti) self-dual Weyl tensors, etc. Remarkably, the $4d$ electro-magnetic duality transformation and its extension to all higher spins[15]

$$\begin{aligned} c(\mathcal{X})_{a_1\dots a_{2s}} &\to \exp[2si\varphi]c(\mathcal{X})_{a_1\dots a_{2s}}\,, \\ \bar{c}(\mathcal{X})_{\dot{a}_1\dots\dot{a}_{2s}} &\to \exp[-2si\varphi]c(\mathcal{X})_{\dot{a}_1\dots\dot{a}_{2s}}\,, \end{aligned} \tag{112}$$

acquires a purely geometric origin in the generalized space-time $\mathcal{M}_4$, being a part of the SL_4 generalized Lorentz transformations with the group element of the form

$$a^a{}_b=\exp[i\varphi]\delta^a{}_b\,,\qquad a^{\dot{a}}{}_{\dot{b}}=\exp[-i\varphi]\delta^{\dot{a}}{}_{\dot{b}}\,,\qquad a^a{}_{\dot{b}}=a^{\dot{a}}{}_b=0\,. \tag{113}$$

This transformation belongs to the generalized $SO(M)$ space rotation because it leaves invariant the time-matrix $T^{a\dot{b}}=\delta^{a\dot{b}}$. Moreover, the duality transformation leaves invariant all space-time coordinates $\mathcal{X}^{a\dot{a}}$ of $R\times\sigma$. This is why from the perspective of the local Cauchy surface it acts only on the spin indices. (All other generalized Lorentz transformations affect the space-time coordinates and therefore contain derivatives of the $4d$ dynamical fields in the transformation lows.) Thus, the approach developed in Ref. 1 and in this paper incorporates dualities in a natural geometric way as particular generalized space-time symmetry transformations.

6 Quantization

Although Lagrangian formulation for the dynamical systems described by the equations (21) and (22) is yet lacking, the form of the general solutions (75) and (76) suggests a natural quantization prescription. The key property of the general solutions (75) and (76) is that they admit a well-defined decomposition into positive and negative frequency parts thus allowing for the definition of the creation and annihilation operators $b^+(\xi)$, $f^+(\xi)$ and $b^-(\xi)$, $f^-(\xi)$, respectively.

Let us now give more precise definitions. To make Gaussian integrals over svector integration variables well-defined we introduce complex coordinates

$$Z^{\alpha\beta} = Y^{\alpha\beta} + iX^{\alpha\beta} \,. \tag{114}$$

The imaginary part of $Z^{\alpha\beta}$ is identified with the coordinates in the generalized space-time $X^{\alpha\beta}$. The real part $Y^{\alpha\beta}$ is required to be positive definite. It is treated as a regulator that makes the Gaussian integrals well-defined. Physical quantities are obtained in the limit $Y^{\alpha\beta} \to 0$.

The expressions (75) and (76) are to be understood as

$$b(Z,\bar Z) = \frac{1}{\pi^{\frac{M}{2}}} \int d^M\xi \left(b^+(\xi) \exp -\xi_\alpha\xi_\beta \overline{Z}^{\alpha\beta} + b^-(\xi) \exp -\xi_\alpha\xi_\beta Z^{\alpha\beta} \right), \tag{115}$$

$$f_\alpha(Z,\bar Z) = \frac{1}{\pi^{\frac{M}{2}}} \int d^M\xi\, \xi_\alpha \left(f^+(\xi) \exp -\xi_\gamma\xi_\beta \overline{Z}^{\gamma\beta} + f^-(\xi) \exp -\xi_\gamma\xi_\beta Z^{\gamma\beta} \right). \tag{116}$$

The positive and negative frequency parts identify with the holomorphic and antiholomorphic parts of the quantum field

$$b(Z,\bar Z) = b^+(\overline{Z}) + b^-(Z)\,, \qquad f_\alpha(Z,\bar Z) = f^+_\alpha(\overline{Z}) + f^-_\alpha(Z)\,, \tag{117}$$

$$b^\pm(Z) = \frac{1}{\pi^{\frac{M}{2}}} \int d^M\xi\, b^\pm(\xi) \exp -\xi_\alpha\xi_\beta Z^{\alpha\beta}, \tag{118}$$

$$f^\pm_\gamma(Z) = \frac{1}{\pi^{\frac{M}{2}}} \int d^M\xi\, \xi_\gamma f^\pm(\xi) \exp -\xi_\alpha\xi_\beta Z^{\alpha\beta}. \tag{119}$$

For the fields $b(Z,\bar Z)$ and $f_\alpha(Z,\bar Z)$ to be real, $b^+(\xi)$ and $f^+(\xi)$ have to be complex conjugated to $b^-(\xi)$ and $f^-(\xi)$, respectively. The quantum operators $b^\pm(\xi)$ and $f^\pm(\xi)$ are required to have definite oddness according to (77).

We now interpret $b^+(\xi)$, $f^+(\xi)$ and $b^-(\xi)$, $f^-(\xi)$ as hermitian conjugated bosonic and fermionic creation and annihilation operators subject to the commutation relations

$$\left[b^\pm(\xi_1), b^\pm(\xi_2)\right] = 0\,, \quad \left[b^-(\xi_1), b^+(\xi_2)\right] = \frac{1}{2}\left[\,\delta(\xi_1-\xi_2) + \delta(\xi_1+\xi_2)\right], \tag{120}$$

$$\big[f^{\pm}(\xi_1), f^{\pm}(\xi_2)\big]_+ = 0\,, \quad \big[f^{-}(\xi_1), f^{+}(\xi_2)\big]_+ = \frac{1}{2}\big[\,\delta(\xi_1-\xi_2)-\delta(\xi_1+\xi_2)\big]\,, \tag{121}$$

where $[\,,\,]_+$ denotes anticommutator. The vacuum state is defined to satisfy

$$b^{-}(\xi)|0\rangle = 0\,, \qquad f^{-}(\xi)|0\rangle = 0\,, \tag{122}$$

$$\langle 0|b^{+}(\xi) = 0\,, \qquad \langle 0|f^{+}(\xi) = 0\,. \tag{123}$$

$$\langle 0|0\rangle = 1\,. \tag{124}$$

One-particle states are

$$\int d^M\xi B(\xi)b^{+}(\xi)\,|0\rangle\,, \qquad \int d^M\xi F(\xi)f^{+}(\xi)\,|0\rangle\,, \tag{125}$$

with arbitrary complex functions $B(-\xi) = B(\xi)$ and $F(-\xi) = -F(\xi)$.

The states are normalizable provided that $B(\xi)$ and $F(\xi)$ belong to L^2

$$\int d^M\xi \overline{B(\xi)}B(\xi) < \infty\,, \qquad \int d^M\xi \overline{F(\xi)}F(\xi) < \infty\,. \tag{126}$$

A useful basis is provided by the functions of the form

$$B(\xi) = P(\xi)\exp -T^{\alpha\beta}\xi_\alpha\xi_\beta\,, \qquad F(\xi) = Q(\xi)\exp -T^{\alpha\beta}\xi_\alpha\xi_\beta\,, \tag{127}$$

where $P(\xi)$ and $Q(\xi)$ are polynomials of ξ_α and $T^{\alpha\beta}$ is the positive definite matrix associated with the time arrow. This basis is equivalent to the unitary Fock module over the higher spin conformal symmetries, that was conjectured in Refs. 3 and 1 to be equivalent to the space of quantum states in the corresponding quantum field theory. The formulas (125) thus prove this conjecture.

