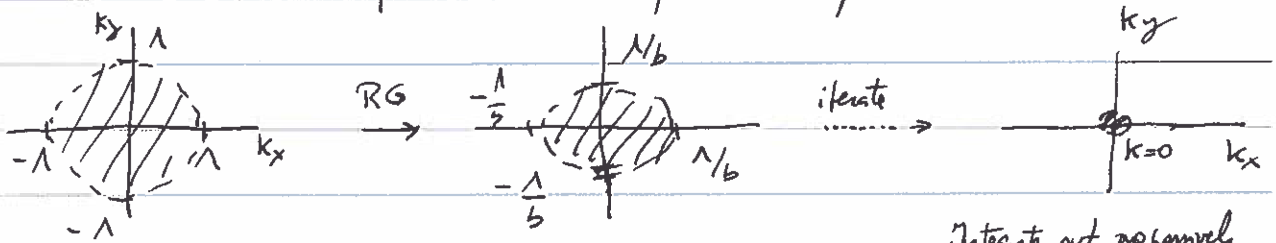
ϵ expansion - approximate.-

Imagine writing the order parameter $\psi(x) = \int_{|k|<\Lambda} \frac{d^d k}{(2\pi)^d} e^{ik\cdot x} \hat{\psi}_k$

where the cut-off is $\Lambda = \frac{\pi}{a}$, a large number.

The renormalization transformation entails partial integration:



Scale transformation by a factor b

Integrate out progressively so that after iteration only the modes in the vicinity of |k|=0 remain.

In order to accomplish this, one formally writes.

$$\psi(x) = \bar{\psi}(x) + \delta\psi(x)$$

$\psi(x)$: all modes $|k| < \Lambda$ included

$\bar{\psi}(x)$: degrees of freedom for $|k| < \Lambda/b$

$\delta\psi(x)$: degrees of freedom from "fast" modes $\Lambda/b < |k| < \Lambda$

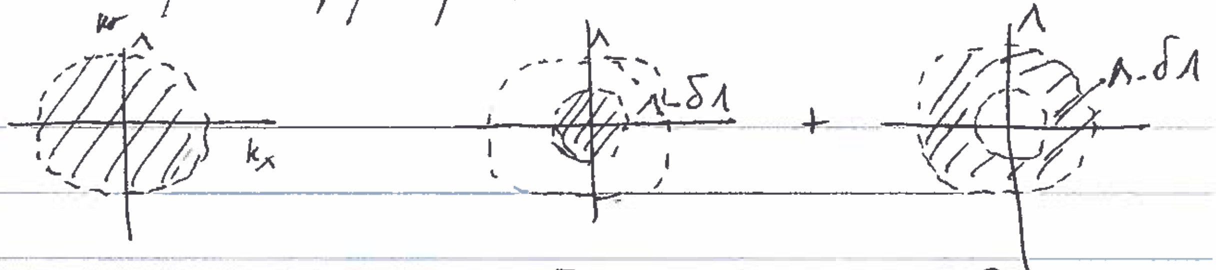
In other words:

$$\bar{\psi}(x) = \int_{|k|<\Lambda-\delta\Lambda} \frac{d^d k}{(2\pi)^d} \hat{\psi}_k e^{ik\cdot x}$$

$$\delta\psi(x) = \int_{\Lambda-\delta\Lambda<|k|<\Lambda} \frac{d^d k}{(2\pi)^d} \hat{\psi}_k e^{ik\cdot x}$$

where $\Lambda - \delta\Lambda = \frac{\Lambda}{b}$, b the scale factor. $\left(b = \frac{\Lambda}{\Lambda-\delta\Lambda} \geq 1\right)$.

this can be represented graphically as:



$$\psi(x) \quad = \quad \bar{\psi}(x) \quad + \quad \delta\psi(x)$$

and we define a transformation that involves partial integration over $\delta\psi(x)$ while keeping the slightly more coarse grained $\bar{\psi}(x)$ fixed. We start from

$$E\{\psi\} = \int d^dx \left\{ \frac{1}{2}|\nabla\psi|^2 + \frac{r_0}{2}\psi^2 + u_0\psi^4 \right\} =$$

$$= \int d^dx \left\{ \frac{1}{2}|\nabla(\bar{\psi}+\delta\psi)|^2 + \frac{r_0}{2}(\bar{\psi}+\delta\psi)^2 + u_0(\bar{\psi}+\delta\psi)^4 \right\}$$

We now proceed to integrate the terms containing $\delta\psi$. We expand:

N: # points in the ~~[illegible]~~ ~~[illegible]~~ shell.

$$\delta\psi(x) = \sum_{i=1}^{N} c_i\varphi_i(x)$$

φ_i have short characteristic wavelengths order $\sim 1/\Lambda$, inside the circle outlined above.

x in the entire system. Shell only in k.

