Chapter 7

Equilibrium correlation functions and Scattering

Equilibrium fluctuations can be understood as local deviations in a subsystem that is part of a larger system in equilibrium. We study in this chapter how to describe this equilibrium fluctuations.

Consider as an example a system of N classical and indistinguishable particles at constant temperature. The partition function is

$$Z(N,V,T) = \frac{1}{N!} \int \frac{d^N \vec{r} d^N \vec{p}}{h^{3N}} e^{-\beta E(\vec{r}^N,\vec{p}^N)}.$$

It is useful to divide the energy into kinetic and potential, and the joint probability of space and momentum,

$$p(\vec{r}^N, \vec{p}^N) = \phi(\vec{p}^N) P(\vec{r}^N)$$

with

$$P(\vec{r}^{N}) = \frac{e^{-\beta V(\vec{r}^{N})}}{\int d^{N} \vec{r} e^{-\beta V(\vec{r}^{N})}}$$
(7.1)

where V is the potential energy function. By definition, P is the probability of observing the system in any specific spatial configuration while it evolves in configuration space in equilibrium. For an interacting system, P does not factorize in simpler distribution functions involving a single particle, and hence one often resorts to defining marginal probability distribution functions. The function P itself has too much information to be of any use if it were possible to compute it.

The first function is

$$\rho_1(\vec{r}) = N \int d\vec{r}_2 \dots d\vec{r}_N P(\vec{r}^N).$$
(7.2)

The integral is the probability that particle 1 is at \vec{r} . Since all particles are equivalent, the prefactor N leads to ρ_1 being the probability that *any* particle is at \vec{r} . The definition is such that if the system is uniform $P(\vec{r}^N) = 1/V^N$, and $\rho_1 = N/V = \rho$, the equilibrium density.

Analogously, one defines

$$\rho_2(\vec{r}_1, \vec{r}_2) = N(N-1) \int d\vec{r}_3 \dots d\vec{r}_N P(\vec{r}^N), \tag{7.3}$$

which is the joint probability of finding any one particle at $\vec{r_1}$ and any other particle at $\vec{r_2}$. If the system is uniform, then $\rho_2 = N(N-1)/V^2 \simeq \rho^2$. In the case of the density ρ , it is customary to define the correlation function g as

$$g(\vec{r}_1, \vec{r}_2) = \frac{\rho_2(\vec{r}_1, \vec{r}_2)}{\rho^2}.$$
(7.4)

This function has an easy interpretation from probability theory. Recall that P(A, B) = P(A|B)P(B). The joint probability distribution (for a system with translational invariance) would be $P(A, B) = \rho_2(0, \vec{r})$, and since $P(B) = \rho$, then

$$P(A|B) = \frac{P(A,B)}{P(B)} = \rho g(\vec{r}).$$
(7.5)

That is, $\rho g(\vec{r})$ is the conditional probability of finding a particle at \vec{r} given that there is a particle at the origin.

7.1 Measurement of $g(\vec{r})$ by diffraction

Consider a radiation scattering experiment as indicated in Fig. 7.1. Scattering experiments involve



Figure 7.1:

the change in momentum of some incident particle or radiation by a sample. The radiation of choice depends on the scale of the fluctuations that one wishes to probe. In general, the wavelength of the radiation must be of the same order of magnitude as the spatial scale of the fluctuations.

The scattering wave following a scattering event in the sample is a radial outgoing wave. Its electric field is

$$E \sim E_0 \frac{e^{i\vec{k}_f \cdot \vec{r}}}{r}.$$

where \vec{k}_f is the outgoing wavevector, and ω the angular frequency of the wave. We consider here only elastic scattering in which the frequency of the outgoing wave is the same as the frequency of the incident wave (inelastic scattering is also used as a diagnostic tool, but we will not address it here). The wave that arrives at the detector after scattering off an element of volume in the sample at \vec{r}_j is,

$$E \sim E_0 \frac{1}{|\vec{R}_D - \vec{r}_j|} e^{i\vec{k}_i \cdot \vec{r}_j + i\vec{k}_f \cdot (\vec{R}_D - \vec{r}_j)} e^{-i\omega t}.$$
(7.6)

considering the path that the radiation has traveled from the source, to the sample, and to the detector. We now define the scattering wavevector as $\vec{k} = \vec{k}_f - \vec{k}_i$ and rewrite,

$$E \sim E_0 \frac{1}{|\vec{R}_D - \vec{R}_S|} e^{i\vec{k}_f \cdot \vec{R}_D} e^{-i\vec{k} \cdot \vec{r}_j} e^{-i\omega t},$$

where we have assumed that the detector is far from the sample and hence $|\vec{R}_D - \vec{r}_j| \approx |\vec{R}_D - \vec{R}_S|$, where \vec{R}_S is the location of the sample.

