This article is dedicated to the memory of Michael S. Marinov,
a great scientist and a close friend

GAUGE THEORIES ON NONCOMMUTATIVE SPACES

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A formalism is presented where gauge theories for nonabelian groups can be constructed on a noncommutative algebra.

Contents

1 Introduction

One of our basic assumptions is that the laws of physics are based on the space time structure of $\mathbb{R}^n$. The coordinates form a differential manifold on which gauge theories are built. These gauge theories describe all known physics from the theory of gravity to the standard model of elementary particle physics. All known experiments support this assumption, nevertheless we are obliged to ask if this will still be true at ever higher energies. This is motivated by the fact that the rules of local quantum field theory that make gauge theories in particle physics so successful are not successful for the theory of gravity. Quantum gravity is plagued by too many infinities, which can be traced to the short distance behaviour of the theory.

The question arises if at very short distances the theory can be changed, possibly by changing our notion of space-time. We recall that the structure of a differential manifold implies strong information on the topology of the space. Points and their neighbourhoods are defined as well as the distance between points. Is this really the true structure of space-time at very short distances? Or do we have to replace the concept of a differential manifold by weaker assumptions? A very weak assumption is that the coordinates form an algebra[1] and that the rules of this algebra, addition and multiplication, are the only relevant information to start with. Thus in the chain coordinates – differential manifold – gauge theory we could try to replace differential manifold by algebra and see how far we can develop such a theory.

This idea is not new.[2,3] It was already expressed in a letter by Heisenberg to Ehrenfest in the year 1930. There noncommutative coordinates that lead to an uncertainty relation for the coordinates themselves were proposed. But Heisenberg could not formulate these ideas mathematically. The idea, however, propagated and led to a publication by Hartland S. Snyder in the year 1947. In this paper Snyder discussed quantized space time for a model with a Lie algebra structure for the coordinates. In a preceding letter to Pauli (1946) Snyder discussed the interpretation of a theory with noncommutative space-time in a fully satisfactory way. This discussion was based on the rules of quantum mechanics, treating the selfadjoint operators for space and time as physical observables. In a letter to Bohr (1947), Pauli mentioned the work of Snyder as a mathematically ingeneous proposal which, however, seemed to Pauli a failure for reasons of physics.

In the meantime our mathematical skill, also concerning algebraic properties, has seen a strong development, the same is true for the art of designing new experiments at ever higher energies. It seems to be natural to try anew the noncommutative coordinates approach and to see if it still meets Pauli's objections. Finally, experiment will have to decide, but at any energy reached at new accellerators the question on the causal structure of space-time has to be asked again, noncommutative space-time only being a particular version of a modification.

In this article I try to show that a gauge theory based on an algebra can be constructed and that definite physical consequences can be derived and tested experimentally. In this construction, coordinates more or less lose their meaning as physical observables and only play a role as parameters in the formulaton of the theory.

2 The algebra

Let me first exhibit the algebraic structure of $\mathbb{R}^n$ and then generalise to noncommutative coordinates. The coordinates $x^1 \dots x^n \in \mathbb{R}^n$ are considered as elements of an associative algebra over $\mathbb{C}$. The algebra, freely generated by these elements, will be denoted by $\mathbb{C}\left[[x^1, \dots, x^n]\right]$. The two brackets indicate that formal power series are allowed in the algebra.

