

Analytical determination of the bifurcation thresholds in stochastic differential equations with delayed feedback

Mathieu Gaudreault,¹ François Drolet,^{1,2} and Jorge Viñals¹

¹*Department of Physics, McGill University, Montreal, Quebec, Canada H3A 2T8*

²*FPIInnovations, 570 Saint-Jean Boulevard, Pointe Claire, Quebec, Canada H9R 3J9*

(Received 16 July 2010; published 17 November 2010)

Analytical expressions for pitchfork and Hopf bifurcation thresholds are given for a nonlinear stochastic differential delay equation with feedback. Our results assume that the delay time τ is small compared to other characteristic time scales, not a significant limitation close to the bifurcation line. A pitchfork bifurcation line is found, the location of which depends on the conditional average $\langle x(t)|x(t-\tau) \rangle$, where $x(t)$ is the dynamical variable. This conditional probability incorporates the combined effect of fluctuation correlations and delayed feedback. We also find a Hopf bifurcation line which is obtained by a multiple scale expansion around the oscillatory solution near threshold. We solve the Fokker-Planck equation associated with the slowly varying amplitudes and use it to determine the threshold location. In both cases, the predicted bifurcation lines are in excellent agreement with a direct numerical integration of the governing equations. Contrary to the known case involving no delayed feedback, we show that the stochastic bifurcation lines are shifted relative to the deterministic limit and hence that the interaction between fluctuation correlations and delay affect the stability of the solutions of the model equation studied.

DOI: [10.1103/PhysRevE.82.051124](https://doi.org/10.1103/PhysRevE.82.051124)

PACS number(s): 05.70.Ln, 02.30.Ks, 05.10.Gg, 87.16.Yc

I. INTRODUCTION

We present an approximate analytic determination of the bifurcation threshold of a model stochastic differential equation with delayed feedback, in the limit in which the delay time τ is small compared with the characteristic time scale of evolution of the dynamical variable $x(t)$. This condition is expected to be generally valid near the bifurcation. The bifurcation diagram is determined from the stationary probability distribution function $p(x)$, and the results obtained agree well with an earlier numerical estimate, both for pitchfork and Hopf bifurcations.

Differential delay equations naturally appear in many contexts that range from applied mathematics to economics [1]. More recently, stochastic extensions have been introduced in the study of both natural and synthetic gene regulation networks [2,3]. A discrete model for protein degradation with delayed feedback has also been introduced [4], and its system size dependence has been studied in [5]. However, these models are rarely tractable analytically due to their non-Markovian nature. Exceptions include the derivation of a two-time Fokker-Planck equation in [6] and results on the bifurcation of the first and second moments of a linear equation with delayed feedback [7,8]. We extended the results in [7,8] to a full nonlinear model in [9] through numerical integration of the governing equation. As is the case for models without feedback, explicit consideration of nonlinearities removes an apparent dependence of the stochastic bifurcation threshold on the order of the statistical moment under consideration. We develop here an approximate treatment that allows the analytic determination of the bifurcation diagram given in [9].

We consider a minimal model that incorporates delay, stochasticity, and nonlinearity, and that leads to a bifurcation diagram that displays both direct and oscillatory instabilities. Given a dynamical variable $x(t)$ the model is defined as

$$\dot{x}(t) = ax(t) + bx(t - \tau) - x^3(t) + x(t)\xi(t), \quad (1)$$

where the constant a plays the role of control parameter, the constant b is the intensity of a feedback loop of delay $\tau > 0$, and $\xi(t)$ is a Gaussian white noise with mean $\langle \xi(t) \rangle = 0$ and correlation $\langle \xi(t)\xi(t') \rangle = 2D\delta(t-t')$, where D is the intensity of the noise. Equation (1) is interpreted in the Stratonovich sense of stochastic calculus. We consider parametric noise in a here due to external sources of fluctuations, although it is possible to include noise also in b or τ .

The bifurcation diagram of the deterministic limit of Eq. (1) ($\xi=0$) is known [7]. A pitchfork bifurcation separates exponentially decaying and growing solutions when $b\tau > -1$, whereas a Hopf bifurcation is found when $b\tau < -1$. The two bifurcation lines intersect at $(a, b) = (1/\tau, -1/\tau)$. The bifurcation threshold of Eq. (1) without delay ($b=0$) is also known [10–12]. Direct linearization of the stochastic equation leads to the conclusion that the statistical moment $\langle x^n(t) \rangle$ bifurcates at a value of the control parameter that depends on the order of the moment n . When a saturating nonlinearity is introduced, the stationary probability distribution function of the process can be determined, and with it the threshold for instability [10]. In this case, the bifurcation point is unique for all moments of x . The conclusion of this analysis is that despite the existence of parametric fluctuations, the bifurcation threshold of the stochastic equation is at $a_c=0$, the same value as in the deterministic case. The question therefore arises as to whether parametric fluctuations do induce a change in the bifurcation line when delay is introduced ($b \neq 0$).