The quantization prescription (120) and (121) allows us to write the conserved charges associated with the generators of the $sp(2M)$ transformations (23) and (24)

$$P_{\alpha\beta} = \int d\xi^M \Big(b^{+}(\xi)\xi_\alpha\xi_\beta b^{-}(\xi) + f^{+}(\xi)\xi_\alpha\xi_\beta f^{-}(\xi)\Big)\,, \tag{128}$$

$$L_\beta{}^\alpha = -\frac{i}{2}\int d\xi^M \tag{129}$$

$$\times\,\Big(b^{+}(\xi)(\xi_\beta\frac{\partial}{\partial\xi_\alpha} + \frac{\partial}{\partial\xi_\alpha}\xi_\beta)b^{-}(\xi) + f^{+}(\xi)(\xi_\beta\frac{\partial}{\partial\xi_\alpha} + \frac{\partial}{\partial\xi_\alpha}\xi_\beta)f^{-}(\xi)\Big)\,,$$

$$K^{\alpha\beta} = -\frac{1}{4}\int d\xi^M \Big(b^{+}(\xi)\frac{\partial^2}{\partial\xi_\alpha\partial\xi_\beta}b^{-}(\xi) + f^{+}(\xi)\frac{\partial^2}{\partial\xi_\alpha\partial\xi_\beta}f^{-}(\xi)\Big)\,. \tag{130}$$

The supergenerators are

$$Q_\alpha = \int d\xi^M \Big(b^+(\xi)\xi_\alpha f^-(\xi) + f^+(\xi)\xi_\alpha b^-(\xi)\Big)\,, \tag{131}$$

$$S^\alpha = \frac{i}{2}\int d\xi^M \Big(b^+(\xi)\frac{\partial}{\partial\xi_\alpha} f^-(\xi) + f^+(\xi)\frac{\partial}{\partial\xi_\alpha} b^-(\xi)\Big)\,, \tag{132}$$

with the obvious anticommutation relations

$$\{Q_\alpha, Q_\beta\} = 2P_{\alpha\beta}\,, \qquad \{S^\alpha, S^\beta\} = 2K^{\alpha\beta}\,, \qquad \{Q_\alpha, S^\beta\} = -2L_\alpha{}^\beta\,. \tag{133}$$

Note that from these relationships it follows that averages of the operators $P_{\alpha\beta}$ and $K^{\alpha\beta}$ form some positive semi-definite matrices as it is also obvious from (128) and (130). In particular, the energy operator $E = (1/M)T^{\alpha\beta}P_{\alpha\beta}$ is positive semi-definite for any positive definite matrix $T^{\alpha\beta}$ associated with the chosen time direction. Note also that bosonic and fermionic vacuum energies in (128)-(130) cancel out as a consequence of supersymmetry.

Let us now define the $\mathcal{D}$ functions as

$$\mathcal{D}^-(Z) = \frac{i}{\pi^M}\int d^M\xi\ \exp -\xi_\alpha\xi_\beta Z^{\alpha\beta}\,, \tag{134}$$

$$\mathcal{D}^+(\overline{Z}) = -\mathcal{D}^-(\overline{Z}) = \overline{\mathcal{D}^-(Z)}\,. \tag{135}$$

We have

$$\begin{aligned}\big[b^-(Z_1), b^+(\overline{Z}_2)\big] &= -i\mathcal{D}^-(Z_1+\overline{Z}_2)\,,\\ \big[b(Z_1,\overline{Z}_1), b(Z_2,\overline{Z}_2)\big] &= -i\mathcal{D}(Z_1+\overline{Z}_2, \overline{Z}_1+Z_2)\,,\end{aligned}$$

where

$$\mathcal{D}(Z,\overline{Z}) = \mathcal{D}^-(Z) + \mathcal{D}^+(\overline{Z}) = \mathcal{D}^-(Z) - \mathcal{D}^-(\overline{Z})\,. \tag{136}$$

By construction, the functions $\mathcal{D}^-(Z)$, $\mathcal{D}^+(Z)$ and $\mathcal{D}(Z,\overline{Z})$ solve the equations of motion (21). A rotation of the contour of integration over the variables ξ_α in the complex plane gives the following result

$$\mathcal{D}^-(Z)\Big|_{Y\to 0} = \frac{i}{\pi^{\frac{M}{2}}}\exp -\frac{i\pi I_X}{4}\frac{1}{\sqrt{|det(Z)|}}\Big|_{Y\to 0}\,. \tag{137}$$

Here I_X is the inertia index of the matrix $X^{\alpha\beta}$.

$$I_X = n_+ - n_-\,, \tag{138}$$

where n_+ and n_- are, respectively, the numbers of positive and negative eigenvalues of $X^{\alpha\beta}$. The formula (137) is in accordance with the explicit check in section 2 that $|det(X)|^{-\frac{1}{2}}$ provides a solution of the field equations away from singularities. The integral representation (134) provides the precise definition of the regularized expression in the complex plane.

From (137) it follows that

$$\mathcal{D}^+(Z)\Big|_{Y\to 0} = \frac{1}{i\pi^{\frac{M}{2}}}\exp\frac{i\pi I_X}{4}\frac{1}{\sqrt{|det(Z)|}}\Big|_{Y\to 0}\,, \tag{139}$$

and, therefore,

$$\mathcal{D}(Z)\Big|_{Y\to 0} = \frac{2}{\pi^{\frac{M}{2}}}\sin(\frac{\pi I_X}{4})\frac{1}{\sqrt{|det(Z)|}}\Big|_{Y\to 0}\,. \tag{140}$$

Extension of the suggested quantization scheme to the fermionic case of svector field $f_\alpha(X)$ is straightforward

$$\{f^-_\alpha(Z_1), f^+_\beta(\overline{Z}_2)\} = -i\mathcal{D}^-_{\alpha\beta}(Z_1+\overline{Z}_2)\,, \tag{141}$$

$$\{f_\alpha(Z_1,\overline{Z}_1), f_\beta(Z_2,\overline{Z}_2)\} = -i\mathcal{D}_{\alpha\beta}(Z_1+\overline{Z}_2,\overline{Z}_1+Z_2)\,, \tag{142}$$

where

$$\mathcal{D}_{\alpha\beta}(Z,\overline{Z}) = \mathcal{D}^-_{\alpha\beta}(Z)+\mathcal{D}^+_{\alpha\beta}(\overline{Z})\,, \qquad \mathcal{D}^+_{\alpha\beta}(Z) = \mathcal{D}^-_{\alpha\beta}(Z)\,, \tag{143}$$

$$\mathcal{D}^-_{\alpha\beta}(Z) = -\frac{\partial}{\partial Z^{\alpha\beta}}\mathcal{D}^-(Z)\,. \tag{144}$$