The elements of this algebra are then subject to relations that make them commutative:

$$\mathcal{R}: \quad x^i x^j - x^j x^i = 0 \ . \tag{1}$$

These relations generate a two-sided ideal I_R; it consists of all the elements of the algebra $\mathbb{C}\left[[x^1, \dots, x^n]\right]$ that can be obtained from the relation (1) by multiplying (1) from the left and the right by all possible products of the coordinates. We factor out this ideal and obtain the desired algebra:

$$\mathcal{A}_x = \frac{\mathbb{C}\left[[x^1, \dots, x^n]\right]}{I_{\mathcal{R}}} \ . \tag{2}$$

The elements of this algebra are the polynomials and the formal power series in the commuting variables $x^1, \dots, x^n \in \mathbb{R}$,

$$\begin{aligned} f(x^1, \dots, x^n) &\in \mathcal{A}_x \ , \\ f(x^1, \dots, x^n) &= \sum_{r_i=0}^{\infty} f_{r_1 \dots r_n} (x^1)^{r_1} \cdot \dots \cdot (x^n)^{r_n} \ . \end{aligned} \tag{3}$$

Multiplication in this algebra is the pointwise multiplication of these functions.

This algebraic concept can be easily generalized to noncommutative coordinates. We consider algebras, freely generated by elements $\hat{x}^1, \dots \hat{x}^n$, again we call these elements coordinates, but now they are supposed to satisfy relations that make them noncommutative:

$$\mathcal{R}_{\hat{x},\hat{x}}: \quad [\hat{x}^i, \hat{x}^j] = i\theta^{ij}(\hat{x}) \ . \tag{4}$$

Following L. Landau, noncommutativity carries a hat. Again the relations (4) generate an ideal and we define our algebra $\hat{\mathcal{A}}_{\hat{x}}$ as follows:

$$\mathcal{A}_{\hat{x}} = \frac{\mathbb{C}[[\hat{x^1}, \ldots, \hat{x^n}]]}{I_{\mathcal{R}_{\hat{x},\hat{x}}}} , \qquad \hat{f} \in \quad \hat{\mathcal{A}}_{\hat{x}} \, . \tag{5}$$

We impose one more condition on the algebra. The vectorspace of the homogeneous polynomials of degree m, $\hat{V}^m_{\hat{x}}$ should have the same dimension as V^m_x. Algebras of this type are said to have the Poincare-Birkhoff-Witt property. In the following we shall consider such algebras only.

3 The $*$ product

The vectorspaces V^m_x and $\hat{V}^m_{\hat{x}}$ are finite-dimensional, thus they are isomorphic. To establish an isomorphism we map a given basis of one space into a given basis of the other space. This then defines a vectorspace isomorphism between the vectorspaces $\hat{V}_{\hat{x}}$ and V_x .

We now change the algebra $\mathcal{A}_x$ to extend the above vector space isomorphism to an algebra isomorphism. For this purpose we have to change the multiplication law in $\mathcal{A}_x$. When we multiply two elements in $\hat{\mathcal{A}}_{\hat{x}}$ we can compute from the multiplication law in $\hat{\mathcal{A}}_{\hat{x}}$ the coefficient function of the product in a given basis. We define the product in the vectorspace V_x to be the element with the same coefficient function as it was calculated in $\hat{\mathcal{A}}_{\hat{x}}$. This multiplication rule we call $*$ (star) product and this defines the algebra $*\mathcal{A}_x$. The algebras $\hat{\mathcal{A}}_{\hat{x}}$ and $*\mathcal{A}_x$ are isomorphic.

It is natural to use the elements of $*\mathcal{A}_x$ as objects in physics. The pointwise product has to be replaced by the $*$ product. In all the cases of interest the $*$ product can be expressed with the help of a differential operator. This makes it possible to extend the $*$ product to functions without referring to power series expansion. Thus we treat the elements $\mathcal{A}_x$ like ordinary fields but replace the pointwise product by the $*$ product. This would be the starting point of deformation quantization. As we have based the concept on associative algebras, associativity of the $*$ product is guaranteed.