A numerical determination of the bifurcation diagram of Eq. (1) has been given in [9]. It is shown that the bifurcation threshold in the presence of parametric fluctuations is shifted relative to that of the deterministic limit. An analytical approximation for the pitchfork bifurcation line in the limit of short delay was also given by using a method first introduced

by Risken [13] and later used in [14–16]. The results agree well with the numerical estimate. Nevertheless, the method fails for the Hopf branch. We derive new analytical expressions valid for both pitchfork and Hopf bifurcation thresholds by direct expansion of the Langevin equation in small τ . In the case of the pitchfork bifurcation, Eq. (1) is formally integrated in powers of τ , and an approximate expression for the conditional probability $\langle x(t-\tau)|x(t) \rangle$ is obtained to first order in τ . This allows a closed expression for the governing equation for $p(x,t)$ and its stationary solution. Close to the Hopf branch, we introduce the stochastic counterpart of a multiple scale expansion. We decompose $x(t)$ in fast and slow varying components by assuming that the envelopes of the oscillations evolve on a time scale that is slower than the oscillations themselves. The fast components are then eliminated by using the method of stochastic averaging. The resulting Fokker-Planck equation for the slowly varying envelopes yields an analytical expression for the bifurcation threshold. We note that the boundary thus determined coincides with the threshold for the first moment of the linearized equation given in [7].

II. RESULTS

A. Pitchfork bifurcation

We begin by defining the probability distribution of x in Eq. (1) as $p(x,t) = \langle \delta[x(t) - x] \rangle$, where $\langle \cdot \rangle$ stands for the ensemble average over realizations of the stochastic process ξ . We summarize the steps leading to the determination of the Fokker-Planck equation of Eq. (1). A more detailed derivation is shown in [9]. Take the time derivative of $p(x,t)$,

$$\begin{aligned} \frac{\partial}{\partial t} p(x,t) = & - \frac{\partial}{\partial x} [\langle \delta[x(t) - x] [ax(t) + bx(t-\tau) - x^3(t)] \rangle \\ & + \langle \delta[x(t) - x] x(t) \xi(t) \rangle], \end{aligned} \quad (2)$$

and introduce dummy variables $x(t) \rightarrow x$ and $x(t-\tau) \rightarrow x_\tau$. By the Furutsu-Novikov theorem, one has

$$\begin{aligned} \langle \delta[x(t) - x] x(t) \xi(t) \rangle &= x \langle \delta[x(t) - x] \xi(t) \rangle \\ &= Dx \left\langle \frac{\delta \langle \delta[x(t) - x] \rangle}{\delta \xi(t)} \right\rangle. \end{aligned} \quad (3)$$

The second term of the right-hand side of Eq. (2) is then

$$\begin{aligned} - \frac{\partial}{\partial x} \langle \delta[x(t) - x] x(t) \xi(t) \rangle &= D \frac{\partial}{\partial x} \left\{ x \frac{\partial}{\partial x} [xp(x,t)] \right\} \\ &= -D \frac{\partial}{\partial x} [xp(x,t)] + D \frac{\partial^2}{\partial x^2} [x^2 p(x,t)]. \end{aligned} \quad (4)$$

Consider now the first term of the right-hand side of Eq. (2),

$$\begin{aligned} \langle \delta[x(t) - x] [ax(t) + bx(t-\tau) - x^3(t)] \rangle \\ &= \int \int \delta[x(t) - x] (ax + bx_\tau - x^3) p(x_\tau|x) p(x) dx dx_\tau \\ &= p(x,t) \left[ax - x^3 + b \int x_\tau p(x_\tau|x) dx_\tau \right], \end{aligned} \quad (5)$$

where we have used $p(x, x_\tau) = p(x_\tau|x)p(x)$. The last term of Eq. (5) is the conditional value of x at $t-\tau$ given its value at t ,

$$\langle x_\tau|x \rangle = \int x_\tau p(x_\tau|x) dx_\tau. \quad (6)$$

Combining Eqs. (4)–(6) and substituting into Eq. (2) leads to the following Fokker-Planck equation:

$$\begin{aligned} \frac{\partial}{\partial t} p(x,t) = & - \frac{\partial}{\partial x} \{ [(a+D)x - x^3 + b \langle x_\tau|x \rangle] p(x,t) \} \\ & + D \frac{\partial^2}{\partial x^2} [x^2 p(x,t)]. \end{aligned} \quad (7)$$

This equation is not closed because of the presence of the conditional probability $\langle x_\tau|x \rangle$.

In order to compute this conditional probability, we formally integrate Eq. (1) in $[t-\tau, t]$,

$$\begin{aligned} x(t) = & x(t-\tau) + \int_{t-\tau}^t [ax(t') + bx(t'-\tau) - x^3(t')] dt' \\ & + \int_{t-\tau}^t x(t') \xi(t') dt', \end{aligned} \quad (8)$$

where the integrand is approximated by its integral representation in $[t-\tau, t']$,

$$\begin{aligned} x(t') = & x(t-\tau) + \int_{t-\tau}^{t'} [ax(t'') + bx(t''-\tau) - x^3(t'')] dt'' \\ & + \int_{t-\tau}^{t'} x(t'') \xi(t'') dt''. \end{aligned} \quad (9)$$