From (140) it follows that bosonic and fermionic $\mathcal{D}-$functions vanish if $I_X = 4n$, being different from zero otherwise. Since the indices α and β take even number of values M, $I_X = n_+ - n_- = M - 2n_-$ is even. Therefore

$$\mathcal{D}(Z,\overline{Z}) \neq 0 \qquad \text{for} \quad I_X = 4n+2 \quad n\in\mathbf{Z}\,. \tag{145}$$

For the case of $M = 2$ corresponding to the usual $3d$ geometry, this leads to the standard picture that the $\mathcal{D}-$function is zero for the space-like separation X with $I_X = 0$ and is different from zero for the future ($I_X = 2$) and past ($I_X = -2$) cones. For the first sight, the situation for $M > 2$ looks unsatisfactory because the $\mathcal{D}$ functions can be different from zero for the space-like separations with $I_X \neq \pm M$. The point however is that there is no reason to require the $\mathcal{D}-$function to vanish on the global Cauchy surface Σ as a whole, where the fields cannot be localized. The microcausality requires instead the $\mathcal{D}-$function to have usual properties upon restriction to the space-time $R^1 \times \sigma$ with some local Cauchy surface σ as a space manifold. In particular, this is the case for the $4d$ coordinates $\mathcal{X}^{a\dot{a}}$ of (87) at $x^{ab} = \bar{x}^{\dot{a}\dot{b}} = 0$ because the (real) inertia index of matrices of this form can only take values $+4$ (future), 0 (space-like separation) or -4 (past). As a result the corresponding $4d$ $\mathcal{D}-$ function vanishes inside the future and past cones. It is however different from zero at the boundaries of the future and past cones which conclusion respects the microcausality principle and is in accordance with the properties of the usual $\mathcal{D}-$function for $4d$ massless fields known to be localized at the boundary of the cone (see, for instance, Ref. 16). The $4d$ $\mathcal{D}-$function is a distribution localized at zeros of the eigenvalues of $X^{\alpha\beta}$. In section 7 we argue that the microcausality for higher M is also respected provided the local Cauchy surface σ is associated with an appropriate Clifford algebra in which case I_X can only take three values associated with the past, future and space-like separations.

Note that choosing the opposite type of the commutation relations, would replace $\sin(\frac{\pi I_X}{4})$ in (140) by $\cos(\frac{\pi I_X}{4})$. As a result, the corresponding $\mathcal{D}-$function would be different from zero for the space-like arguments with $I_X = 0$ thus violating the microcausality principle. Therefore, analogously to the case of Minkowski space-time, the locality requirement fixes statistics of the fields in the usual way requiring the scalar and svector fields to obey the Bose and Fermi statistics, respectively.

Let $\Theta(X)$ be the characteristic function of the future cone, i.e.

$$\begin{aligned} \Theta(X) &= 1 \qquad \text{if } X \text{ is positive semi-definite,} \\ \Theta(X) &= 0 \qquad \text{otherwise}. \end{aligned} \tag{146}$$

The advanced, retarded and causal Green functions are defined as

$$G^{ret}(X) = \Theta(X)\mathcal{D}(Z,\overline{Z})\Big|_{Y\to 0}, \quad G^{adv}(X) = -\Theta(-X)\mathcal{D}(Z,\overline{Z})\Big|_{Y\to 0}, \tag{147}$$

$$G^{c}(X) = \Big(\Theta(X)\mathcal{D}^{-}(Z) - \Theta(-X)\mathcal{D}^{+}(\overline{Z})\Big)\Big|_{Y\to 0}. \tag{148}$$

7 Towards any M

In the general case, the local Cauchy bundle E is M-dimensional. Although a full analysis of the structure of E is beyond the scope of this paper, as a first step towards the general case we wish to emphasize the role of the Clifford algebras in the generalized space-time geometry.

In the examples of $M = 2$ and $M = 4$ the space coordinates were associated with the set of symmetric matrices $\sigma_n^{\alpha\beta}$

$$x^{\alpha\beta} = x^n \sigma_n^{\alpha\beta} \tag{149}$$

such that the matrices

$$\gamma_i{}^\alpha{}_\beta = \sigma_i{}^{\alpha\gamma} T_{\gamma\beta}\,, \qquad i = 1 \dots d-1\,, \tag{150}$$

satisfy the Clifford algebra relationships

$$\gamma_i{}^\alpha{}_\gamma \gamma_j{}^\gamma{}_\beta + \gamma_j{}^\alpha{}_\gamma \gamma_i{}^\gamma{}_\beta = 2\eta_{ij}\delta_\beta^\alpha\,, \tag{151}$$

where η_{ij} is some positive definite symmetric form (for example, one can choose a basis with $\eta_{ij} = \delta_{ij}$). There are several reasons why the base space manifold σ is likely to be associated with the Clifford algebras for the general case.

An immediate consequence of (151) is that the matrices $\sigma_n^{\alpha\beta}$ are traceless

$$\sigma_n^{\alpha\beta} T_{\alpha\beta} = 0\,, \tag{152}$$

whenever $d \geq 3$, thus belonging to the global Cauchy surface. Another important property is that the momenta

$$k_n(\xi) = \sigma_n^{\alpha\beta} \xi_\alpha \xi_\beta \tag{153}$$

map the cone R^M/Z_2 on R^{d-1}, i.e. varying real twistor parameters ξ_α it is possible to get arbitrary values of $k_n(\xi)$. This results from the invariance of the construction under the space rotations $SO(d-1)$ generated by

$$M_{nm} = \frac{1}{4}[\gamma_n, \gamma_m]\,. \tag{154}$$

By a space rotation one aligns a vector $k_n(\xi)$ along any direction and then normalizes it arbitrarily by a rescaling of ξ_α. That momenta $k_n(\xi)$ span R^{d-1} allows for localization of the fields in $d-1$ space x^n coordinates dual to k_n, i.e., by means of integration over k_n, one can reach the delta-functional initial data $\delta(x^n - x_0^n)$ localized at any point of the physical space R^{d-1}.

Because the square of any linear combination of γ matrices is proportional to the unit matrix, for any vector a^n there exists such a basis in the space of ξ_α that

$$T^{\alpha\beta} = \delta^{\alpha\beta}\,, \qquad a^n \sigma_n^{\alpha\beta} = \sqrt{a^2}\, Y^{\alpha\beta}\,, \qquad a^2 = a^n a^m \eta_{nm}\,, \tag{155}$$

where

$$Y = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{156}$$

with all four blocks being $\frac{M}{2} \times \frac{M}{2}$ matrices (M is assumed to be even). As a result, the (non-degenerate) space-time matrix coordinates of the form

$$X^{\alpha\beta} = tT^{\alpha\beta} + x^n \sigma_n^{\alpha\beta} \tag{157}$$

can only have three values of the inertia index (138)

$$\begin{aligned} I_X &= M \qquad \text{for} \quad t > \sqrt{x^2}\,, \\ I_X &= -M \qquad \text{for} \quad t < -\sqrt{x^2}\,, \\ I_X &= 0 \qquad \text{for} \quad t^2 < x^2\,, \end{aligned} \tag{158}$$

where

$$x^2 = x^n x^m \eta_{nm}\,. \tag{159}$$

This corresponds to the standard space-time picture with the future cone ($I_X > 0$), past cone ($I_X < 0$) and the space-like region ($I_X = 0$). In accordance with the consideration of section 6 this property of space Clifford coordinates implies microcausality in the space-time $R \times \sigma$.