4 Gauge theory

In this context it is possible to formulate a gauge theory.[4] We start from a Lie algebra:

$$[T^a, T^b] = i f^{ab}_c T^c \, . \tag{6}$$

In a usual gauge theory on commutative spaces the fields will span a representation of this Lie algebra and they will transform under the usual gauge transformation with Lie algebra valued parameters:

$$\delta_{\alpha^0}\psi(x) = i\alpha^0(x)\psi(x) \; . \tag{7}$$

The commutator of two such transformations remains Lie algebra valued.

$$\begin{aligned}&(\delta_{\alpha^0}\delta_{\beta^0} - \delta_{\beta^0}\delta_{\alpha^0})\psi = -(\beta^0\alpha^0 - \alpha^0\beta^0)\psi = i(\alpha^0 \times \beta^0)\psi = \delta_{\alpha^0\times\beta^0}\psi \; , \\ &\qquad \alpha^0 \times \beta^0 \equiv \alpha^0_a \beta^0_b f^{ab}_c T^c.\end{aligned} \tag{8}$$

For a theory on non-commutative spaces we start with fields that are elements of $*\mathcal{A}_x$. Gauge transformations have to be defined with the $*$ product:

$$\delta_\alpha \psi(x) = i\alpha(x) * \psi(x) \tag{9}$$

The star product of functions is not commutative. The commutator of two Lie algebra valued transformations does not reproduce a Lie algebra valued parameter. Thus we shall assume that the infinitesimal transformation parameters are enveloping algebra valued:[1]

$$\alpha(x) = \alpha^0_a(x)T^a + \alpha^1_{ab}(x) : T^aT^b : + \cdots + \alpha^{n-1}_{a_1\ldots a_n}(x) : T^{a_1}\cdots T^{a_n} : + \cdots \tag{10}$$

We have adopted the :: notation for a basis in the enveloping algebra of the Lie algebra. Completely symmetrized products could serve as a basis:

$$: T^a := T^a \, , \quad : T^aT^b := \frac{1}{2}(T^aT^b + T^bT^a) \, , \qquad \text{etc.} \tag{11}$$

The commutator of two transformations is certainly enveloping algebra valued.

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi = [\alpha \stackrel{*}{,} \beta] * \psi \; . \tag{12}$$

The disadvantage of this approach is that infinitely many parameters $\alpha^n(x)$ have to be introduced.

It is a surprise that it is possible to define gauge transformations where all the parameters $\alpha^n(x)$ depend on the finite set of parameters $\alpha^0(x)$ (Lie algebra valued) and in addition on the gauge potential $a(x)$ of a usual gauge theory and on their derivatives. The gauge potential $a(x)$ has the usual transformation properties:

$$\begin{aligned}\delta a_i &= \partial_i \alpha^0 + i[\alpha^0, a_i] \; , \\ \delta a_{i,a} &= \partial_i \alpha^0_a - \alpha^0_b f^{bc}_a a_{i,c} \; .\end{aligned} \tag{13}$$

We will call the new type of transformation parameters $\Lambda_{\alpha^0}(x)$. The new transformations are supposed to close under a commutator into a transformation characterized by $(\alpha^0 \times \beta^0)$,

$$\begin{aligned} \delta_{\alpha^0}\psi(x) &= i\Lambda_{\alpha^0(x)}(x) * \psi(x)\,, \\ (\delta_{\alpha^0}\delta_{\beta^0} - \delta_{\beta^0}\delta_{\alpha^0})\psi &= \delta_{\alpha^0\times\beta^0}\psi\,, \\ (\alpha^0 \times \beta^0)_a &= \alpha^0_b \beta^0_c f_a^{bc}\,. \end{aligned} \tag{14}$$

These equations define $\Lambda_{\alpha^0}(x)$. We shall see that all $\alpha_n(x)$ in (10) can be defined in terms of $\alpha^0(x)$ and the gauge potential $a(x)$. The transformation property (14) then holds as a consequence of (13). The solution of this problem, however, is not unique, this will be seen in the following.

As a consequence of the a dependence of Λ^0_α we have to transform Λ^0_α under the second variation in the commutator. This changes equation (12) and this is the reason why the new approach works.