Approximate now the integrands in Eq. (9) by their values at the lower bound, i.e., $x(t'') \approx x(t-\tau)$, $x(t''-\tau) \approx x(t-2\tau)$, and $x^3(t'') \approx x^3(t-\tau)$ to write

$$\begin{aligned} x(t') = & x(t-\tau) + [ax(t-\tau) + bx(t-2\tau) - x^3(t-\tau)] \\ & \times [t' - (t-\tau)] + x(t-\tau) \int_{t-\tau}^{t'} \xi(t'') dt''. \end{aligned} \quad (10)$$

Perform then the integration of Eq. (10) in $[t-\tau, t]$,

$$\begin{aligned} \int_{t-\tau}^t x(t') dt' = & x(t-\tau) \tau + [ax(t-\tau) + bx(t-2\tau) - x^3(t-\tau)] \\ & \times \frac{\tau^2}{2} + x(t-\tau) \int_{t-\tau}^t \int_{t-\tau}^{t'} \xi(t'') dt' dt''. \end{aligned} \quad (11)$$

Note that only the first term in Eq. (11) is first order in τ . Repeat the same method on the other integrands, keeping only terms that are first order in τ ,

$$\int_{t-\tau}^t x(t'-\tau) dt' \approx x(t-2\tau) \tau. \quad (12)$$

Using $x^3(t') \approx x^3(t-\tau) + 3x^2(t)[x(t') - x(t-\tau)]$,

$$\int_{t-\tau}^t x^3(t') dt' \approx x^3(t-\tau)\tau, \quad (13)$$

and finally

$$\int_{t-\tau}^t x(t') \xi(t') dt' \approx x(t-\tau) \int_{t-\tau}^t \int_{t-\tau}^{t'} \xi(t') \xi(t'') dt' dt'', \quad (14)$$

where we have used Eq. (9).

Combine Eqs. (11)–(14) and substitute into Eq. (8),

$$\begin{aligned} x(t) = & x(t-\tau) + \tau[ax(t-\tau) + bx(t-2\tau) - x^3(t-\tau)] \\ & + x(t-\tau) \int_{t-\tau}^t \int_{t-\tau}^{t'} \xi(t') \xi(t'') dt' dt''. \end{aligned} \quad (15)$$

Let $t \rightarrow t-\tau$ in Eq. (15) so that $x(t-\tau) = x(t-2\tau) + O(\tau)$. Approximate then $x(t-2\tau) \approx x(t-\tau)$ and take the ensemble average on both sides of Eq. (15) given the value of $x(t-\tau)$,

$$\begin{aligned} \langle x(t)|x(t-\tau) \rangle = & [1 + \tau(a+b)]x(t-\tau) - \tau x^3(t-\tau) \\ & + x(t-\tau) \int_{t-\tau}^t \int_{t-\tau}^{t'} \langle \xi(t') \xi(t'') \rangle dt' dt''. \end{aligned} \quad (16)$$

Using

$$\int_{t-\tau}^t \int_{t-\tau}^{t'} \langle \xi(t') \xi(t'') \rangle dt' dt'' = D\tau, \quad (17)$$

Eq. (16) finally leads to

$$\langle x(t)|x(t-\tau) \rangle = [1 + \tau(a+b+D)]x(t-\tau) - \tau x^3(t-\tau). \quad (18)$$

However, since in the statistical stationary state this conditional probability can only be a function of the time difference τ , we also have that under stationary conditions,

$$\langle x(t-\tau)|x(t) \rangle = [1 + \tau(a+b+D)]x(t) - \tau x^3(t). \quad (19)$$

Substitution of Eq. (19) in Eq. (7) leads to a closed form for the stationary Fokker-Planck equation,

$$-(1+b\tau) \frac{\partial}{\partial x} \{[(a+b+D)x - x^3]p(x)\} + D \frac{\partial^2}{\partial x^2} [x^2 p(x)] = 0. \quad (20)$$

This equation always has as a normalizable stationary solution $p(x) = \delta(x)$. This is the trivial state, the stability of which is being sought. Equation (20) has an additional solution,

$$p(x) \sim |x|^{\alpha} e^{-[(1+b\tau)/2D]x^2}, \quad (21)$$

with

$$\alpha = \frac{(1+b\tau)(a+b+D)}{D} - 2, \quad (22)$$

which is only normalizable (and hence physical) when $\alpha > -1$. Since Eq. (21) allows for finite moments of x , the point $\alpha_c = -1$ is defined to be the bifurcation threshold of

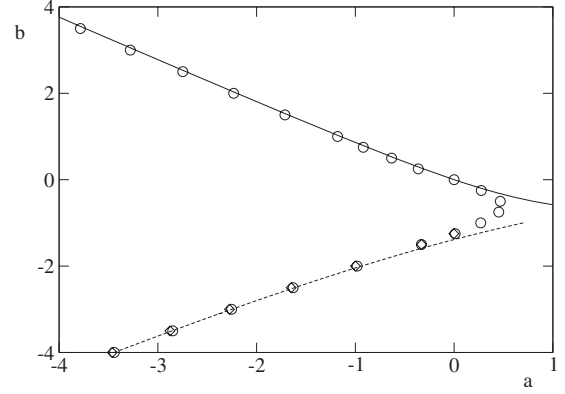


FIG. 1. Bifurcation diagram for Eq. (1) with $\tau=1$ and $D=0.3$. The upper solid curve is the pitchfork branch in the limit of small time delay [Eq. (23)], whereas the lower dashed curve is the Hopf branch [Eq. (56)]. The symbols are the numerically determined bifurcation thresholds in [9], namely, the point in the (a, b) plane for which the exponent of the power law of the stationary probability distribution function of the solution $p(x)$ (\circ) and of the maximum amplitude of the Fourier transform of the trajectories $p_s(\max[X(\omega)])$ (\diamond) are -1 .