We will assume:

- $\bar{\psi}(x)$ is constant when integrating over $\delta\psi$ in the range $\Lambda-\delta\Lambda<|k|<\Lambda$
- $\int dx\, |\nabla\varphi_i|^2 \approx \Lambda^2$
- Neglect $(\delta\psi)^4$ and higher order.

Also, one takes:

- $\int dx\, (\phi_i)^2 = 1$ $\int dx\, (\phi_i)^n = 0$ if $n =$ odd.
(not to change the symmetry of E after integration).

$\int dx\, \phi_i(x)\,\phi_j(x) \simeq 0$ $(i\neq j)$ orthogonality of ϕ.

OTE: some sloppiness in what integrals are done or left, and volume elements.

- $$\int d^dx \, |\nabla(\bar{\psi}+\delta\psi)|^2 = \int d^dx \left\{ |\nabla\bar{\psi}|^2 + |\nabla\delta\psi|^2 + 2\nabla\bar{\psi}(x) \cdot \nabla\delta\psi(x) \right\}$$

$$\simeq \int d^dx \, |\nabla\bar{\psi}|^2 + \underset{\uparrow}{\underline{\underline{\Lambda}}}^2 \sum_{i=1}^{N} c_i^2 + 0$$

integral over fast

as the last term is odd in both variables

- $$\int d^dx \, (\bar{\psi}+\delta\psi)^2 = \int d^dx \left\{ \bar{\psi}^2(x) + \left(\sum c_i\phi_i(x)\right)^2 + \underline{2\bar{\psi}(x)\sum_i c_i\phi_i(x)} \right\}$$

odd terms again integrate to zero.

$$= \int d^dx \left\{ \bar{\psi}^2(x) \right\} + \sum c_i^2 \}$$

use the normalization of ϕ_i and the approximate orthogonality after one integrates over the fast modes.

We have assumed $\delta\psi = \sum c_i\phi_i$ and normalized $\int d^dx \, |\phi_i|^2 = 1$

$\Rightarrow$ [illegible] $[\phi_i] \sim L^{-d/2}$

This is an unusual dimensionalization and $\{c_i\}$ would be dimensionless.

as it forces $[\delta\psi] \sim [\phi_i] \sim L^{-d/2}$ as well (instead of the normal L^{-d})

Also volume elements of integration over fast variables $(\delta\Lambda)^d$.

- Quadratic term $[r\psi^2][d^dx] = [r]\frac{1}{L^d}L^d \Rightarrow [r] = 1$ dimensionless (units of energy).
- Quartic $[u\psi^4][d^dx] = [u]\frac{1}{L^{2d}}L^d \rightarrow [u] \sim L^d$
- Gradient term: $[\underset{(=\frac{1}{2}\text{ in our units})}{K} |\nabla\psi|^2][d^dx] = [K]\frac{1}{L^2}\frac{1}{L^d}L^d \rightarrow [K] \sim L^2$

not sure what to do with this and units of Λ.

$$\cdot \int d^dx \left(\bar{\varphi}(x) + \delta\varphi\right)^4 = \int d^dx \Big\{ \bar{\varphi}^4 + \cancel{4\bar{\varphi}^3 \delta\varphi} + \cancel{4\bar{\varphi}\,\delta\varphi^3} + 6\bar{\varphi}^2 \delta\varphi^2 \Big\} = \int d^dx \Big\{ \bar{\varphi}^4 + 6\bar{\varphi}^2 \Big(\sum c_i \phi_i\Big)^2 \Big\}$$

remove as they are odd

$$\simeq \int d^dx\, \bar{\varphi}^4 + 6 \sum_i \bar{\varphi}^2(x_i)\, c_i^2$$

point in the support of φ_i

Made all $\phi_i \phi_j$ products zero, and use ϕ_i just to project $\bar{\varphi}(x)$ onto the slow variation $\bar{\varphi}$.

