# Chapter 3

# Self similarity and Fractals

This chapter allows us to explore one situation in which a large object is not comprised of many independent parts. More precisely, by looking at objects that are self similar, we will uncover a class of systems in which spatial correlation functions do not decay exponentially with distance, the hallmark of statistical independence among the parts.

Fractals are mathematical constructs that seem to exist in fractional dimensions between 1 and 2, 2 and 3, etc.. They provide a useful starting point for thinking about systems n which $V/\xi^3 \sim \mathcal{O}(1)$, or if $V = L^3$, $\xi/L \sim \mathcal{O}(1)$. Figure 3.1 schematically shows a fractal object. The perimeter appears

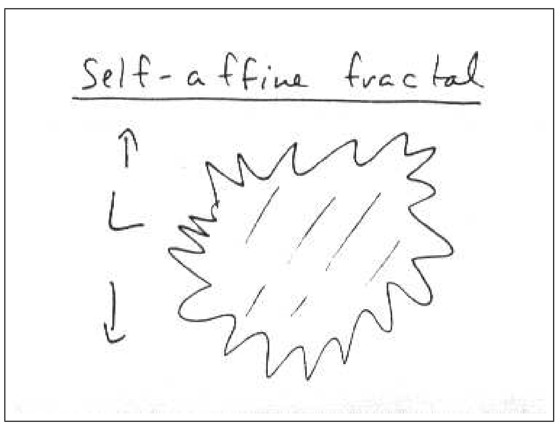


Figure 3.1: Self affine Fractal

very jagged, or as it is said in physics, *rough*. The self-affine fractal has the property that its perimeter length $P$ satisfies,

$$P \sim L^{d_s} \quad \text{as } L \to \infty \tag{3.1}$$

where $d_s > 1$ is the so called self-affine fractal exponent. Note that for a circle of diameter $L$, $P_{circle} = \pi L$, while for a square of side $L$ $P_{square} = 4L$. In fact, $P \sim L^1$ for any run-of-the-mill, non fractal, object in two spatial dimensions. More generally, the surface of a compact (non fractal) object in $d$ spatial dimensions is

$$P_{\text{non fractal}} \sim L^{d-1} \tag{3.2}$$

It is usually the case that

$$d_s > d - 1 \tag{3.3}$$

Another example is given in Fig. 3.2. This object has a distribution of holes such that its mass M satisfies

$$M \sim L^{d_f} \text{ as } L \to \infty, \tag{3.4}$$

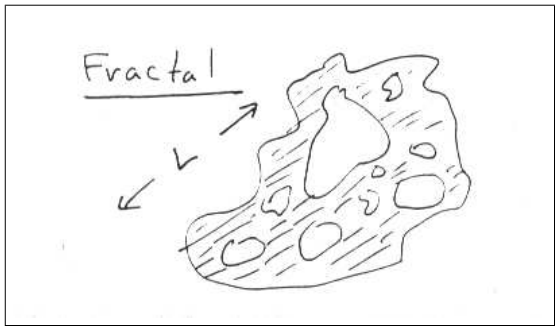


Figure 3.2: Volume Fractal

where $d_f$ is the fractal dimension. Note again that for a circle $M = (\pi/4)\rho L^2$ if $L$ is the diameter, while for a square $M = \rho L^2$. In fact, for run-of-the-mill compact objects

$$M_{\text{non fractal}} \sim L^d \tag{3.5}$$

in $d$ dimensions. Usually

$$d_f < d \tag{3.6}$$

S since the mass density is normally defined as $M/L^d$ in $d$ dimensions, this would imply that

$$\rho_f \sim \frac{1}{L^{d-d_f}} \to 0 \tag{3.7}$$

as $L \to \infty$.

