Correlation function for Ornstein-Zernicke

Define Fourier transforms:

$$\begin{cases} \cancel{\delta(x)}\ \varphi(x) = \displaystyle\int \frac{d^dk}{(2\pi)^d}\, e^{ikx}\, \hat{\varphi}(k) & \text{contin.} \\ \varphi(x) = \displaystyle\frac{1}{L^d} \sum_k e^{ikx}\, \hat{\varphi}_k & \text{discrete.} \end{cases}$$

This implies:

$$\begin{cases} \delta(x) = \displaystyle\int \frac{d^dk}{(2\pi)^d}\, e^{ikx} \\ \delta(x) = \displaystyle\frac{1}{L^d} \sum e^{ikx} \end{cases}$$

Or: $\delta(\vec{k}) = \displaystyle\int \frac{d^dx}{(2\pi)^d}\, e^{ikx}$, $\quad \delta_{\vec{k},0} = \displaystyle\frac{1}{L^d} \int d^dx\, e^{ikx}$

is the Kronecker delta.

- We have: $D(k) = \dfrac{L^d}{(2\pi)^d}$ as the density of states in k.

$\Delta k = \dfrac{2\pi}{L}$

$\sum F(k) = \dfrac{L^d}{(2\pi)^d}$

$\cdot \sum F(k)\, \Delta^d k \simeq \dfrac{L^d}{(2\pi)^d} \displaystyle\int F(k)\, d^dk$

So that

$$\sum_k F(k) = \left(\frac{L}{2\pi}\right)^d \int d^dk\, F$$

or

$$\lim_{L\to\infty} \frac{1}{L^d} \sum_k F(k) = \int \frac{d^dk}{(2\pi)^d}\, F(k)$$

for any function $F(k)$.

We now have:

$$E = \int d^dx \left[\frac{K}{2} (\nabla\varphi)^2 + \frac{r}{2}\varphi^2 \right] \qquad T > T_c$$

$$E = \int d^dx \left[\frac{K}{2} (\nabla \Delta\varphi)^2 + |r|\, \Delta\varphi^2 \right] \qquad T < T_c$$

and we wish to calculate the correlation function

$$C(x) = \langle \varphi(0)\, \varphi(x) \rangle$$

We conduct the calculation in Fourier space:

$$\int d^dx\, \frac{K}{2}(\nabla\phi)^2 = \frac{K}{2}\int d^dx \left[\frac{\partial}{\partial x}\int \frac{d^dk}{(2\pi)^d}\, e^{ikx}\,\hat\phi(k)\right]^2 =$$

$$= \frac{K}{2}\int d^dx \int\frac{d^dk}{(2\pi)^d}\int\frac{d^dk'}{(2\pi)^d}\,(i\vec k)(i\vec k')\,\hat\phi(k)\,\phi(k')\, e^{i(k+k')x} =$$

$$= \frac{K}{2}\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dk'}{(2\pi)^d}\,(-\vec k\cdot\vec k')\,\hat\phi(k)\,\hat\phi(k')\,\underbrace{\int d^dx\, e^{i(k+k')x}}_{(2\pi)^d\,\delta(k+k')}$$

$$= \frac{K}{2}\int\frac{d^dk}{(2\pi)^d}\,|\vec k|^2\,\hat\phi(k)\,\hat\phi(-k) =$$

$$= \frac{K}{2}\int\frac{d^dk}{(2\pi)^d}\,k^2\,|\hat\phi(k)|^2$$

A similar calculation can be conducted with $\frac{r}{2}\phi^2$ leading to.

$$E = \int\frac{d^dk}{(2\pi)^d}\left(\frac{K}{2}k^2 + \frac{r}{2}\right)|\hat\phi(k)|^2 \quad \text{in the continuum case}$$

$$E = \frac{1}{L^d}\sum_k\left(\frac{K}{2}k^2 + \frac{r}{2}\right)|\hat\phi_k|^2 \quad \text{in the discrete case.}$$

In order to compute the ~~f~~ partition function, we have to sum over all states. Now: $\phi(x)$ is real, whereas $\hat\phi$ is complex ($\phi^*(k) = \phi(-k)$) so we only need to sum over half the spectrum of independent modes (otherwise we will count every state twice).

only half the spectrum

$$\sum_{\text{states}} \to \prod_x \int \cancel{dh}\, d\phi(x) = \int \prod_k d^2\hat h_k$$

and since $\hat h_k$ is complex $\quad d^2\hat h_k = \underset{\text{real part}}{\underset{\uparrow}{dh'_k}}\ \underset{\text{imaginary part}}{\underset{\uparrow}{dh''_k}}$