7.1. MEASUREMENT OF $G(\vec{R})$ BY DIFFRACTION

If one now adds the scattering from all elements of volume in the sample, the radiation that arrives at the detector is,

$$E \sim E_0 \frac{1}{|\vec{R}_D - \vec{R}_S|} e^{i\vec{k}_f \cdot \vec{R}_D} \left(\sum_j e^{-i\vec{k} \cdot \vec{r}_j} \right) e^{-i\omega t}.$$

We now compute the intensity of the radiation $I(\vec{k}) = E^*(\vec{k})E(\vec{k})$ and compute the statistical average, to find,

$$I(\vec{k}_f) \sim \frac{1}{|\vec{R}_D - \vec{R}_S|^2} NS(\vec{k}), \tag{7.7}$$

where we have defined the so called structure factor as,

$$S(\vec{k}) = \frac{1}{N} \langle \sum_{j=1}^{N} \sum_{m=1}^{N} e^{i\vec{k} \cdot (\vec{r}_m - \vec{r}_j)} \rangle.$$
(7.8)

The scattering intensity is a function of \vec{k} , the scattering wavevector.

We now show how to relate the structure factor to the correlation function Write,

$$S(\vec{k}) = \frac{1}{N} \langle \sum_{l=1,l=j}^{N} 1 + \sum_{l\neq j}^{N} e^{i\vec{k} \cdot (\vec{r}_l - \vec{r}_j)} \rangle = \frac{1}{N} N + \frac{1}{N} N(N-1) \langle e^{i\vec{k} \cdot (\vec{r}_l - \vec{r}_j)} \rangle$$

By writing the thermal average explicitly, we find,

$$S(\vec{k}) = 1 + \frac{N(N-1)}{N} \frac{\int d^N \vec{r} e^{-\beta V(\vec{r}^N)} e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)}}{\int d^N \vec{r} e^{-\beta V(\vec{r}^N)}} = 1 + \frac{N(N-1)}{N} \frac{\int d\vec{r}_1 \int d\vec{r}_2 e^{i\vec{k}\cdot(\vec{r}_1 - \vec{r}_2)} \int d\vec{r}_3 \dots d\vec{r}_N e^{-\beta V(\vec{r}^N)}}{\int d^N \vec{r} e^{-\beta V(\vec{r}^N)}}$$

The last integral in the right hand side and the numerator allow us to introduce the correlation function ρ_2 ,

$$S(\vec{k}) = 1 + \frac{1}{N} d\vec{r_1} d\vec{r_2} \rho_2(\vec{r_1}, \vec{r_2}) e^{i\vec{k} \cdot (\vec{r_1} - \vec{r_2})}.$$

Change $\vec{r_2} = \vec{r_1} + \vec{r}$, assume translational invariance, and integrate over $\vec{r_1}$ to obtain our final result

$$S(\vec{k}) = 1 + \frac{1}{N} \int d\vec{r}_1 \int d\vec{r} \rho^2 g(\vec{r}) e^{i\vec{k}\cdot\vec{r}} = 1 + \rho \int d\vec{r} g(\vec{r}) e^{i\vec{k}\cdot\vec{r}}.$$
(7.9)

In short, the scattered intensity is proportional to the structure factor as shown in Eq. (7.7), and the structure factor is the Fourier transform of the par correlation function, Eq. (7.9). Therefore the scattered intensity measured in a diffraction experiment is directly the Fourier transform of the pair correlation function.

In practice, experiments measure the diffraction intensity as a function of the scattering angle θ , the angle between the incident wavevector k_i and the outgoing wavevector k_f . Since the scattering wavevector $\vec{k} = \vec{k}_f - \vec{k}_i$ and since we focus on elastic scattering $|\vec{k}_i| = |\vec{k}_f|$, we then have $k = k_i + \sin(\theta/2) + k_f \sin(\theta/2)$ or

$$k = \frac{4\pi}{\lambda_i} \sin \frac{\theta}{2}.$$

Hence changing the location of the detector relative to the sample (and hence changing θ produces the scattered intensity for different wavevectors \vec{k} .