## 3.1 Von Koch Snowflake

We construct in this section a specific example of a fractal object, and compute its fractal dimension. In order to construct a von Koch fractal (or snowflake), follows the steps outlined in Fig. 3.3. In

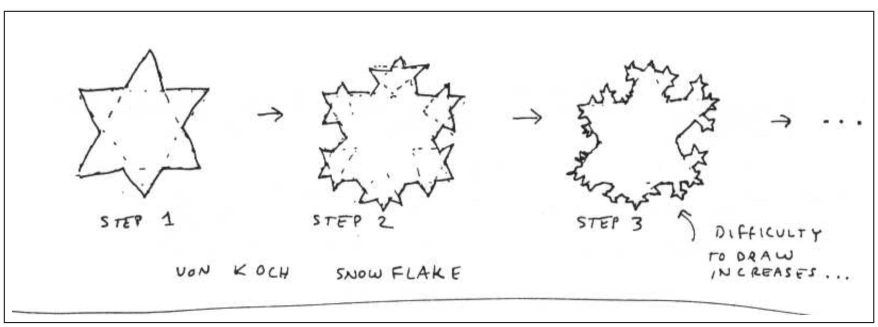


Figure 3.3: Von Koch Snowflake

order to find the self-affine or fractal exponent for the perimeter, imagine we had drawn one of these fractals recursively. Say the smallest straight element has length $l$. We calculate the perimeter $P$ as we go through the recursion indicated in Fig. 3.4.

From the figure, we find that the perimeter $P_n = 4^n l$ after $n$ iterations, whereas the end to end distance after $n$ iterations is $L_n = 3^n l$. By taking the logarithm, we find $\ln(P/l) = n \ln 4$ and $\ln(L/l) = n \ln 3$. Hence,

$$\frac{\ln P/l}{\ln L/l} = \frac{\ln 4}{\ln 3} \quad \text{or} \quad P/l = (L/l)^{d_s}$$

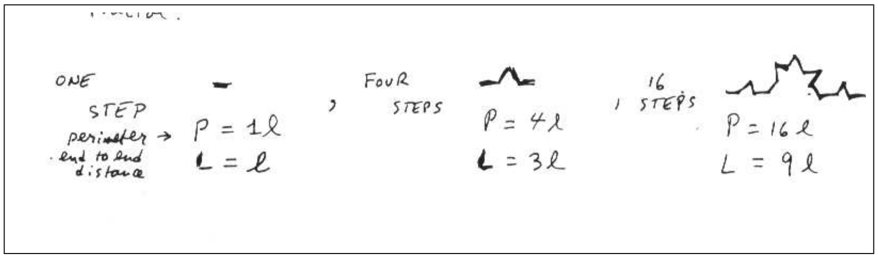


Figure 3.4:

with

$$d_s = \frac{\ln 4}{\ln 3} \approx 1.26 > d - 1 = 1. \tag{3.8}$$

## 3.2 Sierpinskii Triangle

The second example that we discuss is the Sierpinskii triangle. It can also be constructed recursively by following the steps outlined in Fig. 3.5. The iteration to construct this fractal is also shown in

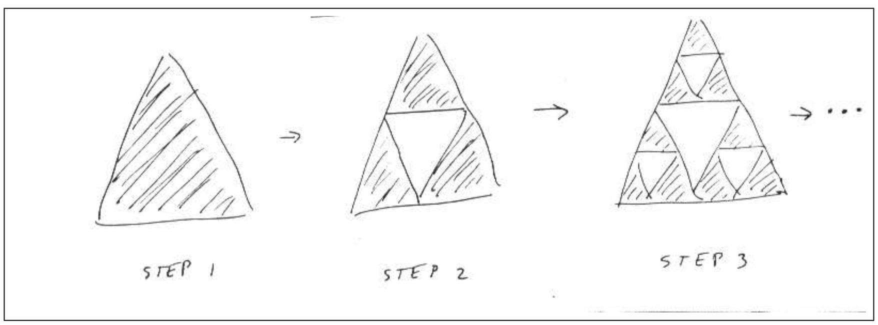


Figure 3.5: Sierpinskii Triangle

Fig. 3.6. Define the mass of the smallest filled triangle as $m$ and its volume as $v$. Successive masses are given by $M_n = 3^n m$ after $n$ iterations, with a volume $V_n = 4^n v$. Since $V_n = \frac{1}{2}L_n^2$, and $v = \frac{1}{2}l^2$