Note that the condition that I_X can take in the linear space of matrices of the form (157) only maximal value M (future), minimal value $-M$ (past) or some fixed intermediate value $I_X = I_{space} \neq \pm M$ for all values of t and x^n can be taken as an alternative definition leading to the Clifford algebra relations (151). Actually, it is true if for any x^n the matrix $x^n \sigma_n^{\alpha\beta} T_{\beta\gamma}$ has just two different eigenvalues. Since the sign change of x^n maps $I_{space} \to -I_{space}$, the only consistent choice is $I_{space} = 0$. Assuming that the time t is defined so that the matrices $\sigma_n^{\alpha\beta}$ associated with the space coordinates are T-traceless this implies that the matrix $x^n \sigma_n^{\alpha\beta} T_{\beta\gamma}$ has $\frac{M}{2}$ eigenvalues μ and $\frac{M}{2}$ eigenvalues $-\mu$. Equivalent statement is that, for any x^n, the matrix $x^n \sigma_n^{\alpha\beta} T_{\beta\gamma}$ is traceless and its matrix square is proportional to the unit matrix that is equivalent to the Clifford algebra definition (151).

These properties indicate that the Clifford algebra realization of space is closely related to the concept of locality and microcausality. In other words, the generalized space-time $\mathcal{M}_M$ is visualized via Clifford algebras. Let us note that the space metric η_{nm} appears in the theory just by identification of an appropriate Clifford algebra (151).

The Clifford realization of the space-time $R \times \sigma$ guarantees usual conformal symmetry in d dimensions. The ordinary space rotation symmetry $o(d-1)$ is generated in the standard way as the subalgebra of the generalized Lorentz symmetry sl_M spanned by the generators (154). Its extension to the Lorentz subalgebra $o(d-1,1) \subset sl_M$ is achieved by boosts realized as

$$l_n = \gamma_n\,. \tag{160}$$

The Lorentz algebra extends to the Poincare algebra by the transformations (23), (24) with the parameters

$$\epsilon^{\alpha\beta} = a^0 T^{\alpha\beta} + a^k \sigma_k^{\alpha\beta}. \tag{161}$$

Its further extension to the conformal algebra is achieved via the generalized special conformal transformations (23), (24) with the parameters

$$\epsilon_{\alpha\beta} = b^0 T_{\alpha\beta} + b^k \sigma_{k\,\alpha\beta}\,. \tag{162}$$

Recall that the dilatation appears as the central component in the gl_M extension of the generalized Lorentz transformations, i.e. it is generated by the unit element of the Clifford algebra.

The (classical) generalized electro-magnetic duality group identifies with such a subgroup of the generalized Lorentz transformations SL_M that leaves invariant the time matrix $T^{\alpha\beta}$ and the space coordinates of the local Cauchy surface. By definition, such defined duality group acts on the fiber of the local Cauchy bundle E, that is on the indices of the usual space-time fields in $R\times\sigma$.

For $M = 2^p$ the algebra of real matrices Mat_{2^p} is isomorphic to a particular real Clifford algebra. Despite the system was shown to be Lorentz covariant, this does not necessarily mean that the set of space γ_n matrices associated with the local Cauchy surface admits an extension by a matrix γ_0

$$\gamma_0{}^\alpha{}_\gamma \gamma_0{}^\gamma{}_\beta = -\delta^\alpha_\beta\,, \tag{163}$$

that anticommutes to the space-like matrices γ_i. Leaving details for a future publication,[17] let us just mention that γ_0 exists when the Clifford algebra has antisymmetric charge conjugation matrix ($p = 1$ or $2 \; mod \;\; 4$) but does not exist otherwise ($p = 0$ or $3 \; mod \;\; 4$). This fact has an important interpretation. Namely, the cases that do not allow Lorentz invariant extension of the Clifford algebra are chiral, i.e. svector indices correspond to left or right real spinors (depending on the definition of γ in view of the automorphism $\gamma \to -\gamma$). Indeed, when γ_0 exists, the boost generators can be identified as usual with $\gamma_k\gamma_0$. The operator $\Gamma = \gamma_0\gamma_1\ldots\gamma_{d-1}$ then allows to define chirality in the standard way, thus giving rise to left and right svectors (may be complex conjugated to each other). If γ_0 does not exist one has to use the realization (160) for boosts implying that the svector representation forms an irreducible (and, therefore, chiral) representation of the Lorentz algebra. As a result, the corresponding theory as a whole turns out to be chiral, describing irreducible (anti)self-dual conformal fields in d dimensions. Recall that conformal fields are described by scalar, spinor and, for d even, by massless fields associated with the representations of the little group $o(d-2)$ described by rectangular Young diagrams[18,19] of maximal height $\frac{1}{2}(d-2)$.

Let us consider the important example of $M=8$. The corresponding real Clifford algebra is defined by the relationships

$$\{\psi_A\,,\psi_B\} = -\delta_{AB}\,, \qquad A,B = 1-6\,. \tag{164}$$

The charge conjugation matrix $C_{\alpha\beta} = C_{\beta\alpha}$ is symmetric and positive definite. It can therefore be identified with the time-defining matrix $T_{\alpha\beta}$. The matrices

$$\psi_{A_1\ldots A_n\,\alpha\beta} = \psi_{[A_1}\ldots\psi_{A_n]\alpha\beta} \tag{165}$$

turn out to be symmetric in α,β for $n=0,3$ and 4 and antisymmetric for $n=1,2,5,6$ in an irreducible representation. Here the indices α and β are raised and lowered by the charge conjugation matrix $C_{\alpha\beta}$ identified with $\psi_{A_1\ldots A_n\,\alpha\beta}$ at $n=0$. Being antisymmetric, the matrices ψ_A cannot serve themselves as a basis for coordinates of a local Cauchy surface in $\mathcal{M}_M$. One can however choose five symmetric matrices

$$\gamma_n = \psi_{n56} \quad \text{for} \quad n = 1\ldots 4\,, \qquad \gamma_n = \psi_{1234} \quad \text{for} \quad n=5\,, \tag{166}$$

that satisfy the Clifford algebra relations

$$\{\gamma_n\,,\gamma_m\} = 2\delta_{nm}\,, \qquad n,m = 1\ldots 5\,. \tag{167}$$

These matrices are traceless and can therefore be identified with a particular basis of space coordinates. Along with the time matrix $C_{\alpha\beta} = T_{\alpha\beta}$ this defines a $6d$ space-time. The matrices (165) do not contain a matrix γ_0 that anticommutes to γ_n. Eight-component svector therefore identifies with a chiral $6d$ spinor. The properties of the $\mathcal{M}_8$ model make it reminiscent of the $6d$ (super)conformal theory proposed by Hull.[20] We expect that the $M=8$ theory is a higher spin extension of the $6d$ conformal self-dual gravity theory studied by Hull.[20] For supersymmetrization of the model for any M see Ref. 1 and section 8 of this paper. In Ref. 1 it was also explained for the example of $4d$ (i.e. $M=4$) theory how one can truncate the model to a particular (lower spin, if desirable) irreducible supermultiplet by virtue of certain auxiliary non-commutative scalar fields.