In order to compute $C(x)$ we define $\langle \hat{\varphi}(k)\hat{\varphi}(k')\rangle$.

$$\langle \hat{\varphi}(k)\hat{\varphi}(k')\rangle = \int d^dx \int d^dx' \, e^{-ikx} e^{-ik'x'} \langle \varphi(x)\varphi(x')\rangle$$

if translational invariance holds

$$\stackrel{\downarrow}{=} \int d^dx \int d^dx' \, e^{-ikx} e^{-ik'x'} C(x-x')$$

Define $y = x - x'$, $y' = \frac{1}{2}(x+x')$

and $d^dx \, d^dx' = d^dy \, d^dy'$

(or $x = y' + \frac{1}{2}y$, $x' = y' - \frac{1}{2}y$)

$$\langle \hat{\varphi}(k)\hat{\varphi}(k')\rangle = \int d^dy \, d^dy \, e^{-i(k-k')\frac{y}{2}} G(y).$$

$$\cdot \underbrace{\int dy' e^{-i(k+k')\cdot y}}_{(2\pi)^d \, \delta(k+k')}$$

$\Rightarrow$ $\langle \hat{\varphi}(k)\hat{\varphi}(k')\rangle =$

$$= \underbrace{\int d^dy \, e^{-iky} G(y)}_{\hat{G}(k)} (2\pi)^d \, \delta(k+k')$$

$$\Rightarrow \langle \hat{\varphi}(k)\hat{\varphi}(k')\rangle = \hat{G}(k)(2\pi)^d \, \delta(k+k') \quad \text{continuous}$$

$$= G_k \, L^d \, \delta_{k+k',0} \quad \text{discrete.}$$

$\Rightarrow$ the only terms we need to calculate in Fourier space is:

$$\langle \hat{\varphi}_k \hat{\varphi}_{-k}\rangle = \langle |\hat{\varphi}_k|^2\rangle.$$

By definition:

$$\langle |\hat{\varphi}_k|^2\rangle = \prod_{k'}{}' \int d\varphi_{k'} \, |\varphi_k|^2 \, e^{-\frac{1}{k_BTL^d}\sum_{k''}\left(\frac{K}{2}k''^2 + \frac{r}{2}\right)|\varphi_{k''}|^2}$$

$$\overline{\prod_{k'}{}' \int d^2\varphi_{k'} \, e^{-\frac{1}{k_BTL^d}\sum_{k''}\left(\frac{K}{2}k''^2 + \frac{r}{2}\right)|\varphi_{k''}|^2}}$$

All products in numerator and denominator cancel ~~expe~~ except for k.

$$\langle |\psi_k|^2 \rangle = \frac{\int d^2\hat{\psi}_k \, |\hat{\psi}_k|^2 e^{-\frac{2}{k_B T L^d}\left(\frac{K}{2}k^2 + \frac{r}{2}\right)|\hat{\psi}_k|^2}}{\int d^2\hat{\psi}_k \, e^{-\frac{2}{k_B T L^d}\left(\frac{K}{2}k^2 + \frac{r}{2}\right)|\hat{\psi}_k|^2}}$$

→ the factor of two in the exponent comes from $\psi_k \psi_{-k} + \psi_{-k}\psi_k$ both contributing to the sum. This is unrelated to the Π' restriction. For any of the unrestricted k's, we have two terms in the sum.

The integrals that are left are Gaussian. Write $\hat{\psi}_k = Re^{i\theta}$

so that $$\langle |\hat{\psi}_k|^2 \rangle = \frac{\int_0^\infty R\,dR\, R^2 e^{-R^2/a}}{\int_0^\infty R\,dR\, e^{-R^2/a}}$$ with $a = \dfrac{L^d \, k_B T}{\frac{K}{\cancel{2}}k^2 + \frac{r}{\cancel{2}}}$

The integral leads to: $\langle |\psi_k|^2 \rangle = a$

$$\Rightarrow \boxed{\langle |\hat{\psi}_k|^2 \rangle = \frac{k_B T L^d}{\frac{K}{\cancel{2}}k^2 + \frac{r}{\cancel{2}}}}$$

or $$\boxed{G_k = \frac{k_B T}{\frac{K}{\cancel{2}}k^2 + \frac{r}{\cancel{2}}}}_{T > T_c}$$ as the Fourier transform of the correlation function

Below the critical point:

$$C(r) = \langle \Delta\psi(0)\, \Delta\psi(r) \rangle = \langle \psi(0)\psi(r) \rangle - \frac{|r|}{u}$$

and a similar calculation leads to: $$\boxed{G_k = \frac{k_B T}{K k^2 + 2|r|} - \frac{|r|}{u}(2\pi)^d \delta(k)}$$