So the energy after partial integration reads:

$$E\{\varphi\} \simeq E\{\bar{\varphi}\} + \underbrace{\sum_{i=1}^{N} \left(\frac{1}{2}\left(\Lambda^2 + r_0\right) + 6u_0 \bar{\varphi}^2(x_i)\right) c_i^2}$$

contribution to the energy from the slowly varying modes.

NOTE : it is quadratic in c_i (and diagonal) so the remaining contribution to the partition function can be exactly integrated out.

We now define a transformed energy $E^{(1)}\{\bar{\varphi}\}$ through integration over the remains of the slow variables:

$$e^{-E^{(1)}\{\bar{\varphi}\}} = \prod_{i=1}^{N} \int_{-\infty}^{\infty} dc_i\, e^{-E\{\varphi\}} = e^{-E\{\bar{\varphi}\}} \prod_{i=1}^{N} \int_{-\infty}^{\infty} dc_i\, e^{-\frac{1}{2}\left(\Lambda^2 + r_0\right) + \overset{6u_0\bar{\varphi}^2}{[illegible]})}$$

$$= e^{-E\{\bar{\varphi}\}} \prod_{i=1}^{N} \sqrt{\frac{\pi}{2}} \left(\Lambda^2 + r_0 + 12 u_0 \bar{\varphi}^2\right)^{-1/2}$$

where we had taken $\bar{\varphi}$ as fixed.

[Algebra checked] (13.86) in Huang's book.

or: $$E^{(1)}\{\bar{\varphi}\} = E\{\bar{\varphi}\} + \frac{1}{2} \sum_{i=1}^{N} \ln \left[\left(\frac{2\Lambda^2}{\pi}\right)\left(1 + \frac{r_0}{\Lambda^2} + \frac{12u_0}{\Lambda^2} \bar{\varphi}^2(x_i)\right) \right]^{\vee}$$

$b = 1 + \delta$

$\ln b \simeq \delta$

$\delta\Lambda = \Lambda - \frac{\Lambda}{b}$

$= \Lambda\left(1 - \frac{1}{b}\right)$

$= \Lambda\left(1 - \frac{1}{1+\delta}\right) =$

$\simeq \Lambda\left(1 - (1-\delta)\right)$

$= \Lambda\delta = \Lambda \ln b$

$\frac{\delta\Lambda}{\Lambda} \simeq \ln b$

As in the Gaussian approximation, we rewrite the energy in the same form for $\bar{\psi}$ and absorb the effects of the transformation in the coefficients.

- We will first approximate the sum by an integral:

$$\sum_{i=1}^{N} = S^{d}(\Lambda)\,\delta\Lambda \cdot D(k) \text{ times the implied integration over the slow variable.}$$

← density of states $\frac{V}{(2\pi)^3}$ in 3d.

$$= C_d \Lambda^{d-1} \delta\Lambda \cdot \int d^d x$$

← for the volume.

a sphere of radius Λ, with the surface coefficient known:

$$= C_d \Lambda^{d-1} \Lambda \ln b \int d^d x = C_d \Lambda^{d} \ln b \int d^d x \qquad C_d = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}$$

We therefore rewrite:

$$E^{(1)}\{\bar{\psi}\} = E\{\bar{\psi}\} + \frac{1}{2} C_d \Lambda^{d} \ln b \int d^d x \left[\ln\left[\left(\frac{2\Lambda^2}{n}\right)\left(1 + \frac{r_0}{\Lambda^2} + \frac{12u_0}{\Lambda^2}\bar{\psi}^2\right)\right]\right]$$

In the vicinity of the critical point r_0 and $\bar{\psi}$ are small; also Λ is a large quantity. We use $\ln(1+x) \simeq x - \frac{1}{2}x^2 \ldots$ and keep terms up to second order in r_0 and u_0:

$$\ln\left(1 + \frac{r_0}{\Lambda^2} + \frac{12u_0}{\Lambda^2}\bar{\psi}^2\right) \simeq \frac{r_0}{\Lambda^2} + \frac{12u_0}{\Lambda^2}\bar{\psi}^2 - \frac{1}{2}\left(\frac{r_0}{\Lambda^2} + \frac{12u_0}{\Lambda^2}\bar{\psi}^2\right)^2 \simeq$$

$$\simeq \frac{r_0}{\Lambda^2} + \frac{12u_0}{\Lambda^2}\bar{\psi}^2 - \frac{1}{2}\left(\frac{r_0^2}{\Lambda^4} + \frac{144u_0^2\bar{\psi}^4}{\Lambda^4} + \frac{24r_0u_0}{\Lambda^4}\bar{\psi}^2\right)$$

$$= \frac{r_0}{\Lambda^2} - \frac{r_0^2}{2\Lambda^4} + \left(\frac{12u_0}{\Lambda^2} - \frac{12r_0u_0}{\Lambda^4}\right)\bar{\psi}^2 - \frac{72u_0^2\bar{\psi}^4}{\Lambda^4}$$