5 Constant θ

To illustrate this approach we restrict it to the algebra where $\theta^{\mu\nu}$ is a constant. In this case we obtain in a fully symmetrized basis the following $*$ product:[5]

$$\begin{aligned} (f * g)(x) &= e^{\frac{i}{2}\frac{\partial}{\partial x^i}\theta^{ij}\frac{\partial}{\partial y^j}} f(x)g(y)\Big|_{y\Rightarrow x} \\ &= \int d^n y\ \delta^n(x-y) e^{\frac{i}{2}\frac{\partial}{\partial x^i}\theta^{ij}\frac{\partial}{\partial y^j}} f(x)g(y). \end{aligned} \tag{15}$$

We expand in θ,

$$\Lambda_{\alpha^0} = \alpha^0_a T^a + \theta^{ij}\Lambda^1_{\alpha^0,ij} + \cdots\,. \tag{16}$$

The $*$ product has to be expanded as well. Finally we expand the defining equation for Λ_{α^0},

$$\begin{aligned} (\delta_{\alpha^0}\delta_{\beta^0} - \delta_{\beta^0}\delta_{\alpha^0})\psi &= i(\delta_{\alpha^0}\Lambda_{\beta^0} - \delta_{\beta^0}\Lambda_{\alpha^0}) * \psi + [\Lambda_{\alpha^0} \overset{*}{,} \Lambda_{\beta_0}] * \psi\,, \\ &= \delta_{\alpha^0\times\beta^0}\psi = i\Lambda_{\alpha^0\times\beta^0} * \psi\,. \end{aligned} \tag{17}$$

The zeroeth order in the expansion of (17) defines α^0 as Lie algebra valued.

In first order we obtain

$$\begin{aligned} \theta^{ij}\big((\delta_{\alpha^0}\Lambda^1_{\beta^0,ij} &- \delta_{\beta^0}\Lambda^1_{\alpha^0,ij}) - i([\alpha^0, \Lambda^1_{\beta^0,ij}] \\ &- [\beta^0, \Lambda^1_{\alpha^0,ij}])\big) + \frac{1}{2}\partial_i\alpha^0_a\partial_j\beta^0_b : T^a T^b := \theta^{ij}\Lambda^1_{\alpha^0\times\beta^0,ij}\,. \end{aligned} \tag{18}$$

A closer look shows that this is an inhomogeneous linear equation for Λ^1. The inhomogeneous term is known, it contains α^0 and β^0 only. A particular solution of (18) is

$$\theta^{ij}\Lambda^1_{\alpha^0,ij} = \frac{1}{2}\theta^{ij}(\partial_i\alpha^0_a)a_{j,b} : T^aT^b : \; . \tag{19}$$

Any solution of the homogeneous part of (18) can be added to (19).

We can proceed order by order in θ, the structure of the equations will always be the same. It will be an inhomogeneous linear equation, the homogeneous remains the same, the inhomogeneous part will contain known quantities only. This way we obtain Λ_{α^0} in a θ expansion,

$$\Lambda_{\alpha^0} = \alpha^0_a T^a + \frac{1}{2}\theta^{ij}(\partial_i\alpha^0_a)a_{j,b} : T^aT^b : + \cdots \; . \tag{20}$$

Such a construction of the transformation parameter first occured in the context of the Seiberg-Witten map.[6]

6 Covariant coordinates

In a usual gauge theory we would procede with the definition of covariant derivatives. Derivatives, however, are not a natural concept for algebras. It is more natural to introduce covariant coordinates. Based on such a concept gauge theories can be developed as well.