One can distinguish two cases for $\hat{\rho}(q)$: (i) it corresponds to a local density or local concentration that is uniform in equilibrium, and (ii), it corresponds to a more general order parameter that may not be uniform in equilibrium (e.g., periodic), In the first case, Fourier transformation of a uniform density gives

$$S(\vec{k}) \propto \delta(\vec{k}).$$
 (7.10)



Figure 7.2:



7.2. SCATTERING OFF A FLAT INTERFACE

The cartoon in Fig. 7.2 may make this clearer. The second case is more subtle. An example is the sub lattice concentration of a binary alloy. Say atoms of the two species are A and B, and assume that locally A wants to be neighbors with B but not with another A. This makes the equilibrium state look like a checker-board as shown in Fig. 7.3.

Assuming that the scattered wave is different from atoms A or B atoms this equilibrium structure can be seen by analyzing the structure factor S(k). Note that the A atoms occupy a sub lattice, which has twice the lattice spacing of the original lattice as shown in Fig. 7.4. Hence the density



(Sublatice 2 is unshaded)

Figure 7.4:

of A atoms is uniform on this sub lattice with twice the lattice constant of the original system, and the scattering will show peaks *not* at $\vec{k} = 0$ but at $\vec{k} = \vec{k_0}$ corresponding to that structure,

$$S(\vec{k}) \propto \sum_{\vec{k}_0} \delta(\vec{k} - \vec{k_0}) \tag{7.11}$$

where $k_0 = 2\pi/(2a)$, and a is the original lattice spacing. The same scattered intensity is observed



Figure 7.5:

for a crystal in which Bragg peaks form at specific positions in \vec{k} space. By monitoring the height and width of such peaks, the degree of order can be determined.

A further complication is that usually the sample will not be a single crystal perfectly aligned with the incident radiation so that one obtains spots on the k_x and k_y axes as shown. Instead they appear at some random orientation (Fig. 7.6). This might seem trivial (one could just realign the sample), except that often a real crystal is a poly crystal, that is, it is comprised of many single crystal domains of different orientations that scatter simultaneously. In this case, all the orientations indicated above are smeared into a ring of radius k_0 .

7.2 Scattering off a flat interface

The two examples just given concern systems that are single phase. Consider now the case of a flat interface -without roughening- separating two uniform phases as shown in Fig. 7.8. For simplicity







Figure 7.7: Average of Many Crystallites



Figure 7.8:

7.3. SCATTERING FROM A LINE DEFECT

let us focus on an interface embedded in dimension d = 2. If the density is uniform in the bulk phases, we can write, $\rho(x, y) = A \ \theta(y) + B$, where $\theta(y)$ is the Heaviside step function, and A and B are constants. For simplicity, we will let A = 1 and B = 0 from now on, $\rho(x, y) = \theta(y)$

We now proceed to compute the scattering intensity that follows from this configuration. In order to do so, we first need to determine the Fourier transform of the Heaviside function θ . We first note that $\partial_y \rho(y) = \delta(y)$. Hence $ik_y \hat{\rho}(k_y) = 1$. We therefore find $\hat{\rho}(k_y) = 1/ik_y$. This result is correct up to a constant: $\partial_y(\rho(y) + C) = \delta(y)$ also satisfies the equation for any arbitrary constant C. Fourier transforming, we find $ik_y(\hat{\rho}(k_y) + C\delta(y)) = 1$. In order to determine the constant C we note that

$$\theta(x) + \theta(-x) = 1$$

Let us compute,

$$\hat{\theta}(k_y) = \int_{-\infty}^{\infty} dy \theta(y) e^{-ik_y y} = \int_{-\infty}^{-\infty} (-dy) \theta(-y) e^{-ik_y (-y)} = -\int_{-\infty}^{-\infty} dy \left(1 - \theta(y)\right) e^{-ik_y (-y)}$$

and hence,

$$\hat{\theta}(k_y) = \int_{-\infty}^{\infty} e^{-i(-k_y)y} - \int_{-\infty}^{\infty} dy \theta(y) e^{-i(-k_y)y} = \delta(k_y) - t\hat{heta}(-k_y).$$