Let us note that, excluding the fifth coordinate associated with γ_5 from the set of space coordinates allows one to introduce the $5d$ time-like matrix γ_0 by identifying it with ψ_{123456}. In fact, one can treat such a model as a (non-chiral) result of compactification of the original $6d$ chiral model on S^1 upon an appropriate identification in the momentum space.

The matrices γ_n provide a basis for the coordinates of the $M = 8$ local Cauchy surface R^5. Let us now show that the rest three coordinates of the local Cauchy bundle are associated with the group manifold[f] $SO(3)$, i.e. $E = R^5 \times SO(3)$. The point is that the five generalized space momenta

$$k_n(\xi) = \gamma_{n\alpha\beta}\xi^\alpha\xi^\beta \tag{168}$$

are invariant under the $SU(2)$ rotations of ξ_α generated by ψ_5, ψ_6 and ψ_{56}. The coordinates of $SU(2)$ are analogous to the cyclic coordinate ϕ in the $M = 4$ case. Because we are only interested in bilinear combinations in the twistor momenta ξ_α the $SU(2)$ reduces to $SO(3)$. Analogously to the $4d$ case, the modes on $SO(3)$ are expected to be associated with various $6d$ (generalized) higher spin massless fields. Moreover, $SO(3)$ is expected to be a (classical) electro-magnetic duality group of the $M = 8$ model because, by construction, it is a subgroup of the generalized Lorentz transformations that leaves invariant the time coordinate and the coordinates of the local Cauchy surface σ. It would be interesting to compare its component action with the $6d$ higher spin duality transformations discussed by Hull.[15]

For the $M = 16$ case with generalized conformal symmetry $Sp(32)$ the situation is analogous to $M = 8$. The Clifford basis elements (165) are symmetric for $n = 0, 3, 4, 7, 8$ and antisymmetric for $n = 1, 2, 5, 6$. The maximal number of 9 Clifford space coordinates can be identified with $\psi_{A_1 \ldots A_n\,\alpha\beta}$ $(A = 1 \ldots 8)$ at $n = 7$ and $n = 8$. This implies a nine-dimensional local Cauchy surface σ and, therefore, ten-dimensional space-time. Again, the corresponding $10d$ theory is chiral because the corresponding Clifford algebra associated with the space coordinates does not admit an extension to the $10d$ Minkowski case. It becomes a non-chiral relativistic $9d$ theory upon compactification of one of the space dimensions. The $M = 16$ local Cauchy bundle is expected to have a seven-dimensional fiber. The world line analysis of the twistor dynamics suggests[21] that the relevant choice of the $M = 16$ Cauchy bundle may be $E = \sigma \times S^7$, $\sigma = R^9$.

The $M = 32$ model possessing the generalized conformal symmetry $Sp(64)$ is a relativistic $11d$ theory and, as such, can be related to M theory. (Note that the relevance of 64 supercharges to M-theory was discussed in Ref. 5).

A detailed analysis of $M > 4$ models requires some technicalities on the Clifford algebra realization of $\mathcal{M}_M$ and associated symmetries and will be given elsewhere.[17] One hard issue is to analyze higher M analogues of the changes of variables (79) and (89), (90).

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[f] Note that the idea that the $M = 8$ twistor space identifies with $R^5 \times S^3$ was also suggested in the context of the analysis of the world-particle models by Lukierski.[21]

8 Extended supersymmetry

The unfolded form of the superfield extended supersymmetry generalization of the equations (21) and (22) was discussed in Ref. 1. Here we would like to discuss a slightly different, although equivalent, formulation of an extended supersymmetric system that exhibits the supersymmetry $osp(2L, 2M)$. Namely, let us introduce the generalized supercoordinates X^{AB} with $A = (\alpha, i)$, $\alpha = 1 \dots M$, $i = 1 \dots L$. Let $\pi(A) = 0$ for $A = \alpha$ and $\pi(A) = 1$ for $A = i$. The coordinates X^{AB} are required to be graded symmetric

$$X^{AB} = (-)^{\pi(A)\pi(B)} X^{BA}. \tag{169}$$

The coordinates $X^{\alpha\beta} = X^{\beta\alpha}$ and $X^{ij} = -X^{ji}$ are even (commuting) while $X^{\alpha i} = X^{i\alpha}$ are odd (i.e., anticommuting elements of Grassmann algebra). Note that the anticommuting supercoordinates can be identified with a half of the superspace coordinates $\theta^{\alpha i}$ of section 7.3 of Ref. 1. The formulation presented here can be thought of as a sort of a chiral superfield formulation compared to that of Ref. 1. The coordinates X^{ij} are new.

The straightforward generalization of the equation (21) is

$$\frac{\partial^2}{\partial X^{AB} \partial X^{CD}} \Phi(X) = (-1)^{\pi(C)\pi(B)} \frac{\partial^2}{\partial X^{AC} \partial X^{BD}} \Phi(X) \,. \tag{170}$$

This equation is invariant under the straightforward extension of the $sp(2M)$ transformations (1)-(3) to $osp(2L, 2M)$ with appropriate grading-dependent signs inserted. Generic solution of (170) is analogous to (75)

$$\Phi(X) = \frac{1}{\pi^{\frac{M}{2}}} \int d^{M,L}\eta \left(\Phi^+(\eta) \exp i\eta_A \eta_B X^{AB} + \Phi^-(\eta) \exp -i\eta_A \eta_B X^{AB} \right), \tag{171}$$

where $\eta_A = (\xi_\alpha, \psi_i)$ with ξ_α and ψ_i being, respectively, even and odd $(\psi_i \psi_j = -\psi_j \psi_i)$ integration variables. We require

$$\Phi^\pm(-\eta) = (-1)^{M+L} \Phi^\pm(\eta) \,, \tag{172}$$

relaxing the condition that M is even. As a result, the $osp\,(2L, 2M)$ invariant system (170) turns out to be equivalent to the set of the fields $b(\xi, \psi)$ and $f(\xi, \psi)$ in $\mathcal{M}_M$ satisfying

$$b(\xi, -\psi) = (-1)^{L+M} b(\xi, \psi) \,, \quad f(\xi, -\psi) = -(-1)^{L+M} f(\xi, \psi) \,.$$

In other words, a single field $\Phi(X)$ in the generalized superspace contains a set of bosons and fermions described by all antisymmetric tensors of even (odd) and odd (even) ranks, respectively, for $M + L$ even (odd). Note that the generators of $osp(2L, 2M)$ can be realized as quantum operators analogous to (128)-(130) associated with various bilinear combinations of η_A and $\frac{\partial}{\partial \eta_A}$.