It is obvious that coordinates do not commute with gauge transformations, it is also natural to introduce covariant coordinates in analogy to covariant derivatives:

$$\begin{aligned} X^i &= x^i + A^i(x)\,, \qquad (21)\\ \delta_{\alpha^0} X^i * \psi &= i\Lambda_{\alpha^0} * X^i * \psi\,. \end{aligned}$$

This leads to the following transformation law for the gauge potential:

$$\delta A^i = -i[x^i \stackrel{*}{,} \Lambda_{\alpha^0}] + i[\Lambda_{\alpha^0} \stackrel{*}{,} A^i]\,. \tag{22}$$

To satisfy such a transformation law we have again to assume that $A(x)$ is enveloping algebra valued. In general, this would imply infinitely many gauge fields. For the restricted gauge transformations Λ_{α^0} it is possible to construct a gauge potential that depends on the Lie algebra valued potential $a(x)$ and its derivatives only. The transformation law (13) for $a(x)$ will imply the transformation law for $A(x)$. This is the main achievement of the Seiberg-Witten map, see Eq. (28).

The construction of gauge fields that transform to tensorial follows the usual concept as we know it from covariant derivatives. An obvious definition is

$$X^\mu X^\nu - X^\nu X^\mu - i\theta^{\mu\nu}(X) = \hat{F}^{\mu\nu} \,. \tag{23}$$

It is chosen in such a way that $\tilde{F}^{\mu\nu}$ vanishes for a vanishing gauge potential A^μ.

The tensorial transformation law of $\tilde{F}^{\mu\nu}$ follows directly from (22),

$$\delta_\alpha \tilde{F}^{\mu\nu} = [\Delta_{\alpha 0} * \tilde{F}^{\mu\nu}] \,. \tag{24}$$

It should be noted, however, that the trace in the representation space of the Lie algebra of a tensor is not an invariant because the star product is not commutative.

7 The integral

An invariant action can be constructed only if the integral has its trace property,

$$\int f * g = \int g * f \,. \tag{25}$$

Integration is not a natural concept in an algebra. It is supposed to be a linear map from $\mathcal{A}_{\hat{x}}$ into $\mathbb{C}$,

$$\int : \quad \mathcal{A}_{\hat{x}} \to \mathbb{C} \,, \qquad \int (c_1 \hat{f} + c_2 \hat{g}) = c_1 \int \hat{f} + c_2 \int \hat{g} \,. \tag{26}$$

In addition the trace property is required:

$$\int \hat{f}\hat{g} = \int \hat{g}\hat{f}. \tag{27}$$

This is equivalent to (25).

8 Gauge theory for constant θ

For constant θ the usual integral in x-space will have the trace property. This can be shown by a direct calculation.

Let us have a look at this formalism for constant $\theta^{\mu\nu}$.
The Seiberg-Witten map:

$$\begin{aligned} A^i(x) &= \theta^{ij} V_j \,, \\ V_j(x) &= a_{j,a} T^a - \frac{1}{2}\,\theta^{ln} a_{l,a} (\partial_n a_{j,b} + F_{nj,b} : T^a T^b : + \cdots \,, \\ F_{nj,b} &= \partial_n a_{j,b} - \partial_j a_{n,b} + f_b^{cd} a_{n,c} a_{j,d} \,. \end{aligned} \tag{28}$$

The field strength:

$$\tilde{F}_{ij} = F_{ij,a}T^a + \theta^{ln}(F_{il,a}F_{jn,l} - \frac{1}{2}a_{l,a}(2\partial_n F_{ij,b} + a_{n,c}F_{ij,d}f_e^{cd})) : T^aT^b : + \cdots . \tag{29}$$

The Lagrangian:

$$L = \frac{1}{4}\mathrm{Tr}\, F_{ij} * F^{ij} . \tag{30}$$

The invariant action:

$$W = \frac{1}{4}\int \mathrm{Tr} F_{ij} * F^{ij} = \frac{1}{4}\int \mathrm{Tr} F_{ij} F^{ij} .$$

New coupling terms arise, $\theta^{\mu\nu}$ appears as a coupling constant, it is a Lorentz tensor and the interaction term breaks Lorentz invariance. This was to be expected because the defining relation (1) already breaks Lorentz invariance.