Eq. (1). Alternatively, the bifurcation point as a function of the control parameter is

$$a_c = -\frac{b[1 + \tau(b+D)]}{1+b\tau}. \quad (23)$$

The pitchfork bifurcation threshold [Eq. (23)] is shown in Fig. 1 and compared to the numerically determined threshold given in [9]. Both agree quite well, except close to the multicritical point, located at $(a, b) = (1/\tau - D, -1/\tau)$.

The expansion method presented here captures the interplay between fluctuation correlations and delay that lead to our result for the delayed conditional probability [Eq. (19)]. This non-Markovian term in Eq. (7) is directly responsible for the existence of a bifurcation threshold shift relative to the deterministic case. Note that in the limit $b=0$ we recover the known result that $a_c=0$.

B. Hopf bifurcation

A straightforward expansion of Eq. (1) in powers of τ in the vicinity of the Hopf branch fails. As a matter of fact, a similar expansion already fails in the deterministic limit [17,18]. Close to the Hopf bifurcation line we introduce a multiple scale expansion for x and assume a solution of the form

$$x(t, T) = \epsilon A(T) \cos(\omega t) + \epsilon B(T) \sin(\omega t), \quad (24)$$

where ω is a frequency and $T = \epsilon^2 t$ is a slow time scale for the envelope variables A and B . We neglect any stochastic component on the scale t , so that the only remaining stochastic variation is in the amplitudes A and B . This is expected to be correct near the bifurcation threshold.

Substituting Eq. (24) into Eq. (1) (following change of variable rules appropriate for the Stratonovich interpretation of Eq. (1) [19]) yields

$$\begin{aligned}
& \epsilon^3 \partial_T \epsilon A(T) \cos(\omega t) + \epsilon^3 \partial_T \epsilon B(T) \sin(\omega t) \\
&= \omega [\epsilon A(T) \sin(\omega t) - \epsilon B(T) \cos(\omega t)] + a [\epsilon A(T) \cos(\omega t) \\
&+ \epsilon B(T) \sin(\omega t)] + b [\epsilon A(T - \epsilon^2 \tau) \cos[\omega(t - \tau)] + \epsilon B(T \\
&- \epsilon^2 \tau) \sin[\omega(t - \tau)]] - [\epsilon A(T) \cos(\omega t) + \epsilon B(T) \sin(\omega t)]^3 \\
&+ [\epsilon A(T) \cos(\omega t) + \epsilon B(T) \sin(\omega t)] \xi(t). \quad (25)
\end{aligned}$$

The time delay τ is small in the scale T , and hence we approximate $A(T - \epsilon^2 \tau) \approx A(T)$ and $B(T - \epsilon^2 \tau) \approx B(T)$ to write Eq. (25) as

$$\begin{aligned}
& \epsilon^3 \partial_T A(T) \cos(\omega t) + \epsilon^3 \partial_T B(T) \sin(\omega t) \\
&= \cos(\omega t) [-\epsilon \omega B(T) + \epsilon a A(T) + \epsilon b \cos(\omega \tau) A(T) \\
&- \epsilon b \sin(\omega \tau) B(T) - \epsilon^3 A^3(T) \cos^2(\omega t) \\
&- 3 \epsilon^3 A(T) B^2(T) \sin^2(\omega t) + \epsilon A(T) \xi(t)] + \sin(\omega t) [\epsilon \omega A(T) \\
&+ \epsilon a B(T) + \epsilon b \sin(\omega \tau) A(T) + \epsilon b \cos(\omega \tau) B(T) \\
&- \epsilon^3 B^3(T) \sin^2(\omega t) - 3 \epsilon^3 A^2(T) B(T) \cos^2(\omega t) + \epsilon B(T) \xi(t)]. \quad (26)
\end{aligned}$$

We next eliminate the dependence on the fast scale t . Multiply both sides of Eq. (26) by $(\omega/2\pi) \int_0^{2\pi/\omega} \cos(\omega t) dt$ and perform the integration. Repeat the operation but multiply instead by $(\omega/2\pi) \int_0^{2\pi/\omega} \sin(\omega t) dt$ and perform the integration (a related procedure has been used previously in the literature to obtain the envelope equations of differential delay equations [20,21] and for the van der Pol–Duffing oscillator [22–24] in the Ito interpretation). We obtain two coupled stochastic differential equations:

$$\begin{aligned}
\epsilon^3 \partial_T A(T) &= \epsilon \mu A(T) - \epsilon \nu B(T) - \frac{3}{4} \epsilon^3 A(T) [A^2(T) + B^2(T)] \\
&+ \epsilon A(T) \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \xi(t) dt \\
&+ \epsilon A(T) \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \cos(2\omega t) \xi(t) dt \\
&+ \epsilon B(T) \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \sin(2\omega t) \xi(t) dt, \\
\epsilon^3 \partial_T B(T) &= \epsilon \nu A(T) + \epsilon \mu B(T) - \frac{3}{4} \epsilon^3 B(T) [A^2(T) + B^2(T)] \\
&+ \epsilon B(T) \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \xi(t) dt \\
&- \epsilon B(T) \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \cos(2\omega t) \xi(t) dt \\
&+ \epsilon A(T) \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \sin(2\omega t) \xi(t) dt, \quad (27)
\end{aligned}$$

where we have defined $\mu = a + b \cos(\omega \tau)$ and $\nu = \omega + b \sin(\omega \tau)$ for simplicity. We assume furthermore that these parameters scale as $\mu = \epsilon^2 \tilde{\mu}$ and $\nu = \epsilon^2 \tilde{\nu}$ close to the bifurcation. This statement will be verified latter in the derivation.

We next define the stochasticity over the slow time scale. One can take the advantage of the relation $T = \epsilon^2 t$ to write

$$\xi(t) = \epsilon \xi_0(T), \quad (28)$$

where $\xi_0(T)$ is a Gaussian random variable with mean $\langle \xi_0(T) \rangle = 0$ and correlation $\langle \xi_0(T) \xi_0(T') \rangle = 2D \delta(T - T')$. Note that $\xi_0(T)$ is defined over the slow time scale and is independent of t . We exploit the same idea and define $\cos(2\omega t) \xi(t)$ and $\sin(2\omega t) \xi(t)$ over the slow time scale T . In order to do so, consider their correlations,

$$\langle \cos(2\omega t) \xi(t) \cos(2\omega t') \xi(t') \rangle = \epsilon^2 \langle \cos^2(2\omega t) \rangle \langle \xi(T) \xi(T') \rangle, \quad (29)$$

$$\langle \sin(2\omega t) \xi(t) \sin(2\omega t') \xi(t') \rangle = \epsilon^2 \langle \sin^2(2\omega t) \rangle \langle \xi(T) \xi(T') \rangle, \quad (30)$$

where $\xi_1(T)$ and $\xi_2(T)$ are independent Gaussian white noise with mean $\langle \xi_j(T) \rangle = 0$ and correlation $\langle \xi_j(T) \xi_k(T') \rangle = 2D \delta(T - T')$ if $j=k$ and 0 otherwise, with $j, k = \{1, 2\}$. We replace $\langle \cos^2(2\omega t) \rangle$ and $\langle \sin^2(2\omega t) \rangle$ by their time average values 1/2 in Eqs. (29) and (30). We thus define over the slow time scale,

$$\cos(2\omega t) \xi(t) \rightarrow \frac{\epsilon}{\sqrt{2}} \xi_1(T), \quad (31)$$

$$\sin(2\omega t) \xi(t) \rightarrow \frac{\epsilon}{\sqrt{2}} \xi_2(T). \quad (32)$$

We then have two coupled stochastic differential equations for which the fast time scale t is eliminated,

$$\begin{aligned}
\epsilon^3 \partial_T A(T) &= \epsilon \mu A(T) - \epsilon \nu B(T) - \frac{3}{4} \epsilon^3 A(T) [A^2(T) + B^2(T)] \\
&+ \epsilon^2 A(T) \xi_0(T) + \frac{\epsilon^2}{\sqrt{2}} A(T) \xi_1(T) + \frac{\epsilon^2}{\sqrt{2}} B(T) \xi_2(T), \\
\epsilon^3 \partial_T B(T) &= \epsilon \nu A(T) + \epsilon \mu B(T) - \frac{3}{4} \epsilon^3 B(T) [A^2(T) + B^2(T)] \\
&+ \epsilon^2 B(T) \xi_0(T) - \frac{\epsilon^2}{\sqrt{2}} B(T) \xi_1(T) + \frac{\epsilon^2}{\sqrt{2}} A(T) \xi_2(T). \quad (33)
\end{aligned}$$

The system of Eqs. (33) can then be written in matrix form,

$$\begin{aligned}
\frac{d}{dT} \begin{bmatrix} A \\ B \end{bmatrix} &= \begin{bmatrix} \tilde{\mu} & -\tilde{\nu} \\ \tilde{\nu} & \tilde{\mu} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} - \frac{3}{4} \begin{bmatrix} A(A^2 + B^2) \\ B(A^2 + B^2) \end{bmatrix} \\
&+ \frac{1}{\epsilon} \Sigma_0 \begin{bmatrix} A \\ B \end{bmatrix} \xi_0(T) + \frac{1}{\epsilon} \Sigma_1 \begin{bmatrix} A \\ B \end{bmatrix} \xi_1(T) \\
&+ \frac{1}{\epsilon} \Sigma_2 \begin{bmatrix} A \\ B \end{bmatrix} \xi_2(T), \quad (34)
\end{aligned}$$

with

$$\Sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \Sigma_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \quad \Sigma_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (35)$$