In fact, the relevance of the equations (170) is most obvious from their unfolded form

$$(\frac{\partial}{\partial X^{AB}} - \frac{\partial^2}{\partial Y^A \partial Y^B})\Phi(Y|X) = 0\,, \qquad \Phi(-Y|X) = \Phi(Y|X)\,, \tag{173}$$

where $Y^A = (y^\alpha, \omega^i)$ are auxiliary supercoordinates. Following to the methods of unfolded dynamics (see Refs. 1, 2 and references therein) it is elementary to see that the system (173) is $osp(2L, 2M)$ invariant, is equivalent to (170) for $\Phi(X) = \Phi(0|X)$ and reduces to the system of bosons and fermions in $\mathcal{M}_M$ associated, respectively, with odd and even elements of the Grassmann algebra, generated by ω^i, dual to the Grassmann algebra generated by ψ_i.

9 Geometric origin of the generalized space-time

The group $Sp\,(2M)$ is constituted by the real matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{174}$$

with $M \times M$ blocks $a_\alpha{}^\beta$, $b_{\alpha\beta}$, $c^{\alpha\beta}$ and $d^\alpha{}_\beta$ satisfying relations

$$a_\alpha{}^\gamma b_{\beta\gamma} = a_\beta{}^\gamma b_{\alpha\gamma}\,, \tag{175}$$

$$c^\alpha{}_\gamma d^{\beta\gamma} = c^\beta{}_\gamma d^{\alpha\gamma}, \tag{176}$$

$$a_\alpha{}^\gamma d^\beta{}_\gamma - b_{\alpha\gamma} c^{\beta\gamma} = \delta^\beta_\alpha\,, \tag{177}$$

equivalent to the invariance condition $ACA^t = C$ for the skewsymmetric bilinear form

$$C = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{178}$$

where I is the $M \times M$ unit matrix.

$Sp\,(2M)$ contains the subgroup T constituted by the elements

$$t(X) = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}, \tag{179}$$

with various real generalized coordinates $X^{\alpha\beta}$. The group of translations T is Abelian and has the product law

$$t(X)t(Y) = t(X+Y)\,. \tag{180}$$

The subgroup of generalized Lorentz transformations and dilatations is described by the matrices (174) with $b = c = 0$ and $a_\alpha{}^\gamma d^\beta{}_\gamma = \delta_\alpha^\beta$. The subgroup of special conformal transformations is constituted by the matrices (174) with $a = d = I$, $b = 0$.

Let P_M be the parabolic subgroup of $Sp\,(2M)$ constituted by the matrices (174)-(177) with $b^{\alpha\beta} = 0$, i.e.

$$P_M \ni p = \left(\begin{array}{cc} a & 0 \\ c & d \end{array} \right). \tag{181}$$

The compactified generalized space-time is the coset space

$$\mathcal{CM}_M = Sp\,(2M)/P_M\,, \tag{182}$$

constituted by the elements $h \in Sp\,(2M)$ identified modulo the right action of P_M

$$h \sim h_1 = hp\,, \qquad h \in Sp\,(2M)\,, \quad p \in P_M\,. \tag{183}$$

$\mathcal{CM}_M$ consists of the classes represented by elements $t(X)$ of the group of translations T, which identify with the points of the uncompactified generalized space-time $\mathcal{M}_M$, along with some additional equivalence classes that represent conformal infinity.

Any $Sp\,(2M)$ group element A (174) with a nondegenerate block d is in the class represented by some $t(X) \in T$. Indeed, once $det\Big|d^\alpha{}_\beta\Big| \neq 0$, then $A = A'C'$ with some

$$A' = \left(\begin{array}{cc} a' & b' \\ 0 & d' \end{array} \right), \qquad C' = \left(\begin{array}{cc} I & 0 \\ c' & I \end{array} \right), \tag{184}$$

and then $A' = t(X)\tilde{A}$ where $\tilde{A}$ has only diagonal blocks nonzero. As a result, any element of $Sp\,(2M)$ with $det\Big|d^\alpha{}_\beta\Big| \neq 0$ belongs to some equivalence class associated with the uncompactified generalized space-time $\mathcal{M}_M$.

From (177) it follows that d is non-degenerate for any element p (181) of the parabolic subalgebra P_M. As a result, $rank|d|$ of an element $A \in Sp\,(2M)$ (174) is the same for all Ap, $p \in P_M$. In other words, $rank|d|$ characterizes different types of the equivalence classes, i.e. different subsets of the compactified space-time $\mathcal{CM}_M$. The subset of elements with $det|d| = M$ identifies with $\mathcal{M}_M$. Those with $rank|d| = m$, $m = 0, 1, 2, \ldots M-1$ describe the conformal infinity strata mentioned in section 1.

The inversion R is now a well-defined transformation in $\mathcal{CM}_M$. Consider the following element of $Sp\,(2M)$

$$\tilde{R} = \left(\begin{array}{cc} 0 & I \\ -I & 0 \end{array} \right), \tag{185}$$

i.e. $\tilde{R} = A$ (174) with ${a_\alpha}^\beta = 0$, ${d^\alpha}_\beta = 0$, $b_{\alpha\beta} = \delta_{\alpha\beta}$, $c^{\alpha\beta} = -\delta^{\alpha\beta}$ (note that the conditions (175)-(177) are satisfied). It follows that

$$\tilde{R}\, t(X) = \begin{pmatrix} 0 & I \\ -I & -X \end{pmatrix} . \tag{186}$$

Choose $p \in P_M$ in the form

$$p(X) = \begin{pmatrix} -X & 0 \\ I & -X^{-1} \end{pmatrix} . \tag{187}$$

Then

$$\tilde{R}\, t(X)\, p(X) = \begin{pmatrix} I & -X^{-1} \\ 0 & I \end{pmatrix} . \tag{188}$$

Thus $\tilde{R}$ maps a non-degenerate X to $-X^{-1}$. Up to a sign, this is the inversion (13). If X is degenerate, $\tilde{R}\, t(X)$ is also well-defined in $\mathcal{CM}_M$, mapping X to some element of the conformal infinity classes.

More generally, it is easy to see that according to the definition (182), the action of a general element (174) in $\mathcal{M}$ is described for nondegenerate $(cX+d)$ by the matrix fraction-linear transformation

$$A(X) = (aX+b)(cX+d)^{-1} . \tag{189}$$

This formula for the action of $Sp\,(2M)$ on the space of symmetric matrices was used in particular in Refs. 6 and 7. It reproduces (9), (10), (11) and (188) as particular cases.

Note that the minus sign in the transformation low (188) is not occasional. The group $Sp\,(2M)$ does not contain the PT reflection $X^{\alpha\beta} \to -X^{\alpha\beta}$. For PT reflection to be included, $Sp\,(2M)$ has to be extended to $Sp\,(2M) \times Z_2$ which can be defined as the group that leaves the form C invariant up to a sign. This is equivalent to replacing (177) by

$${a_\alpha}^\gamma {d^\beta}_\gamma - b_{\alpha\gamma} c^{\beta\gamma} = \pm \delta_\alpha^\beta . \tag{190}$$

Simultaneously, P_M has to be extended to $P_M \times Z_2$ with ${a_\alpha}^\gamma {d^\beta}_\gamma = \pm\delta_\alpha^\beta$. The PT reflection is represented by

$$PT = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} . \tag{191}$$

It acts properly on the coordinates $X^{\alpha\beta}$ because $PT\, t(X)\, PT = t(-X)$. The true inversion (13) $R \in Sp(2M) \times Z_2$ is then represented by

$$R = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} . \tag{192}$$

The central element of $Sp\,(2M)$

$$F = \begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix} \tag{193}$$

acts trivially in $\mathcal{CM}_M$. However, according to (17) it acts nontrivially on the sections of the corresponding fiber bundles over $\mathcal{CM}_M$ if fermions are present carrying odd numbers of the svector indices. In other words, F is the boson-fermion parity operator.