Finally then $\hat{\theta}(k_y) + \hat{\theta}(-k_y) = \delta(k_y)$. This relation applied to ρ gives,

$$\hat{\rho}(k_y) + \hat{\rho}(-k_y) = \delta(k_y).$$

Substituting the expression for $\hat{\rho}$ above, we have,

$$\frac{1}{ik} - C\delta(k_y) + \frac{1}{-ik_y} - C\delta(k_y) = \delta(k_y), \quad \text{or} \quad C = -\frac{1}{2}.$$

The final result for the Fourier transform of the configuration with an interface is

$$\hat{\rho}(k_y) = \frac{1}{ik_y} + \frac{1}{2}\delta(k_y).$$
(7.12)

More properly in two dimensions, we should write

$$\hat{\rho}(\vec{k} = \frac{1}{ik_y}\delta(k_x) + \frac{1}{2}\delta(\vec{k}), \qquad (7.13)$$

as the system is uniform in the x direction. Therefore

$$S(k) = |\hat{\rho}(k)|^2 \propto \frac{1}{k_y^2} \delta(k_x) \tag{7.14}$$

where we have omitted the delta function from bulk scattering.

Assume now more generally that the unit vector normal to the interface is \hat{n} , and that the unit tangential vector is is \hat{n}_{\perp} , (Fig. 7.9), then

$$S(k) \propto \frac{1}{(\vec{k} \cdot \hat{n})^2} \delta(\vec{k} \cdot \hat{n}_\perp)$$
(7.15)

Then the scattered intensity would appear as shown in Fig. 7.10: a streak pointed in the direction of \hat{n} (\hat{y} in Fig. 7.10).

7.3 Scattering from a line defect

Assume an idealized model of a line defect (density inhomogeneity) given by,



Figure 7.9:



Figure 7.10: Scattering from interface y = 0







Figure 7.12: Scattering from a line defect at y = 0

7.4. SCATTERING FROM ROUGH INTERFACES

$$\rho(x,y) = \delta(y). \tag{7.16}$$

Then $\hat{\rho} = \delta(k_x)$ and $S(k) \propto \delta(k_x)$. If the unit vector normal to the surface of the defect is \hat{n} , and the tangential vector is \hat{n}_{\perp} , the proper expression is $S(k) \propto \delta(\vec{k} \cdot \hat{n}_{\perp})$ for a line defect.

For a defect of small (but finite) extent, we can imagine a situation in which the density within the defect region ρ is not uniform, but it rather corresponds to a modulated phase of the type discussed above for a binary alloy. We then have to incorporate the wavenumber $\vec{k_0}$ corresponding to the modulated phase. One finds,

$$S(k) \propto \delta((\vec{k} - \vec{k}_0) \cdot \hat{n}_\perp) \tag{7.17}$$

A similar argument can be advanced for the case of an interface now separating modulated phases. One would find,

$$S(k) \propto \frac{1}{|(\vec{k} - \vec{k}_0) \cdot \hat{n}|^2} \delta((\vec{k} - \vec{k}_0) \cdot \hat{n}_\perp).$$
(7.18)

7.4 Scattering from rough interfaces

So far we have focused on idealized situations in which the interface is simply a line of discontinuity, and ignored any possible structure.

Before addressing the case of a rough interface, let us consider a diffuse interface in which the density profile is smooth instead of discontinuous, as shown in Fig. 7.13. Although the precise



Figure 7.13: A diffuse interface, with no roughness

functional dependence does not matter, let us consider the following case:

$$\rho(x,y) \sim \tanh(y/\xi),\tag{7.19}$$

with ξ assumed small determining the width of the interface. In this case, one can show that

$$S(k) = |\hat{\rho}(k)|^2 \sim \frac{\delta(k_x)}{k_y^2} (1 + \mathcal{O}(\xi k_y)^2)$$
(7.20)

for an interface at y = 0. Diffuseness only effects scattering at $k_x = 0$ and for large k_y of order $1/\xi$





(again, assuming that ξ is small so that ξk_y is comparable to 1). The resulting scattering is sketched in Fig. 7.14. This result is general, and does not depend on our choice of tanh for ρ .