These new terms in the Lagrangian will give rise to new interactions. Due to the breaking of Lorentz invariance interaction terms will occur that are forbidden in a Lorentz invariant theory. A good example is the $Z^0 \to \gamma\gamma$ decay. From (31) we find the following interaction terms that contribute to this decay if the gauge theory is based on the standard model:[7]

$$\begin{aligned} \mathcal{L}_{Z_{\gamma\gamma}} &= \frac{g'}{\sqrt{(g^2+g'^2)^3}} \left(\alpha g^2 + \beta(g'^2 - 2g^2)\right) \theta^{kl} \\ &\times \Big(2(-\partial_i Z_k + \partial_k Z_i)\partial_j A_l(\partial^i A^j - \partial^j A^i) \\ &+ (\partial_i A_k \partial_j A_l + \partial_k A_i \partial_l A_j - 2\partial_k A_i \partial_j A_l)(-\partial^i Z^j + \partial^j Z^i) \\ &+ (-2\partial_k Z_i \partial_l A_j + 2\partial_j Z_l \partial_k A_i + 2\partial_i Z_j \partial_k A_l + \partial_k Z_l \partial_i A_j)(\partial^i A^j - \partial^j A^i)\Big) . \end{aligned} \tag{31}$$

This expression is gauge invariant under the usual Lie algebra valued gauge transformation. It contributes to the branching ratio of the Z^0 decay.

We still have to learn how the gauge potential couples to the matter fields. This will be done via covariant derivatives,

$$\begin{aligned} \mathcal{D}_i * \psi &= (\partial_i - iV_i) * \psi , \\ \delta_{\alpha^0}\mathcal{D}_i * \psi &= i\Lambda_{\alpha^0} * \mathcal{D}_i * \psi . \end{aligned} \tag{32}$$

9 Derivatives

First we have to define derivatives. In general, the star product will depend on the coordinates, when we differentiate it the coordinate dependence of the

$*$ product will contribute as well. Nevertheless, we demand a Leibniz rule of the type

$$\partial_\mu * (f * g) = (\partial_\mu f) * g + \mathcal{O}_\mu^\nu(f) * \partial_\nu g\,. \tag{33}$$

From the associativity of the $*$ product follows that $O_\mu^\nu(f)$ has to be an algebra homomorphism.

It is easier to define derivatives for $\hat{\mathcal{A}}_{\hat{x}}$. A general procedure was outlined in Ref. 8. We first extend the algebra by algebraic elements $\hat{\partial}$ and consider the algebra $\mathbb{C}\left[[\hat{x}^1,\ldots,\hat{x}^n,\ldots,\hat{\partial}^1,\ldots,\hat{\partial}^n]\right]$. This algebra has to be divided by the ideal $I_{\hat{x},\hat{x}}$ as before. Then we have to construct a derivative, based on a Leibniz rule that is a map in $\mathbb{C}\left[[\hat{x}^1,\ldots,\hat{x}^n,\ldots,\hat{\partial}^1,\ldots,\hat{\partial}^n]\right]/I_{\hat{x},\hat{x}}$. This leads to consistency relations for the Leibniz rule The Leibniz rule can now be interpreted as a relation and the respective ideals can be constructed and factored out. Finally this has to be supplemented by $\hat{\partial},\hat{\partial}$ relations. We treat these relations as usual and after dividing by the respective ideal we arrive at an algebra that we call $\hat{\mathcal{A}}_{\hat{x},\hat{\partial}}$.

In more detail the generalized Leibniz rule is supposed to have the form:

$$\hat{\partial}_i(\hat{f}\hat{g}) = (\hat{\partial}_i\hat{f})\hat{g} + O_i^l(\hat{f})\hat{\partial}_l\hat{g}\,. \tag{34}$$

From the law of associativity in $\hat{\mathcal{A}}_{\hat{x}}$ follows that the map 0 has to be an algebra homomorphism

$$O_j^i(\hat{f}\hat{g}) = O_l^i(\hat{f})O_j^l(\hat{g})\,. \tag{35}$$

If we define the Leibniz rule on the linear coordinates we can generalize it to all elements.