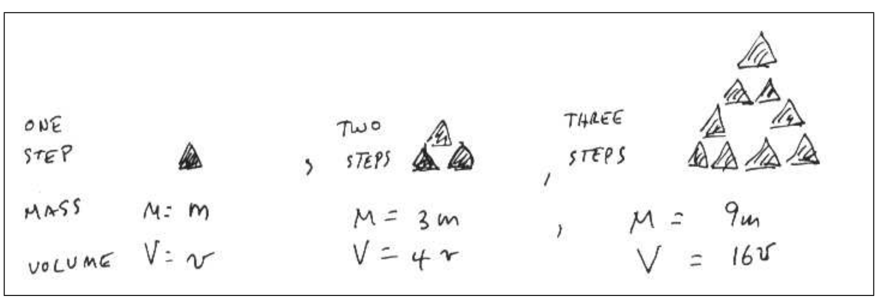


Figure 3.6:

we have $L_n = 2^n l$ Taking logarithms, we obtain $\ln(M/m) = n \ln 3$ and $\ln(L/l) = n \ln 2$, taking the ratio

$$\frac{\ln(M/m)}{\ln(L/l)} = \frac{\ln 3}{\ln 2} \qquad \frac{M}{m} = (\frac{L}{l})^{d_f},$$

with

$$d_f = \frac{\ln 3}{\ln 2} \approx 1.585 < d = 2. \tag{3.9}$$

## 3.3 Correlations in Self-Similar Objects

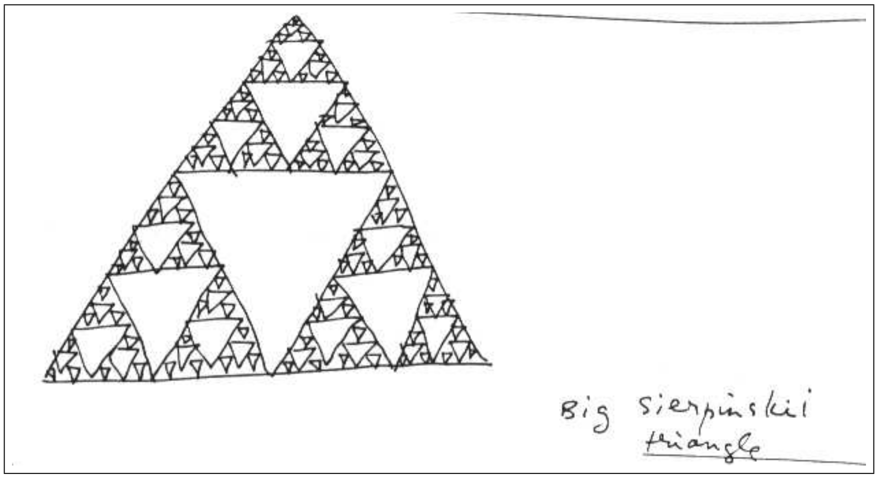


Figure 3.7:

The defining feature about a fractal is that it “looks the same” on different length scales: If you take the fractal in Fig. 3.7, select a small portion of it and magnify it, the magnified structure will be identical to the original fractal. This is the case irrespective of the size of the portion that is being magnified. Although this is true in this particular example, we will focus in what follows in a less restrictive definition: “statistical self similarity”. In this case, configurations and their magnified counterparts are geometrically similar only in a statistical sense. More specifically, any parameter of the structure that is invariant under uniform magnification has to be, on average, independent of scale.

This statement can be quantified by introducing the correlation function $C(r)$ of the density $n(r)$ in Fig. 3.7,

$$C(r) = \langle \Delta n(r)\ \Delta n(0) \rangle$$

The average is, say, over all orientations in space for convenience. In the case of a physical system, the average would be the usual statistical average. If the structure is uniformly magnified by a factor $\lambda$: $r \to \lambda r$, then statistical self-similarity is defined as

$$C(\lambda r) = \lambda^{-p} C(r), \tag{3.10}$$

where the exponent $p$ is a constant. Both the magnified $C(\lambda r)$ and the original $C(r)$ are functionally the same, except for a constant scale factor $\lambda^p$.