The Fokker-Planck equation corresponding to Eq. (34) is [19]

$$\begin{aligned} \frac{\partial}{\partial T} p(A, B, T) = & -\frac{\partial}{\partial A} \left\{ \left[\tilde{\mu}A - \tilde{\nu}B - \frac{3}{4}A(A^2 + B^2) \right] p(A, B, T) \right\} \\ & -\frac{\partial}{\partial B} \left\{ \left[\tilde{\nu}A + \tilde{\mu}B - \frac{3}{4}B(A^2 + B^2) \right] p(A, B, T) \right\} \\ & + \frac{D}{\epsilon^2} \left\{ A^2 \frac{\partial^2}{\partial A^2} + 2AB \frac{\partial^2}{\partial A \partial B} + B^2 \frac{\partial^2}{\partial B^2} \right. \\ & \left. + 5 \left(A \frac{\partial}{\partial A} + B \frac{\partial}{\partial B} \right) + 4 \right\} p(A, B, T) \\ & + \frac{D}{2\epsilon^2} \left\{ A^2 \frac{\partial^2}{\partial A^2} - 2AB \frac{\partial^2}{\partial A \partial B} + B^2 \frac{\partial^2}{\partial B^2} \right. \\ & \left. + \left(A \frac{\partial}{\partial A} + B \frac{\partial}{\partial B} \right) \right\} p(A, B, T) \\ & + \frac{D}{2\epsilon^2} \left\{ B^2 \frac{\partial^2}{\partial A^2} + 2AB \frac{\partial^2}{\partial A \partial B} + A^2 \frac{\partial^2}{\partial B^2} \right. \\ & \left. + \left(A \frac{\partial}{\partial A} + B \frac{\partial}{\partial B} \right) \right\} p(A, B, T). \quad (36) \end{aligned}$$

The intensity of the noise scales as $D = \epsilon^2 \tilde{D}$ and all terms of Eq. (36) are on the same order. In order to find the stationary solution of this Fokker-Planck equation, we change to polar coordinates. Let $A = r \cos(\theta)$ and $B = r \sin(\theta)$. Under this change of variable, the probability distribution function transforms as $\tilde{p}(r, \theta, T) = r p(A, B, T)$, where r is the Jacobian of the transformation. The drift terms of Eq. (36) are

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial T} \tilde{p}(r, \theta, T) \sim & -\frac{1}{r} \frac{\partial}{\partial r} \left[\left(\tilde{\mu}r - \frac{3}{4}r^3 \right) \tilde{p}(r, \theta, T) \right] \\ & - \frac{1}{r} \frac{\partial}{\partial \theta} [\tilde{\nu} \tilde{p}(r, \theta, T)]. \quad (37) \end{aligned}$$

On the other hand, terms in brackets proportional to \tilde{D} in Eq. (36) transform as

$$\begin{aligned} \left\{ A^2 \frac{\partial^2}{\partial A^2} + 2AB \frac{\partial^2}{\partial A \partial B} + B^2 \frac{\partial^2}{\partial B^2} + 5 \left(A \frac{\partial}{\partial A} + B \frac{\partial}{\partial B} \right) \right. \\ \left. + 4 \right\} p(A, B, T) = \left\{ r^2 \frac{\partial^2}{\partial r^2} + 5r \frac{\partial}{\partial r} + 4 \right\} \frac{\tilde{p}(r, \theta, T)}{r}, \quad (38) \end{aligned}$$

whereas the sum of the terms in brackets proportional to $\tilde{D}/2$ in Eq. (36) transforms as

$$\begin{aligned} \left\{ (A^2 + B^2) \left[\frac{\partial^2}{\partial A^2} + \frac{\partial^2}{\partial B^2} \right] + 2 \left(A \frac{\partial}{\partial A} + B \frac{\partial}{\partial B} \right) \right\} p(A, B, T) \\ = \left\{ r^2 \frac{\partial^2}{\partial r^2} + 3r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} \right\} \frac{\tilde{p}(r, \theta, T)}{r}. \quad (39) \end{aligned}$$

Combining Eqs. (37)–(39), the Fokker-Planck equation in polar coordinates is

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial T} \tilde{p}(r, \theta, T) = & -\frac{1}{r} \frac{\partial}{\partial r} \left[\left(\tilde{\mu}r - \frac{3}{4}r^3 \right) \tilde{p}(r, \theta, T) \right] \\ & - \frac{1}{r} \frac{\partial}{\partial \theta} [\tilde{\nu} \tilde{p}(r, \theta, T)] + \frac{\tilde{D}}{2} \left(3r^2 \frac{\partial^2}{\partial r^2} + 13r \frac{\partial}{\partial r} + 8 \right. \\ & \left. + \frac{\partial^2}{\partial \theta^2} \right) \frac{\tilde{p}(r, \theta, T)}{r}. \quad (40) \end{aligned}$$