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Figure 7.15: A very rough interface, with no diffuseness

A rough interface is not planar, but rather described by some profile y = h(x) in, say d = 2. If the rough interface is not diffuse, that is, if it can be locally approximated by a surface of discontinuity, the density satisfies,

$$\rho(x,y) = \theta(y - h(x)). \tag{7.21}$$

If h(x) involves only small distortions, then by expanding in Taylor series we can write,

$$\rho \approx \theta(y) - h(x)\delta(y) + \dots \tag{7.22}$$

since $d\theta/dy = \delta(y)$. Taking the Fourier transform and squaring gives

$$S(k) = \langle |\hat{\rho}(k)|^2 \rangle \sim \frac{\delta(k_x)}{k_y^2} + \langle |\hat{h}(k_x)|^2 \rangle, \qquad (7.23)$$

where we have omitted unimportant delta functions at the origin. We obtain the rather remarkable result that the intensity of the scattered radiation depends on to the spectrum of surface fluctuations $\langle |\hat{h}(k_x)|^2 \rangle$. (In earlier cases in this chapter we did not consider thermal averages as the configuration of the interface was unique and fixed. For a rough surface we can consider all possible equilibrium configurations that the system adopts during the duration of the scattering experiment).



Extra scattering due to roughness

Figure 7.16:

7.5 (*) Scattering from many planar and randomly oriented surfaces

Evidently the scattering from many randomly oriented surfaces gives a pinwheel like cluster if streaks as shown in Fig. 7.17. Angularly averaging over all orientations of the interface gives

$$S = \frac{\int d\hat{n} |\hat{\rho}_k|^2}{\int d\hat{n}} \tag{7.24}$$

where \hat{n} is the unit vector normal to the surface. We write

$$\hat{n} = -\sin\theta\hat{x} + \cos\theta\hat{y} \tag{7.25}$$



Figure 7.17:

and

$$\hat{n}_{\perp} = \cos\theta\hat{x} + \sin\theta\hat{y} \tag{7.26}$$

with

$$\vec{k} = k\cos\beta\hat{x} + k\sin\beta\hat{y} \tag{7.27}$$

Then, for scattering from a surface where $|\hat{\rho}_k|^2 = \frac{\delta(\vec{k}\cdot\hat{n}_{\perp})}{|\vec{k}\cdot\hat{n}|^2}$ we have (in d = 2 and assuming a uniform distribution of orientations θ)

$$S = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{\delta[k\cos\beta\cos\theta + k\sin\beta\sin\theta]}{|-k\cos\beta\sin\theta + k\sin\beta\cos\theta|^2}$$
$$= \frac{1}{2\pi} \frac{1}{k^3} \int_0^{2\pi} d\theta \frac{\delta[\cos(\theta - \beta)]}{|\sin(\theta - \beta)|^2}$$
$$= \frac{1}{2\pi} \frac{1}{k^3} \int_0^{2\pi} d\theta \frac{\delta[\cos\theta]}{|\sin\theta|^2}$$

But $\cos \theta = 0$ at $\theta = \pi/2$, $3\pi/2$ where $|\sin \theta|^2 = 1$, so that

$$S(k) = \frac{1}{\pi} \frac{1}{k^3}$$
(7.28)

Similarly, in arbitrary dimension one has

$$S(k) \propto \frac{\delta(\vec{k} \cdot \hat{n}_{\perp})}{|\vec{k} \cdot \hat{n}|^2} \sim \frac{\delta k^{d-1} n_{\perp}}{k^2} \sim \frac{1}{k^{2+d-1}} = \frac{1}{k^{d+1}}.$$
(7.29)

We have used the facts that the tangent space to the surface is d-1 dimensional, and that $\delta(ax) = \delta(x)/a$. for scattering from a surface.

Scattering from a line defect which has $|\hat{\rho}(k)|^2 \propto \delta(k_x)$, one similarly obtains

$$S(k) \propto \frac{1}{k^{d-1}} \tag{7.30}$$

We shall call these results, Porod's Law. They involve only geometrical considerations about the interface.

The case where ρ is not the order parameter, but a modulated structure exists instead is more subtle. If the wavenumber of the modulated phase is $\vec{k_0}$ then this introduces a new angle via

$$\vec{k}_0 = k_0 \cos \alpha \hat{x} + k_0 \sin \alpha \hat{y} \tag{7.31}$$

Now there are two possible averages

- 1. α or β , average over crystallite orientation, or, angles in a non angularly resolved detector. These are equivalent (look at the figure for a few seconds), and so we only need to do one of these two averages.
- 2. θ , averages over surface orientation.





Figure 7.19:

If we average over α or β , for a fixed \hat{n} , it is easy to anticipate the answer as shown in fig. 7.19, where in the cartoon the regions around $(k_x = k_0, k_y = 0)$ retain the singularity of the original $\delta(k_x - k_0)/(k_y - k_0)^2$.