The first two terms are just a constant to the energy and can be ignored. One has:

$$\ln\left(1+\frac{r_0}{\Lambda^2}+\frac{12u_0}{\Lambda^2}\bar{\psi}^2\right)\simeq 12\left(\frac{u_0}{\Lambda^2}-\frac{r_0u_0}{\Lambda^4}\right)\bar{\psi}^2-\frac{72u_0^2}{\Lambda^4}\bar{\psi}^4$$

to linear or quadratic order in the coupling coefficients.

Therefore we write:

$$E^{(1)}\{\bar{\psi}\}=E\{\bar{\psi}\}+\frac{1}{2}C_d\Lambda^d\ln b\int d^dx\left\{12\left(\frac{u_0}{\Lambda^2}-\frac{r_0u_0}{\Lambda^4}\right)\bar{\psi}^2-\frac{72u_0^2}{\Lambda^4}\bar{\psi}^4\right\}$$

and where have also drop the additive constant $\frac{1}{2}C_d\Lambda^d\ln b\ln\left(\frac{2\Lambda^2}{\pi}\right)$. We then have:

Eq 8.99) in Huang.

$$E^{(1)}\{\bar{\psi}\}=E\{\bar{\psi}\}+C_d\Lambda^d\ln b\int d^dx\left\{6\left(\frac{u_0}{\Lambda^2}-\frac{r_0u_0}{\Lambda^4}\right)\bar{\psi}^2-\frac{36u_0^2}{\Lambda^4}\bar{\psi}^4\right\}$$

We now introduce $E\{\bar{\psi}\}$ explicitly and redefine the coefficients:

$$E^{(1)}\{\bar{\psi}\}=\int d^dx\left\{\frac{1}{2}|\nabla\bar{\psi}|^2+\frac{r_0}{2}\bar{\psi}^2+u_0\bar{\psi}^4+6C_d\Lambda^d\ln b\left(\frac{u_0}{\Lambda^2}-\frac{r_0u_0}{\Lambda^4}\right)\bar{\psi}^2\right.$$
$$\left.-C_d\Lambda^d\ln b\,\frac{36u_0^2}{\Lambda^4}\bar{\psi}^4\right\}$$

We define $\tilde{r}_0=r_0+12C_d\ln b\left[u_0\Lambda^{d-2}-r_0u_0\Lambda^{d-4}\right]$

$$\tilde{u}_0=u_0-36C_d\ln b\,u_0^2\Lambda^{d-4}$$

These are the modifications to the effective coefficients after having integrated out the fast degrees of freedom associated with $\delta\psi$.

Step 2. We have integrated out a shell. We next rescale variables to return the rescaled system to the original scale. By coarse graining:

$x' = x/b$ as we have increased the unit of length by a factor of b – as the high k sector of the spectrum has been integrated out.

$$\to d^d x = b^d d^d x'$$

This would induce a transformation of the gradient term:

$$\nabla\bar{\psi} = \frac{1}{b}\nabla'\psi \Rightarrow \frac{1}{2}|\nabla\bar{\psi}|^2 = \frac{1}{2}\frac{1}{b^2}|\nabla'\bar{\psi}|^2$$

Or:
$$d^d x \frac{1}{2}|\nabla\bar{\psi}^2| = d^d x' \frac{1}{2}\frac{b^d}{b^2}|\nabla'\bar{\psi}|^2 = d^d x' \frac{1}{2} b^{d-2}|\nabla'\bar{\psi}|^2$$

including now the Jacobian of the transformation.

