$\langle A|B\rangle = \mathrm{Tr}\left(\frac{1}{Z} e^{-\beta H} A^* B\right)$

## Projection operators and nonlinear systems.—

In linear response, for a conserved variable:

$$\partial_t \phi_\alpha + \nabla\cdot\vec{J}_\alpha = 0$$

$$\cancel{\delta\langle \vec{J}_\alpha\rangle_{ne}(x,\omega) = L_{\alpha\beta}(\omega)\nabla}$$

$$\delta\langle J_{\alpha i}\rangle_{ne}(x\omega) = L_{ij}^{\alpha\beta}(\omega)\,\partial_j h_\beta(x\omega)$$

where $h_\beta$ is the conjugate variable to $\phi_\beta$.

$\delta\langle m_i(rt)\rangle = 2i\int_0^t dt'\int dr' \sum_{ij}\chi''_{ij}(r-r',t-t')\,h_j(r',t')$

$= 2i\int_0^t dt'\int dr' \sum_{ij}\frac{\chi''_{ij}(r-r',t-t')}{\chi_{ij}}\langle\delta m_j(r',t')\rangle$

or in real time:

$$\delta\langle J_{\alpha i}\rangle_{ne}(xt) = \int_{-\infty}^{t} L_{ij}^{\alpha\beta}(t-t')\,\partial_i h_\beta(x,t')\,dt'$$

(of course, it is also possible to have a nonlocal memory kernel).

→ One can rationalize all we have done so far as projecting the many body dynamics onto a few variables – more explicitly projecting only onto linear level variables: $\phi$, not $\phi^2$, etc. There is the underlying assumption of length and time scales so that only low k, low ω variables are "somehow" retained. The introduction of projection operators can rationalize this a bit.

Consider QM: $\partial_t\phi_i = i[H,\phi_i] = iL\phi_i$ — Heisenberg equation of motion for operator $\phi$. — where $L$ is an operator on $|\phi_i\rangle$

CM $\partial_t\phi_i = \{H,\phi_i\} = iL\phi_i$

Formally: $\phi_i(t) = e^{iLt}\phi_i(0)$

arbitrary phase space function. We will restrict them to slow phase space variables.

Let $\phi_i$ be the slow variables ~~and $\phi_i$~~, and define a ~~to~~ projection operator $P|\phi_i\rangle = |\phi_i\rangle$ for a slow variable, and define $Q = \mathbb{I} - P$, as the complementary projection.

One defines: $$P = \sum_{jk} |\phi_j\rangle \frac{1}{\langle\phi_j|\phi_k\rangle} \langle\phi_k|$$

~~defined~~ all $\langle\cdot\rangle$ defined to be the equilibrium average.

For a general observable $A$:

$$P\cdot A = \sum_{jk} |\phi_j\rangle \frac{1}{\langle\phi_j|\phi_k\rangle} \langle\phi_k|A\rangle$$

and in equilibrium: $$\langle\phi_j|\phi_i\rangle = \frac{1}{\beta^2}\frac{\partial^2 \ln Z}{\partial h_j \cdot \partial h_i} = \frac{1}{\beta}\frac{\partial\langle\phi_j\rangle}{\partial h_i} = \chi_{ji}\frac{1}{\beta}$$

$$\Rightarrow PA = \sum_{jk} |\phi_j\rangle \cancel{\beta} \frac{\partial h_k}{\partial\langle\phi_j\rangle} \cdot \frac{1}{\cancel{\beta}} \frac{\partial\langle A\rangle}{\partial h_k}$$

$$= \sum_j |\phi_j\rangle \frac{\partial\langle A\rangle}{\partial\langle\phi_j\rangle}$$

Two properties: a) $\langle\phi_i|L\phi_j\rangle = \langle L\phi_i|\phi_j\rangle$

Follows from (time translation invariance) $\langle\phi_i(t)|\phi_j(0)\rangle = $
$= \langle\phi_i(0)|\phi_j(-t)\rangle$

b) Since $\partial_t|\phi_i\rangle = iL|\phi_i\rangle \rightarrow \langle\dot{\phi}_i|\phi_j\rangle = -\langle\phi_i|\dot{\phi}_j\rangle$

We begin by writing: $\quad |\phi_i(t)\rangle = e^{itL}|\phi_i\rangle \qquad\qquad |\phi_i\rangle \equiv |\phi_i(0)\rangle$

$$\Rightarrow \frac{\partial}{\partial t}|\phi_i(t)\rangle = iL\, e^{itL}|\phi_i\rangle = e^{itL}(P+Q)\, iL|\phi_i\rangle =$$

$$= e^{itL}P\, iL|\phi_i\rangle + e^{itL}Q\, iL|\phi_i\rangle$$

The first term in the right hand side:

$$e^{itL}P\, iL|\phi_i\rangle = e^{itL}\sum_{jk}|\phi_j\rangle \frac{1}{\langle\phi_j|\phi_k\rangle}\langle\phi_k|\, iL\phi_i\rangle$$