This is basically a detector problem, so instead we will consider averaging θ first. In fact it is obvious what the result of such an average is, it must be the previous result, but shifted from $\vec{k} = 0$



Figure 7.20:

to $\vec{k} = \vec{k}_0$, that is

$$S_{surface}(k) \propto \frac{1}{|\vec{k} - \vec{k}_0|^{d+1}}$$

$$S_{line}(k) \propto \frac{1}{|\vec{k} - \vec{k}_0|^{d-1}}$$
(7.32)

If a further average is done over crystallite orientation, we can take these expressions and conduct an angular average over the angle between \vec{k} and $\vec{k_0}$. Clearly after averaging we will have

$$\bar{S} = S(\frac{|k| - |k_0|}{|k_0|}) \equiv S(\Delta k)$$
(7.33)

First, let

$$\phi = \beta - \alpha \tag{7.34}$$

so that

$$(\vec{k} - \vec{k}_0)^2 = k^2 - 2kk_0\cos\phi + k_0^2$$

or, on using

$$k \equiv k_0(\Delta k + 1) \tag{7.35}$$

we have

$$(\vec{k} - \vec{k}_0)^2 = k_0^2 \{ (\Delta k)^2 + 2(1 - \cos \phi)(\Delta k + 1) \}$$
(7.36)

Hence in two dimensions, for a surface, we have

$$\bar{S}(\Delta k) = \frac{1}{2\pi k_0^3} \int_0^{2\pi} d\phi \frac{1}{|(\Delta k)^2 + 2(1 - \cos\phi)(\Delta k + 1)|^{3/2}}$$
(7.37)

which is a complex integral. We will work out its asymptotic limits $\Delta k >> 1$ and $\Delta k << 1$. First, if $\Delta k >> 1$, then

If $\Delta \kappa >> 1$, then

$$\bar{S}(\Delta k) = \frac{1}{2\pi k_0^3} \frac{1}{(\Delta k)^3} \int_0^{2\pi} d\phi + \dots$$

or (in arbitrary d)

$$\bar{S}(\Delta k) \sim \frac{1}{(\Delta k)^{d+1}}, \, \Delta k \gg 1$$
(7.38)

If $\Delta k \ll 1$, consider

$$\bar{S}(\Delta k \to 0) \stackrel{?}{=} \frac{1}{2\pi k_0^3} \int_0^{2\pi} d\phi \frac{1}{|2(1-\cos\phi)|^{3/2}}$$

which diverges due to the $\phi = 0$ (and $\phi = \pi$) contribution. So we keep a small Δk . For convenience, we will also expand ϕ around 0.

$$\bar{S}(\Delta k)\approx \frac{1}{2\pi k_0^3}\int_0^{2\pi} d\phi \frac{1}{|(\Delta k)^2+\phi^2|^{3/2}}+\dots$$

Now let $u = \phi/\Delta k$

$$\bar{S}(\Delta k) = \frac{1}{2\pi k_0^3} \frac{1}{(\Delta k)^2} \int_0^{2\pi/\Delta k} du \frac{1}{|1+u^2|^{3/2}}$$
$$\bar{S}(\Delta k) \sim \frac{1}{(\Delta k)^2} \text{ as } \Delta k \to 0$$
(7.39)

for all dimensions d.

If one does this in d dimensions with

$$S = \frac{1}{|\vec{k} - \vec{k}_0|^{\gamma}}$$
(7.40)

then

so,

$$\bar{S} = \begin{cases} \frac{1}{(k-k_0)^{\gamma-(d-1)}}, & \frac{k-k_0}{k_0} << 1\\ \frac{1}{(k-k_0)^{\gamma}}, & \frac{k-k_0}{k_0} >> 1 \end{cases}$$
(7.41)

If $\gamma - (d-1) > 0$. For the case $\gamma = d - 1$, one obtains

$$\bar{S} \sim \begin{cases} -\ln(k-k_0), & \frac{k-k_0}{k_0} << 1\\ \frac{1}{(k-k_0)^{d-1}}, & \frac{k-k_0}{k_0} >> 1 \end{cases}$$
(7.42)

It should be noted that the actual scattering is given by a difficult integral like Eq. 7.37, which is



Figure 7.21:

pretty close to an elliptical integral.