- We now absorb the factor b^{d-2} in $\bar{\psi}$ so that the gradient term appears unchanged by rescaling:

$$\bar{\psi}' = b^{(d-2)/2}\bar{\psi} \quad \text{so that} \quad d^d x \frac{1}{2}|\nabla\bar{\psi}|^2 = d^d x' \frac{1}{2}|\nabla'\bar{\psi}'|^2$$

We rescale the other two terms. The quadratic term:

- $d^d x\, \tilde{r}_0 \bar{\psi}^2 = b^d d^d x' \tilde{r}_0 \frac{\cancel{b}1}{b^{d-2}} \bar{\psi}'^2 = d^d x' \tilde{r}_0 b^2 |\bar{\psi}'|^2$
- $d^d x\, \tilde{u}_0 |\bar{\psi}|^4 = b^d d^d x' \tilde{u}_0 \frac{1}{b^{2d-4}} \bar{\psi}'^4 = d^d x' \tilde{u}_0 b^{4-d} (\bar{\psi}')^4$

In short, we have:

$$E^{(1)}\{\bar{\Psi}'\} = \int d^dx' \left\{ \frac{1}{2}|\nabla\bar{\Psi}'|^2 + \frac{r^{(1)}}{2}|\bar{\Psi}'|^2 + u^{(1)}(\bar{\Psi}')\right.$$

in the same form as the original energy, with:

$$\boxed{\begin{aligned} r^{(1)} &= b^2\left[r_0 + 12\, C_d \ln b \left[u_0 \Lambda^{d-2} - r_0 u_0 \Lambda^{d-4} \right]\right] \\ u^{(1)} &= b^{4-d}\left[u_0 - 36 C_d \ln b\, u_0^2 \Lambda^{d-4} \right] \end{aligned}}$$

These are the recursion relations that we obtain from this momentum shell renormalization transformation.

Step 3. Find the fixed point and the eigenvalues.

One needs to iterate the recursion relation until a fixed point is found. One introduces a few transformations to turn the discrete iteration in the recursion relation into a differential equation.

Start with $b = 1+\delta \qquad b^n = (1+\delta)^n \simeq 1 + n\delta$

Also $\ln b \simeq \delta$

$$\Rightarrow \quad b^n \simeq 1 + n \ln b \quad \text{if } b \simeq 1.$$

$$\Rightarrow \quad r^{(1)} \simeq (1 + 2\ln b)\left[r_0 + 12\, C_d \ln b \left[u_0 \Lambda^{d-2} - r_0 u_0 \Lambda^{d-4} \right]\right]$$

and we keep only terms linear in $\ln b$ $\quad (\ln b \simeq 0)$

$$r^{(1)} \simeq r_0 + 12 C_d \ln b \left(u_0 \Lambda^{d-2} - r_0 u_0 \Lambda^{d-4} \right) + 2 r_0 \ln b$$

Or $$r^{(1)} - r_0 = \ln b \left[2 r_0 + 12 C_d \left(u_0 \Lambda^{d-2} - r_0 u_0 \Lambda^{d-4} \right) \right]$$

(note $\ln b \simeq \delta$ small).

We now introduce the continuous variable $z = \ln b$, and denote $r(z) = r^{(1)}$ and $u(z) = u^{(1)}$ and write:

$$\frac{dr}{dz} = 2r + 12\, C_d \left(u \Lambda^{d-2} - r u \Lambda^{d-4} \right)$$

with initial condition $\begin{cases} r(0) = r_0 \\ u(0) = u_0 \end{cases}$

[C_d is eliminated by redefining Λ and r]

$$\boxed{\frac{dr}{dz} = 2r + 12 \left(u \Lambda^{d-2} - r u \Lambda^{d-4} \right)}$$

Similarly:

$$u^{(1)} = \left(1 + (4-d)\ln b\right)\left[u_0 - 36\, C_d \ln b\, u_0^2 \Lambda^{d-4} \right]$$

$$= u_0 - 36\, C_d \ln b\, u_0^2 \Lambda^{d-4} + (4-d)\ln b\, u_0$$ again up to order ln

$+ (4-d)\ln b\, u_0$

$$u^{(1)} - u_0 = (4-d)\ln b\, u_0 - 36\, C_d \ln b\, u_0^2 \Lambda^{d-4}$$

Introducing differential variables: (and absorbing C_d in Λ).

$$\boxed{\frac{du}{dz} = (4-d)u - 36 \Lambda^{d-4} u^2}$$

The two equations in this page are known as the renormalization group equations.