We first prove this relationship. The assumption of self similarity implies that $C(r) \propto C(r/b)$, or $C(r) = f(b)\, C(r/b)$ with $b = 1/\lambda$. If we take the derivative with respect to $b$, and let $r^* = r/b$ we find,

$$\frac{\partial C(r)}{\partial b} = 0 = \frac{df}{db} C(r^*) + f \frac{dC}{dr^*} \frac{dr^*}{db}$$

or,

$$\frac{df}{db} C(r^*) = +f \frac{r^*}{b} \frac{dC}{dr^*}.$$

Therefore,

$$\frac{d \ln f(b)}{d \ln b} = \frac{d \ln C(r^*)}{d \ln r^*} \tag{3.11}$$

Since the left hand side only depends on $b$, and the right hand side only on $r^*$, they both must equal a constant,

$$\frac{d \ln f(b)}{d \ln b} = \text{ const. } = -p \quad f(b) \propto b^{-p} \tag{3.12}$$

and the recursion relation is

$$C(r) = b^{-p} C(r/b) \tag{3.13}$$

up to a multiplicative constant.

Equation (3.13) is sufficient to determine the functional form of $C(r)$. Since the magnification factor $b$ is arbitrary, let us consider the special value

$$b = 1 + \epsilon, \quad \epsilon << 1 \tag{3.14}$$

Then, by expanding Eq. (3.13) in Taylor series, we have

$$\begin{aligned} C(r) &= (1+\epsilon)^{-p}\, C(\frac{r}{1+\epsilon}) \\ &\approx (1 - p\epsilon)\{C(r) - \epsilon r \frac{\partial C}{\partial r}\} + \ldots \end{aligned}$$

At first order in $\epsilon$ we find,

$$-p\, C(r) = r \frac{\partial C}{\partial r}$$

The solution of which is

$$C \propto \frac{1}{r^p}. \tag{3.15}$$

In summary, self similarity implies that correlation functions are generalized homogeneous functions of their arguments,

$$C(r) = b^{-p}\, C(r/b)$$

relation that immediately leads to a power law dependent of the correlation function on distance,

$$C(r) \propto \frac{1}{r^p} \tag{3.16}$$

Usually, these relations apply in physical systems for large distances. At shorter distances (say the scale of an elemental triangle in the case at hand) self-similarity breaks down, and with it the mathematical relations that we have derived. In addition, for systems of finite size, self similarity also breaks down at scales of the order of the size of the system.

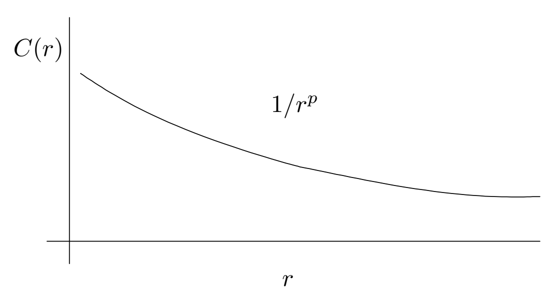


Figure 3.8: Decay of Correlations in a Self similar object

Note that the the dependence of $C(r)$ on $r$ that we have just found is quite different from that which occurs in a system of many independent parts,

$$C(r) \sim e^{-r/\xi}$$

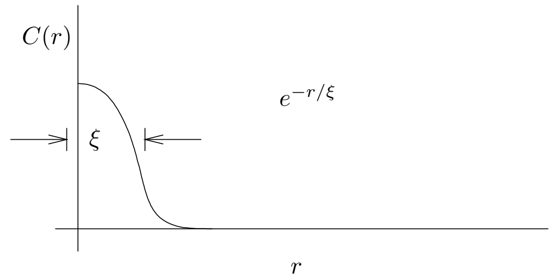


Figure 3.9: Decay of Correlations in a system of many independent parts

In particular, the relation

$$\int d^d r C(r) = \mathcal{O}(\xi^d)$$

in $d$-dimensions that is satisfied for a system comprised of many independent parts is evidently *not* satisfied for a self similar system in which

$$C(r) \sim \frac{1}{r^p}$$

if $d > p$. In this case, the integral diverges! (this formally implies a divergence in the variance of $x$):

$$\int d^d r C(r) \propto \int_a^L r^{d-1} dr \frac{1}{r^p} \sim L^{d-p}, \tag{3.17}$$

where $a$ is a microscopic length (irrelevant here), and $L \to \infty$ the size of the system.

In summary, fractals (self similar or rough objects) do not satisfy the central limit theorem since the integral in Eq. 3.17 diverges. The study of rough surfaces is one of the focuses of the rest of the course.