In the ${}^*\!\mathcal{A}_x$ version of the algebra the Leibniz rule takes the form (34). This rule can be found as follows: $\hat{\partial}$ introduces a map on the basis of $\hat{\mathcal{A}}_{\hat{x}}$, this map defines a map in ${}^*\!\mathcal{A}_x$. This map has finally to be expressed with ordinary x-derivatives. This then leads to (34).

For constant $\theta^{\mu\nu}$ where the $*$ product does not depend on x the $\partial *$ derivatives are just the ordinary x-derivatives.

Covariant derivatives are then defined as usual:

$$\begin{aligned} \mathcal{D}_i * \psi &= (\partial_i - iV_i) * \psi\,, \\ \delta_{\alpha^0}\mathcal{D}_i * \psi &= i\Lambda_{\alpha^0} * \mathcal{D}_i * \psi\,. \end{aligned} \tag{36}$$

The vector potential has to be enveloping algebra valued. Again, it can be expressed in terms of a_μ by a Seiberg-Witten map. Therefore we expect that A_μ and V_μ are related.

For constant $\theta^{\mu\nu}$ we find

$$A^i(x) = \theta^{ij} V_j, . \tag{37}$$

Covariant derivatives exist for $\theta^{\mu\nu} = 0$. From (37) follows that A^μ vanishes in this case, coordinates are already covariant.

10 Gauge couplings to matter fields

The matter field ψ that transforms like

$$\delta_{\alpha^0}\psi(x) = i\Lambda_{\alpha^0(x)}(x) * \psi(x) \tag{38}$$

can be expressed in terms of a field ψ^0 that transforms with a Lie algebra valued parameter and the Lie algebra valued vector potential a. The transformation property (38) will be a consequence of (7) and (13).

For constant θ we find

$$\psi = \psi^0 - \frac{1}{2}\,\theta^{\mu\nu} a^l_\mu T^l \partial_\nu \psi^0 + \dots\ . \tag{39}$$

This now leads to the Lagrangian

$$\begin{aligned}\int \bar{\psi} * (\gamma^\mu D_\mu * - m)\,\psi d^4x &= \int \bar{\psi}^0\,(\gamma^\mu D_\mu - m)\,\psi^0 d^4x \qquad (40)\\ -\frac{1}{4}\,\theta^{\mu\lambda}\int \bar{\psi}^0 F^0_{\mu\lambda}\,(\gamma^\mu D_\mu - m)\,\psi^0 d^4x &- \frac{1}{4}\,\theta^{\sigma\lambda}\int \bar{\psi}^0\gamma^\mu F^0_{\mu\sigma} D_\lambda \psi^0 d^4x + \cdots\ .\end{aligned}$$

The fields ψ^0 and $F^{\mu\nu 0}$ transform like the usual gauge fields with a Lie algebra valued parameter, $F^{\mu\nu 0}$ is just the usual field strength of a gauge theory. Accordingly, $D_\mu\psi^0$ is the usual covariant derivative with the field a_μ as a gauge potential.

11 Conclusion

Such a theory based on noncommutative coordinates should only be relevant for a region with very high energy density, thus for very short distances, i.e. well inside the confinement range. For larger distances we know that physics is described very well with commuting coordinates. $\theta^{\mu\nu}(x)$ will be a complicated function, we treat this function in a power series expansion and start with constant $\theta^{\mu\nu}$. This has a chance to be relevant for processes that take place at very short distances where the constant $\theta^{\mu\nu}$ might be dominant. The higher order contribution on the expansion become relevant at distances where the process has already occured. Such a process will not be sensitive to the functional behaviour of $\theta^{\mu\nu}(x)$ and the constant $\theta^{\mu\nu}$ approximation might be a good approximation.[7] To find such a process demands physical intuition.

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