Introduce dimensionless variables: $x = r/\Lambda^2$, $y = u/\Lambda^{4-d}$

$$\Rightarrow \Lambda^2 \frac{dx}{d\tau} = 2\Lambda^2 x + 12\left(\Lambda^{4-d} y \Lambda^{d-2} - \Lambda^{2} x \cancel{\Lambda^{4-d}} y \cancel{\Lambda^{d-4}}\right)$$

$$= 2\Lambda^2 x + 12\left(\Lambda^2 y - \Lambda^2 xy\right) \Rightarrow \boxed{\frac{dx}{d\tau} = 2x + 12(y - xy)}$$

and $$\Lambda^{4-d}\frac{dy}{d\tau} = (4-d)\Lambda^{4-d} y - 36\Lambda^{d-4}\Lambda^{2(4-d)} y^2$$

$$= (4-d)\Lambda^{4-d} y - 36\Lambda^{-d+4} y^2 \Rightarrow \boxed{\frac{dy}{d\tau} = (4-d)y - 36y^2}$$

Fixed points $\frac{dx}{d\tau} = \frac{dy}{d\tau} = 0$

(a) Trivial (or Gaussian) fixed point $x^* = y^* = 0$. ($r = 0$, $u = 0$ hence the name).

(b) Non trivial fixed point $0 = 2x^* + 12(y^* - x^* y^*)$

$(\epsilon = (4-d))$ $\qquad 0 = \epsilon \cancel{y} - 36 y^{*\cancel{2}} \Rightarrow \boxed{y^* = \epsilon/36}$

$$\rightarrow 0 = 2x^* + 12\left(\frac{\epsilon}{36} - x^*\frac{\epsilon}{36}\right) = 2x^* + \frac{\epsilon}{3} - \frac{\epsilon x^*}{3}$$

$$\Rightarrow 0 = \cancel{2}\, x^*\left(2 - \frac{\epsilon}{3}\right) + \frac{\epsilon}{3} \qquad x^* = \frac{-\epsilon/3}{2 - \epsilon/3}$$

Or if ϵ is small: $x^* = \frac{-\epsilon/3}{2(1-\epsilon/6)} = \frac{-\epsilon}{6} + \cdots$ $\qquad \boxed{x^* = -\frac{\epsilon}{6}}$

NOTE: Since $r = a(T - T_c^{(0)})$, then a negative x^* means a negative r and hence the renormalized critical temperature T_c is lower than $T_c^{(0)}$, the mean field value.

Linearize now around the fixed point $\left\{ \begin{array}{l} x = x^* + \delta x \\ y = y^* + \delta y \end{array} \right.$

$$\frac{d}{dz}\delta x = \underbrace{2\cancel{x^*} + 2\delta x + 12\cancel{y^*} + 12\delta y - 12\cancel{x^*}y^*}_{\text{by definition of fixed point.}} - 12x^*\delta y - 12y^*\delta x$$

$$\frac{d}{dz}\delta y = \underbrace{(4-\cancel{d})y^* + (4-d)\delta y - \cancel{36}y^{*2}}_{\text{by definition of fixed point.}} - 72y^*\delta y$$

Hence

$$\frac{d}{dz}\begin{pmatrix}\delta x \\ \delta y\end{pmatrix} = \begin{pmatrix} 2-12y^* & 12-12x^* \\ 0 & \epsilon - 72y^* \end{pmatrix}\begin{pmatrix}\delta x \\ \delta y\end{pmatrix}$$

Eigenvalues:

$$\begin{vmatrix} 2(1-6y^*) - \lambda & 12(1-x^*) \\ 0 & \epsilon - 72y^* - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - \lambda(\epsilon - 72y^*) - 2\lambda(1-6y^*) + 2(1-6y^*)(\epsilon - 72y^*) = 0$$

$$\lambda^2 - \lambda\left[(\epsilon - 84y^*) + 2\right] + 2(1-6y^*)(\epsilon - 72y^*) = 0$$

(a) At the Gaussian fixed point $x^* = y^* = 0$

$$\lambda^2 - \lambda(\epsilon + 2) + 2\epsilon = 0 \qquad \lambda = \frac{1}{2}\left((\epsilon+2) \pm \sqrt{(\epsilon+2)^2 - 8\epsilon}\right)$$

$$= \frac{1}{2}\left(\epsilon + 2 \pm \sqrt{(\epsilon+2)^2\left[1 - \frac{8\epsilon}{(\epsilon+2)^2}\right]}\right) = \frac{1}{2}\left[(\epsilon+2) \pm (\epsilon+2)\left(1 - \frac{4\epsilon}{(\epsilon+2)^2}\right.\right.$$

$$\Rightarrow \begin{array}{l} (+) \\ (-) \end{array} \boxed{\begin{array}{l} \lambda_1 = \frac{1}{2}(2+2) = 2 \\ \lambda_2 = \epsilon \end{array}}$$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$ unstable