We now make use of the identities

$$\frac{1}{r} \frac{\partial^2}{\partial r^2} \left[r^3 \frac{\tilde{p}(r, \theta, T)}{r} \right] = \left(r^2 \frac{\partial^2}{\partial r^2} + 6r \frac{\partial}{\partial r} + 6 \right) \frac{\tilde{p}(r, \theta, T)}{r}, \quad (41)$$

$$\frac{2}{r} \frac{\partial}{\partial r} \left[r^3 \frac{\partial}{\partial r} \frac{\tilde{p}(r, \theta, T)}{r} \right] = \left(2r^2 \frac{\partial^2}{\partial r^2} + 6r \frac{\partial}{\partial r} \right) \frac{\tilde{p}(r, \theta, T)}{r}, \quad (42)$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left[r^2 \frac{\tilde{p}(r, \theta, T)}{r} \right] = \left(r \frac{\partial}{\partial r} + 2 \right) \frac{\tilde{p}(r, \theta, T)}{r} \quad (43)$$

to rewrite the Fokker-Planck equation as

$$\begin{aligned} \frac{\partial}{\partial T} \tilde{p}(r, \theta, T) = & -\frac{\partial}{\partial r} \left\{ \left[\left(\tilde{\mu} + \frac{5\tilde{D}}{2} \right) r - \frac{3}{4}r^3 \right] \tilde{p}(r, \theta, T) \right\} \\ & + \frac{3\tilde{D}}{2} \frac{\partial^2}{\partial r^2} [r^2 \tilde{p}(r, \theta, T)] - \frac{\partial}{\partial \theta} [\tilde{\nu} \tilde{p}(r, \theta, T)] \\ & + \frac{\tilde{D}}{2} \frac{\partial^2}{\partial \theta^2} \tilde{p}(r, \theta, T). \quad (44) \end{aligned}$$

The stationary Fokker-Planck equation can now be solved by separation of variables. Let $\tilde{p}_s(r, \theta) = p_s(r) p_s(\theta)$. The stationary solution of the angular component, $\dot{p}_s(\theta) = 0$, satisfies

$$0 = -\frac{\partial}{\partial \theta} [\tilde{\nu} p_s(\theta)] + \frac{\tilde{D}}{2} \frac{\partial^2}{\partial \theta^2} p_s(\theta), \quad (45)$$

which leads to

$$p_s(\theta) = \mathcal{N}_\theta \exp\left(\frac{2\tilde{\nu}}{\tilde{D}} \theta\right), \quad (46)$$

where \mathcal{N}_θ is a normalization constant. The stationary solution of the radial component, $\dot{p}_s(r) = 0$, satisfies

$$0 = -\frac{\partial}{\partial r} \left\{ \left[\left(\tilde{\mu} + \frac{5\tilde{D}}{2} \right) r - \frac{3}{4} r^3 \right] p(r) \right\} + \frac{3\tilde{D}}{2} \frac{\partial^2}{\partial r^2} [r^2 p(r)], \quad (47)$$

which leads to

$$p_s(r) = \mathcal{N}_r |r|^{\alpha'} \exp\left(-\frac{r^2}{4\tilde{D}}\right), \quad (48)$$

where \mathcal{N}_r is another normalization constant. We have defined

$$\alpha' = \frac{1}{3} \left(\frac{2\tilde{\mu}}{\tilde{D}} - 1 \right). \quad (49)$$

The stationary probability distribution functions are normalized according to

$$1 = \int_0^{2\pi} \int_0^\infty \tilde{p}_s(r, \theta) dr d\theta = \left[\int_0^{2\pi} p_s(\theta) d\theta \right] \left[\int_0^\infty p_s(r) dr \right], \quad (50)$$

and we choose to normalize both components to 1. Normalization of the radial component leads to

$$\mathcal{N}_r = 2(4\tilde{D})^{-[(\alpha'+1)/2]} \Gamma^{-1} \left(\frac{\alpha'+1}{2} \right). \quad (51)$$

If $\alpha' < -1$, the probability distribution function of the radial component is negative, which is unphysical. The bifurcation threshold is then located at $\alpha' = -1$. This leads to the condition

$$\frac{1}{3} \left(\frac{2\tilde{\mu}}{\tilde{D}} - 1 \right) = -1 \quad (52)$$

or $\tilde{\mu} + \tilde{D} = 0$. Normalization of the angular component yields

$$\mathcal{N}_\theta = \left(\frac{\tilde{\nu}}{\tilde{D}} \right) \frac{\exp\left(-\frac{2\pi\tilde{\nu}}{\tilde{D}}\right)}{\sinh\left(\frac{2\pi\tilde{\nu}}{\tilde{D}}\right)}. \quad (53)$$

The dynamics in the deterministic limit is a limit cycle at threshold. Here, we choose the probability distribution function of the angular component to be uniform at threshold ($\alpha' = -1$) [25]. This occurs at $\tilde{\nu} = 0$ for all θ and the stationary probability distribution function of the angular component is $p_s(\theta) = 1/(2\pi)$. By combining both conditions we find

$$\tilde{\mu} + \tilde{D} = a_c + D + b \cos(\omega\tau) = 0, \quad (54)$$

$$\tilde{\nu} = \omega + b \sin(\omega\tau) = 0. \quad (55)$$

Those conditions [Eqs. (54) and (55)] justify the assumption that the parameters μ , ν and the intensity of the noise D scale as ϵ^2 close to the bifurcation.