Add: $i\Omega$ $\qquad$ Define now the frequency matrix $\boxed{\Omega_{ji} = \sum_k \frac{\langle\phi_k|iL\phi_i\rangle}{\langle\phi_j|\phi_k\rangle}}$

then: $\quad e^{itL}P\, iL|\phi_i\rangle =$

$$= e^{itL}\sum_j \Omega_{ji}|\phi_j\rangle = \sum_j \Omega_{ji}\, e^{itL}|\phi_j\rangle$$

$$= \sum_j \Omega_{ji}|\phi_j(t)\rangle.$$

with $i\Omega$ $\qquad$ This is the first term:

$$\partial_t|\phi_i(t)\rangle = \sum_j \Omega_{ji}|\phi_j(t)\rangle + e^{itL}Q\, iL|\phi_i\rangle$$

~~In~~ particular $\Omega_{ij}$ $\longleftarrow$ Note from the definition $\Omega_{ji}$ has a defined parity as $iL|\phi_i\rangle = \partial_t|\phi_i\rangle$

To ~~dd~~ deal with the second term, one introduces the operator identity:

$$e^{itL} = e^{itQL} + i\int_0^t dt'\, e^{i(t-t')L}PL\, e^{it'QL}$$

Therefore:

$$e^{itL}\, QiL|\phi_i\rangle = \underbrace{e^{itQL}\, QiL|\phi_i\rangle}_{\hat{=}\,|f_i(t)\rangle} + i\int_0^t dt'\, e^{i(t-t')L}\, PL\, e^{it'QL} \cdot QiL|\phi_i\rangle$$

The initial value is $iQL|\phi_i\rangle$ which is the orthogonal projection, and then it evolves according to QL.

$|\phi_i\rangle = |\phi_i(0)\rangle$

We therefore call $\boxed{|f_i(t)\rangle = e^{itQL}\, iQL|\phi_i\rangle}$ a "fast" variable.

(*) Because of the projection Q:

$$\langle\phi_i|f_j(t)\rangle = 0$$

$f_j(t)$ is always orthogonal to the subspace spanned by the $|\phi_j\rangle$.

Collecting terms, we have thus far:

$$\frac{\partial}{\partial t}|\phi_i(t)\rangle = \sum_j \Omega_{ji}|\phi_j(t)\rangle + i\int_0^t dt'\, e^{i(t-t')L}\, PL\, e^{it'QL}\, QiL|\phi_i\rangle + |f_i(t)\rangle$$

We finally consider the integral:

$$i\int_0^t dt'\, e^{i(t-t')L}\, PL\, \underbrace{e^{it'QL}\, QiL|\phi_i\rangle}_{|f_i(t')\rangle} = i\int_0^t dt'\, e^{i(t-t')L}\, PL|f_i(t')\rangle$$

$$= \int_0^t dt'\, e^{i(t-t')L} \sum_{jk} |\phi_j\rangle \frac{1}{\langle\phi_j|\phi_k\rangle} \langle\phi_k|\, iL f_i(t')\rangle$$

$$= \int_0^t dt'\, e^{i(t-t')L} \sum_{jk} |\phi_j\rangle \frac{1}{\langle\phi_j|\phi_k\rangle} \langle -i\underset{\overset{\shortparallel}{L}}{L^{\dagger}}\phi_k|f_i(t')\rangle$$

$$= \int_0^t dt' \, e^{i(t-t')L} \sum_{jk} |\phi_j\rangle \frac{1}{\langle \phi_j | \phi_k \rangle} \langle -iL\phi_k | Q f_i(t') \rangle$$

since $Q$ is the identity on the orthogonal subspace.

I presume that $Q$ is also self-adjoint.

$Q$ self adjoint

$$= \int_0^t dt' \, e^{i(t-t')L} \sum_{jk} |\phi_j\rangle \frac{1}{\langle \phi_j | \phi_k \rangle} \langle \underbrace{-iQL\phi_k}_{f_k(0) \text{ or just simply } f_k} | f_i(t') \rangle$$

$$= -\int_0^t dt' \, e^{i(t-t')L} \sum_{jk} |\phi_j\rangle \frac{\langle f_k | f_i(t') \rangle}{\langle \phi_j | \phi_k \rangle}$$

taking the action of the evolution operator:

$$= -\int_0^t dt' \sum_{jk} \frac{\langle f_k | f_i(t') \rangle}{\langle \phi_j | \phi_k \rangle} |\phi_j(t-t')\rangle$$

The final result:

$$\boxed{\partial_t |\phi_i(t)\rangle = \sum_j \Omega_{ji} |\phi_j(t)\rangle - \int_0^t dt' \, M_{ji}(t') |\phi_j(t-t')\rangle + |f_i(t)\rangle}$$

- $\langle \phi_i | f_j(t) \rangle = 0$
- $$M_{ji}(t') = \sum_k \frac{\langle f_k | f_i(t') \rangle}{\langle \phi_j | \phi_k \rangle}$$

with $i$ $L$ with $M$

which has the form of a Langevin equation with a generalized form of the fluctuation dissipation theorem.