$\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ λ

[illegible] direction $\delta y = 0$

$\delta x + 6\delta y = 0 \quad \delta y = -\frac{1}{6}\delta x$

Gaussian fixed point $x^* = y^* = 0$

$$\lambda_1 = 2 \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ \frac{12}{2-\epsilon} \end{pmatrix}$$

$$\lambda_2 = \epsilon \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- $d<4$ $\quad \epsilon>0$ both directions are unstable. The manifold lines:

$$\vec{v}_1 \cdot (\delta x, \delta y) = \delta x + \frac{12}{2-\epsilon}\,\delta y = 0 \qquad \delta y = -\frac{2-\epsilon}{12}\,\delta x$$

$$\vec{v}_2\,(\delta x, \delta y) = \delta y = 0$$

drawn in the figure.

- $d>4$ $\quad \epsilon<0$ The $\vec{v}_2$ direction is unstable

Critical point (nontrivial) $x^* = -\epsilon/6 \,,\; y^* = \epsilon/36$

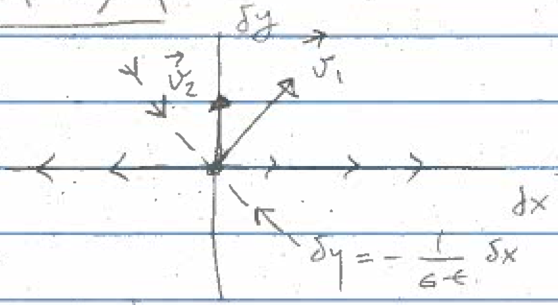
- $\lambda_1 = 2 - \frac{\epsilon}{3} \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ \frac{12(1-x^*)}{2+60y^* - \epsilon} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{12(1+\epsilon/6)}{2+\frac{60\epsilon}{36} - \epsilon} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{12+2\epsilon}{2+\frac{5}{3}\epsilon - \epsilon} \end{pmatrix} =$

$$= \begin{pmatrix} 1 \\ \frac{12+2\epsilon}{2+\frac{2\epsilon}{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{12+2\epsilon}{2(1+\epsilon/3)} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{6+\epsilon}{1+\frac{\epsilon}{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ (6+\epsilon)(1-\frac{\epsilon}{3}) \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 6+\epsilon - \frac{6\epsilon}{3} \end{pmatrix} = \begin{pmatrix} 1 \\ 6-\epsilon \end{pmatrix}$$

$\boxed{\lambda_1 = 2 - \frac{\epsilon}{3} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 6-\epsilon \end{pmatrix}}$ This is the unstable direction

- $\boxed{\lambda_2 = -\epsilon \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$ This is the stable direction



STABLE : $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Lines of constant v_2 : $\delta y = 0$ → unstable flow

$\vec{v}_2 \cdot (\delta x, \delta y) = 0$ ⇗

UNSTABLE :

$\vec{v}_1 = \begin{pmatrix} 1 \\ 6-\epsilon \end{pmatrix}$ Lines of constant v_1 :

$\delta x + (6-\epsilon)\,\delta y = 0 \qquad \delta y = -\frac{1}{6-\epsilon}\,\delta x$

→ on this line the motion is stable

(b) At the nontrivial fixed point :

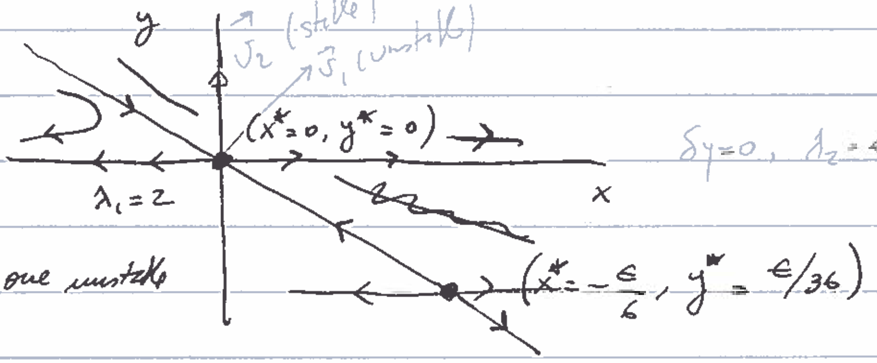
$$\boxed{\begin{aligned}\lambda_1 &= 2 - \frac{\epsilon}{3} \\ \lambda_2 &= -\epsilon\end{aligned}}$$

~~$v_1 = \delta x + (6-\epsilon)$~~ δy ~~$v_2 = \delta y$~~ ~~eigenvectors~~ Scaling fields.