By substituting Eq. (55) into Eq. (54), we find our final result for the Hopf bifurcation line,

$$-\left(\frac{a_c + D}{b}\right) = \cos[\tau\sqrt{b^2 - (a_c + D)^2}]. \quad (56)$$

Interestingly, this condition agrees with the bifurcation threshold of the first moment $\langle x \rangle$ from the linearization of Eq. (1) [7]. However, the bifurcation lines for higher moments of x within a linearized model differ from Eq. (56).

Figure 1 shows our prediction for the Hopf line and compares it to a numerical estimate obtained by direct integration of the original stochastic differential equation for $\tau = 1$. [9]. In the numerical study, the Hopf bifurcation threshold was determined as the point in the (a, b) plane for which the exponent of the power law of the stationary probability distribution $p(x)$ at small x is -1 . The same procedure was repeated for the stationary probability distribution function of the maximum amplitude of the Fourier transform of the trajectories $p(\max[X(\omega)])$ around the Hopf branch. The agreement is excellent, except close to the multicritical point, despite the fact that the delay time $\tau = 1$ is not small. Because characteristic relaxation times diverge near the bifurcation line, we anticipate that the validity of our analytic results is not confined to the range of small τ . In fact, it has been demonstrated that the correlation time of the dynamical variable $x(t)$ diverges at the bifurcation threshold [26]. Therefore, for any large but finite value of the time delay τ , there is an emergent time scale near the bifurcation that is much larger than τ . In this sense, we expect that the approximation of small τ relative to the diverging correlation time of $x(t)$ is not a significant limitation close to the bifurcation line.

We finally mention that our two analytic predictions do not intersect for large a and that the numerical values in the region around the intersection of the pitchfork and Hopf lines are subject to considerable statistical fluctuation.

ACKNOWLEDGMENTS

We are indebted to David Jasnow for useful discussions. This research has been supported by NSERC and the numerical calculations by CLUMEQ, which was supported by the Canada Foundation for Innovation. M.G. acknowledges funding by Le Fonds Québécois de la Recherche sur la Nature et les Technologies (FQRNT).

- [1] J. Belair and M. Mackey, *J. Dyn. Differ. Equ.* **1**, 299 (1989).
 [2] M. B. Elowitz and S. Leibler, *Nature (London)* **403**, 335 (2000).
 [3] J. Hasty, J. Pradines, M. Dolnik, and J. Collins, *Proc. Natl. Acad. Sci. U.S.A.* **97**, 2075 (2000).
 [4] D. Bratsun, D. Volfson, L. Tsimring, and J. Hasty, *Proc. Natl.*

- Acad. Sci. U.S.A.* **102**, 14593 (2005).
 [5] T. Galla, *Phys. Rev. E* **80**, 021909 (2009).
 [6] S. Guillouziec, I. L'Heureux, and A. Longtin, *Phys. Rev. E* **59**, 3970 (1999).
 [7] M. C. Mackey and I. G. Nechaeva, *Phys. Rev. E* **52**, 3366 (1995).

- [8] J. Lei and M. C. Mackey, *SIAM J. Appl. Math.* **67**, 387 (2007).
- [9] M. Gaudreault, F. Lepine, and J. Viñals, *Phys. Rev. E* **80**, 061920 (2009).
- [10] R. Graham and A. Schenzle, *Phys. Rev. A* **25**, 1731 (1982).
- [11] F. Drolet and J. Viñals, *Phys. Rev. E* **57**, 5036 (1998).
- [12] M. San Miguel and R. Toral, *Instabilities and Nonequilibrium Structures VI*, edited by E. Tirapegui, J. Martinez, and R. Tiemann (Kluwer Academic, The Netherlands, 2000).
- [13] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications* (Springer, Berlin, 1987).
- [14] T. D. Frank, *Phys. Rev. E* **72**, 011112 (2005).
- [15] R. Shao and Y. Chen, *Physica A* **388**, 977 (2009).
- [16] H. Zhang, W. Xu, Y. Xu, and D. Li, *Physica A* **388**, 3017 (2009).
- [17] R. D. Driver, *Ordinary and Delay Differential Equations* (Springer-Verlag, New York, 1977).
- [18] T. Saaty, *Modern Nonlinear Equations* (Dover Publications, New York, 1981).
- [19] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, 2nd ed. (Springer-Verlag, Berlin, 1997).
- [20] M. M. Klosek, *Acta Phys. Pol. B* **35**, 1387 (2004).
- [21] M. M. Klosek and R. Kuske, *Multiscale Model. Simul.* **3**, 706 (2005).
- [22] R. Kuske, in *IUTAM Symposium on Nonlinear Stochastic Dynamics*, edited by N. S. Namachchivaya and Y. K. Lin (Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003), p. 492.
- [23] P. H. Baxendale, in *IUTAM Symposium on Nonlinear Stochastic Dynamics*, edited by N. S. Namachchivaya and Y. K. Lin (Kluwer Academic Publishers, Dordrecht, The Netherlands, 2003), p. 492.
- [24] P. H. Baxendale, *Stochastic Proc. Appl.* **113**, 235 (2004).
- [25] F. Baras, M. M. Mansour, and C. van der Broeck, *J. Stat. Phys.* **28**, 577 (1982).
- [26] M. Gaudreault, J. M. Berbert, and J. Viñals, *Phys. Rev. E* (to be published).