Let us note that an alternative definition of $Sp\,(2M)-$invariant $\frac{1}{2}M(M+1)-$dimensional space was given by Fronsdal[7] in terms of isotropic $M-$forms. Although we have not check this explicitly, the two constructions are expected to be equivalent.

The generalization to superspace is straightforward. $Sp\,(2M)$ is extended to $OSp(L,2M)$ constituted by the supergroup elements

$$A = \begin{pmatrix} a & b & e \\ c & d & f \\ g & h & p \end{pmatrix}, \tag{194}$$

that leave invariant the (super)antisymmetric bilinear form

$$C = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & I \end{pmatrix}. \tag{195}$$

The first, second and third rows (columns) in these formulas have, respectively, heights (widths) M, M and L. The superspace with supercoordinates $X^{\alpha\beta}$, θ_i^α, introduced in Ref. 1, corresponds to the coset space $OSp(L,2M)/P_{L,2M}$ where the parabolic supergroup $P_{L,2M}$ is formed be the supergroup elements $A \in OSp(L,2M)$ (194) with $b=0$ and $e=0$.

The superspace of section 8 of this paper results from the decomposition of $OSp\,(2L,2M)$

$$A = \begin{pmatrix} a & b & e & p \\ c & d & f & q \\ g & h & n & r \\ v & u & m & l \end{pmatrix}, \tag{196}$$

with the invariant supersymplectic form

$$C = \begin{pmatrix} 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{pmatrix}, \tag{197}$$

and the parabolic supergroup $P'_{2L,2M}$ formed by the elements $A \in OSp\,(2L, 2M)$ of the form

$$A = \begin{pmatrix} a & 0 & e & 0 \\ c & d & f & q \\ g & 0 & n & 0 \\ v & u & m & l \end{pmatrix}. \tag{198}$$

10 Outlook

It is shown that the equations of motion (21) and (22) in the generalized space-time proposed in Ref. protect1 admit consistent interpretation compatible with causality both at the classical and quantum levels. The coordinates of the generalized space-time $\mathcal{M}_M$ are various symmetric $M \times M$ matrices $X^{\alpha\beta}$, $\alpha, \beta = 1 \dots M$. The future and past cones of the origin of coordinates $X^{\alpha\beta} = 0$ identify with the positive-definite and negative-definite matrices $X^{\alpha\beta}$. The generalized space-time is shown to have only one time coordinate associated with any positive-definite matrix $T^{\alpha\beta}$ (or positive semi-definite for light-like directions). Different choices of $T^{\alpha\beta}$ correspond to different coordinate frames related by generalized Lorentz transformations.

A global Cauchy surface Σ is defined as such a submanifold of $\mathcal{M}_M$ that any two its points are separated by a space-like interval and the set of points that belong to the future and past cones of all points of Σ covers the whole generalized space-time. A particular realization of Σ is provided by matrices $X^{\alpha\beta}$ satisfying $T_{\alpha\beta}X^{\alpha\beta} = 0$, where $T_{\alpha\beta}$ is the inverse of $T^{\alpha\beta}$.

A local Cauchy bundle E is a $M-$dimensional space that provides the full set of unrestricted initial data for the problem. The base space σ of E, called local Cauchy surface, is identified with the usual $d-1$ dimensional space. The difference between the concepts of global Cauchy surface and local Cauchy bundle is due to the fact that the equations (21) and (22) contain constraints that to some extend fix behavior of the fields on the global Cauchy surface. Usual space-time is identified with the fibration $R^1 \times \sigma$ over the local Cauchy surfaces parametrized by the time parameter t. The Cauchy problem in the generalized space-time $\mathcal{M}_M$ is defined in terms of a set of functions on E that allow for localization in terms of distributions on σ. The causality requires the propagation in the space-time $R \times \sigma$ to be microcausal. Remarkably, the coordinates of the local Cauchy surface compatible with microcausality turn out to be associated with the subspace of the symmetric matrices satisfying the Clifford algebra relations. This is how the ordinary Minkowski coordinates and the spinor interpretation of the indices α, β (for $M = 2^p$) reappear. Let us note that the concept of the global Cauchy surface is associated with generalized space-time ($\mathcal{M}_M$ in our case) while the concept of local Cauchy bundle is dependent on a particular dynamical system in $\mathcal{M}_M$. In other words, different dynamical systems can provide different visualizations of the same generalized space-time $\mathcal{M}_M$ via different local Cauchy bundles.

The compactification $\mathcal{CM}_M = Sp\,(2M)/P_M$ of $\mathcal{M}_M$ is introduced, with P_M being an appropriate parabolic subgroup of $Sp\,(2M)$. It is shown how the compactified generalized space-time $\mathcal{CM}_M$ contains infinities of $\mathcal{M}_M$ associated with the singularities of the generalized inversion in $\mathcal{M}_M$. The formulation of dynamics in the generalized space-time provides geometric interpretation of the classical electro-magnetic duality group as the subgroup of the generalized Lorentz transformations $SL_M \in Sp\,(2M)$ that leaves invariant the space-time $R \times \sigma$.

For the lower values of M, namely $M = 2$ and $M = 4$, the Lorentz content of the equations (21) and (22) is completely clear. $\mathcal{M}_2$ identifies with the usual $3d$ space-time $\Sigma = E = \sigma = R^2$. For $\mathcal{M}_4$, $E = \sigma \times S^1$ where $\sigma = R^3$ is the usual three-dimensional space while functions on S^1 parametrize various fields of all spins in the usual $4d$ space-time. The duality group $U(1)$ is the extension of the electro-magnetic duality to all $4d$ higher spins. The $M = 8$ and $M = 16$ models describe some $d = 6$ and $d = 10$ dynamical systems with $\sigma = R^5$ and $\sigma = R^9$, respectively. Their Lorentz field content will be given elsewhere.[17] Presumably, the $M = 8$ theory provides a higher spin extension of the $6d$ superconformal gravity theory by Hull.[20] The $M = 8$ classical duality group inherited from $\mathcal{M}_M$ is $SO(3)$.

The case of $M = 32$ with the generalized conformal symmetry $Sp(64)$ corresponds to some $d = 11$ relativistic theory. To study its possible relationship with M theory is one of the most exciting directions for the future investigation. A related question is to analyze whether for some M there may exist different sets of local Cauchy surfaces in the same model that look like different space-times. Presumably, this could explain duality of different theories as (non-locally equivalent) different local realizations of the same model in $\mathcal{M}_M$.