This completes the calculation of the critical eigenvalues at the nontrivial fixed point.

<u>Stability</u> : • $d > 4$ ~~[illegible]~~ $\epsilon < 0$

$\delta y = -\frac{2-\epsilon}{12}\delta x$
unstable



$\delta y = 0$, $\lambda_2 = \epsilon < 0$, stable

One stable, one unstable direction.

If one starts at the critical point $x \propto r = 0$, then the flow is towards $y \to 0$. The Gaussian fixed point is stable, ~~and~~ (saddle point really), and the physical fixed point for $d > 4$.

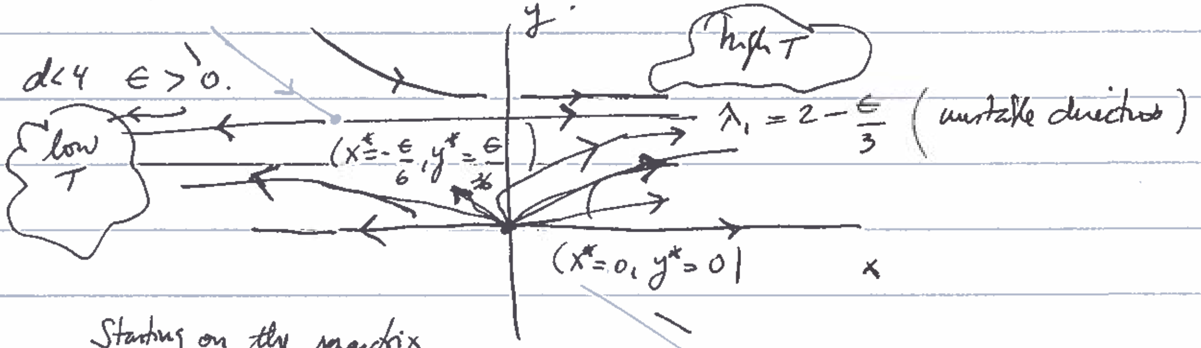
$y^* < 0$, or $u < 0$ which is unphysical as the free energy would decrease by $\phi \to \infty$.

$v_1 = \delta x + (6-\epsilon)\delta y$
$v_2 = \delta y$

$\lambda_2 = -\epsilon < 0$ stable direction $\Rightarrow v_2$ an irrelevant variable.

• $\lambda_1 > 0$, v_1 relevant direction $\Rightarrow v_1 = 0$ is the critical surface.

• $d < 4$ $\epsilon > 0$.



Starting on the separatrix, the flow is towards the critical point.

To the right one flows to high temperature, and ~~to~~ the left one flows to low temperature.

$\lambda_2 = -\epsilon$ (stable direction)

The fixed point is a saddle point with one stable direction if one is ~~at~~ the renormalized critical temperature $x^* = \frac{-\epsilon}{6}$.

Critical exponents : ~~$v_i = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is the eigenvector. Hence~~:

$$\frac{dv_i}{dz} = \lambda_i v_i$$ by definition.

$\rightarrow v_i(z) = v_0 e^{\lambda_i z}$ and since $z = \ln b$, $e^z = b$

$\Rightarrow v_i(z) = v_0 b^{\lambda_i}$ so that the λ_i are directly the critical exponents in any eigendirection.

As seen in plot, λ_1 is the eigenvalue in the unstable eigendirection $\Rightarrow y_{\text{thermal}} = 2 - \frac{1}{3}\epsilon$

and recall that $\nu = \frac{1}{y_t} = \frac{1}{2} + \frac{\epsilon}{12}$

$\rightarrow$ Mean field $\nu_{MF} = 1/2$

Exp $d=3$ $\nu = 0.638$

ϵ expansion ($\epsilon=1$, $\nu = 0.5 + 1/12 = 0.583$

$$\frac{1}{2 - \frac{1}{3}\epsilon} = \frac{1}{2} \frac{1}{1 - \frac{1}{6}\epsilon} = \frac{1}{2}\left(1 + \frac{1}{6}\epsilon\right) = \frac{1}{2} + \frac{\epsilon}{12}$$