The approach proposed in Ref. 1 and further developed in this paper operates in terms of a straightforward generalization of twistors. As a result, solutions of the field equations decompose into positive and negative frequency parts associated with the decomposition of the solution into the holomorphic and antiholomorphic parts in the complex coordinates as in (115) and (116). This allows for a natural quantization prescription in terms of creation and annihilation operators that depend on the twistor variables ξ_α as in (120) and (121). The svector indices $\alpha = 1 \ldots M$ appear in a quite uniform way for all even M. Since the svector indices turn out to be identified with some sets of space-time spinor indices via the Clifford realization of the Minkowski space-time, this may have a number of important improvements in the situations in which the difference between spinor and tensor representations in the Minkowski track plays a role, like in dimensional regularization and supersymmetry. The dynamics in the generalized space admits an extension to generalized supersymmetric models exhibiting $OSp\,(L, 2M)$ supersymmetries realized in the appropriate superspaces. Let us emphasize that only very special Minkowski relativistic models, like, e.g., the model of all massless $4d$ fields, allow for the realization in the generalized space-time $\mathcal{M}_M$ with unbroken generalized symmetries.

One of the lessons of our analysis is that invariant local field equations formulated in a space S can effectively describe propagation in smaller spaces s associated with S. By local observations one can only observe s. However, the full space S manifests itself via symmetries and specific particle spectra of the theory. In the model under consideration $S = \mathcal{M}_M$ and s is some Minkowski space-time. One can say that the usual Minkowski space-time is a visualization of the generalized space-time $\mathcal{M}_M$. It is tempting to speculate that we live in a generalized space-time $\mathcal{M}_M$ which cannot be seen by local observations, but manifests itself via dualities.

An important question is what is a Lagrangian form of the dynamics in the generalized space-time. An interesting option somewhat reminiscent of the group manifold approach[22] is that the Lagrangian is a functional on the submanifold $R^1 \times \sigma$ associated with the usual space-time. To proceed in this direction it is at any rate necessary to develop the formulation of the dynamical equations in terms of potentials rather than in terms of the fields $b(X)$ and $f_\alpha(X)$ which, from the perspective of the usual space-time, are interpreted as generating functions to the field strengths (like Maxwell field strength, Weyl tensor and their further higher spin generalizations). Introducing the generalized gauge field (to contain spin one potential, spin 2 metric tensor etc) will presumably break down the generalized conformal symmetry transformations $Sp\,(2M)$ of the equations (21) and (22) to a smaller Poincare or AdS-type symmetry.

The generalized AdS-like space-time with $\frac{1}{2}M(M+1)$ coordinates was identified in Ref. protect1 with the group manifold $Sp\,(M)$. The AdS-type symmetry algebra associated with the left and right actions of $Sp\,(M)$ on itself is $sp(M) \oplus sp(M)$. Its Lorentz subalgebra $sp^l(M)$ identifies with the diagonal $sp(M)$ while AdS translations belong to the coset space $sp(M) \oplus sp(M)/sp^l(M)$. The conformal symmetries extend $sp(M) \oplus sp(M)$ to $sp(2M)$. For $M = 2$ one recovers the usual $3d$ embedding

$$o(2,2) \sim sp(2) \oplus sp(2) \subset sp(4) \sim o(3,2)\,.$$

The commutation relations of the generalized AdS space-time symmetries are

$$\frac{1}{i}[L_{\alpha\beta}\,, L_{\gamma\delta}] = V_{\beta\gamma}L_{\alpha\delta} + V_{\alpha\gamma}L_{\beta\delta} + V_{\beta\delta}L_{\alpha\gamma} + V_{\alpha\delta}L_{\beta\gamma}\,, \tag{199}$$

$$\frac{1}{i}[L_{\alpha\beta}\,, P_{\gamma\delta}] = V_{\beta\gamma}P_{\alpha\delta} + V_{\alpha\gamma}P_{\beta\delta} + V_{\beta\delta}P_{\alpha\gamma} + V_{\alpha\delta}P_{\beta\gamma}\,, \tag{200}$$

$$\frac{1}{i}[P_{\alpha\beta}\,, P_{\gamma\delta}] = \lambda^2\Big(V_{\beta\gamma}L_{\alpha\delta} + V_{\alpha\gamma}L_{\beta\delta} + V_{\beta\delta}L_{\alpha\gamma} + V_{\alpha\delta}L_{\beta\gamma}\Big)\,, \tag{201}$$

where $V_{\alpha\beta} = -V_{\beta\alpha}$ is a $Sp\,(M)$ invariant symplectic form and λ^2 is a generalized cosmological constant parameter. Note that, in the generalized space-times, the Lorentz-type subalgebra sl_M of the conformal algebra is larger than the Lorentz subalgebra $sp(M)$ of the generalized AdS or Poincare algebras.

The generalized Poincare symmetry results from the limit $\lambda \to 0$ and consists of translations that shift the coordinates as

$$X^{\alpha\beta} \to X'^{\alpha\beta} = X^{\alpha\beta} + a^{\alpha\beta}\,, \tag{202}$$

and $Sp\,(M)$ "Lorentz rotations" that leave invariant the antisymmetric invariant form $V^{\alpha\beta}$. The full list of $\frac{1}{2}M$ independent invariants under the generalized Poincare transformations consists of

$$\chi_n = tr\,X^{2n}\,, \qquad n = 1\ldots\frac{1}{2}M\,, \tag{203}$$

where the matrix $X_\alpha{}^\beta$ is defined with the aid of the symplectic form $V_{\alpha\beta}$

$$X_\alpha{}^\beta = V_{\gamma\alpha}X^{\gamma\beta}\,. \tag{204}$$

(Note that traces of odd powers of $X_\alpha{}^\beta$ vanish by antisymmetry of the symplectic form $V_{\alpha\beta}$.) In particular, χ_1 is bilinear in the coordinates X and can be identified with the Lorentz-like interval of the generalized space-time.

A new point compared to the usual Minkowski geometry is that the metric tensor in the generalized space-time is not an independent object being built from the symplectic form $V_{\alpha\beta}$

$$\eta^{\alpha\beta,\gamma\delta} = \frac{1}{2}\Big(V^{\alpha\gamma}V^{\beta\delta} + V^{\alpha\delta}V^{\beta\gamma}\Big)\,. \tag{205}$$

This metric tensor allows one to single out generalized Poincare invariant Klein-Gordon and Dirac equations from (21) and (22)

$$\eta^{\alpha\beta,\gamma\delta}\frac{\partial^2}{\partial X^{\alpha\beta}\partial X^{\gamma\delta}}b(X) = 0\,, \tag{206}$$

$$\eta^{\alpha\beta,\gamma\delta}\frac{\partial}{\partial X^{\alpha\beta}}f_\gamma(X) = 0\,. \tag{207}$$

The special form of the metric (205) may affect the concepts of general relativity in the generalized space-time: the corresponding Riemannian geometry is expected to be very restricted. Perhaps, this fact may be important for the search of the $M-$interactions in the conformal theories in higher dimensions.[20] Note that although being based on a symplectic form $V^{\alpha\beta}$, a generalized curved geometry is expected to have little to do with the usual symplectic geometry because the space-time coordinates are symmetric tensors rather than vectors.

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