Ornstein-Uhlenbeck process

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(Dated: January 31, 2011)

Abstract

PACS numbers: 02.30.Ks, 05.10.Gg, 05.70.Ln, 87.16.Yc, 87.18.Cf

I. CONTINUOUS LIMIT

The Fokker-Planck equation of an Ornstein-Uhlenbeck process is

$$\frac{\partial}{\partial t}p(x,t) = \lambda\frac{\partial}{\partial x}\left[(x-\mu)p(x,t)\right] + D\frac{\partial^2}{\partial x^2}p(x,t)\ , \tag{1}$$

where μ and λ are constants and where D is the intensity of the noise. The Langevin equation associated to Eq. (1) is

$$dx(t) = -\lambda(x-\mu)dt + dW(t)\ , \tag{2}$$

where $W(t)$ is a Weiner process with mean $\langle W(t)\rangle = 0$ and variance $\langle W^2(t)\rangle = 2Dt$. The stationary solution of Eq. (1) is

$$p_s(x) = \sqrt{\frac{\lambda}{2\pi D}}\exp\left[-\frac{\lambda}{2D}(x-\mu)^2\right]\ . \tag{3}$$

Equation (2) is understood under the Ito interpretation of stochastic calculus. It is integrated numerically by using a first order method,

$$x(t+\Delta t) = x(t) - \left\{\lambda[x(t)-\mu]\right\}\Delta t + \eta(t)\ , \tag{4}$$

where $\eta(t)$ is a gaussian white noise with mean $\langle\eta(t)\rangle = 0$ and correlation $\langle\eta(t)\eta(t')\rangle = 2D\delta(t-t')$.

II. DISCRETE LIMIT

Consider the following chemical reactions in order to model the Ornstein-Uhlenbeck process:

$$\emptyset \xrightarrow{k_+} r \qquad r \xrightarrow{k_-} \emptyset\ . \tag{5}$$

Using the Law of Mass Action, the Master equation corresponding to the above network is,

$$\frac{\partial}{\partial t}p(r,t) = \Omega k_+[\mathbb{E}^{-1}-1]\left\{p(r,t)\right\} + k_-\left[\mathbb{E}-1\right]\left\{rp(r,t)\right\}\ , \tag{6}$$

where Ω is a coarse-grained variable with unit of volume, $\mathbb{E}$ is the raising operator $\mathbb{E}_n f(n) = f(n+1)$, and where k_+ and k_- are reaction rates. We use the van Kampen inverse system size method to solve this equation. Make the change of variable

$$r(t) = \Omega y(t) + \Omega^{1/2}\zeta(t)\ , \tag{7}$$

where the solution $r(t)$ is splited into a deterministic part $y(t)$ and a fluctuating part $\zeta(t)$ around the stationary solution. The operator $\mathbb{E}$ is expanded so that

$$\mathbb{E} = 1 + \Omega^{-1/2}\partial_\zeta + (2\Omega)^{-1}\partial_\zeta^{-2} + \mathcal{O}(\Omega^{-3/2}) \; . \tag{8}$$

Substitute in Eq. (6),

$$\begin{aligned}\partial_t\Pi - \Omega^{1/2}\dot{y}\partial_\zeta\Pi = \Omega k_+ \left[-\Omega^{-1/2}\partial_\zeta + (2\Omega)^{-1}\partial_\zeta^2\right]\Pi + k_- \left[\Omega^{-1/2}\partial_\zeta + (2\Omega)^{-1}\partial_\zeta^2\right] \times \\ \left\{\left(\Omega y + \Omega^{1/2}\zeta\right)\Pi\right\} \; .\end{aligned} \tag{9}$$

Collecting terms that are factor of $\Omega^{1/2}$ leads to a deterministic equation,

$$\dot{y}(t) = k_+ - k_- y(t) \; . \tag{10}$$

This equation has one fixed point located at $y^* = k_+/k_-$. Collect terms that are factors of Ω^0 leads to a Fokker-Planck equation,

$$\partial_t\Pi(\zeta,t) = -k_-\partial_\zeta\left\{\zeta\Pi(\zeta,t)\right\} + 2^{-1}\left[k_+ + y(t)k_-\right]\partial_\zeta^2\Pi(\zeta,t) \; . \tag{11}$$

In the stationary regime, trajectories converge to $y^* = k_+/k_-$. Evaluating Eq. (11) at the fixed point leads to

$$\partial_t\Pi(\zeta,t) = -k_-\partial_\zeta\left\{\zeta\Pi(\zeta,t)\right\} + k_+\partial_\zeta^2\Pi(\zeta,t) \; . \tag{12}$$

Associated to Eq. (12) is a Langevin equation,

$$\dot{\zeta}(t) = k_-\zeta(t) + \sqrt{k_+}\eta(t) \; , \tag{13}$$

where $\eta(t)$ is a Gaussian white noise with mean 0 and correlation $\langle\eta(t)\eta(t')\rangle = 2\delta(t-t')$. Equation (13) represents an Ornstein-Uhlenbeck process. The stationary solution of Eq. (12) is a Gaussian distribution,

$$\Pi_s(\zeta) = \mathcal{N}e^{\frac{k_-}{2k_+}\zeta^2} \; . \tag{14}$$

Time evolution of the n^{th} moment can also be found using Eq. (12),

$$\partial_t\langle\zeta^n(t)\rangle = -nk_-\langle\zeta^n(t)\rangle + n(n-1)k_+\langle\zeta^{n-2}(t)\rangle \; . \tag{15}$$

The stationary moments are thus,

$$\langle\zeta\rangle_s = 0 \; , \tag{16}$$

$$\langle\zeta^2\rangle_s = k_+/k_- \; , \tag{17}$$

The solution of the linear noise approximation is a Gaussian distribution with mean and variance,

$$\begin{aligned} \langle r \rangle_s &= \Omega y^* + \Omega^{1/2}\langle \zeta \rangle_s = \Omega \tfrac{k_+}{k_-} \ , \\ \langle\langle r^2 \rangle\rangle_s &= \Omega \langle\langle \zeta^2 \rangle\rangle_s = \Omega \tfrac{k_+}{k_-} \ , \end{aligned} \tag{18}$$

leading to

$$p_s(r) = \sqrt{\frac{k_-}{2\pi\Omega k_+}} e^{-\frac{\Omega k_-}{2k_+}\left[\frac{r}{\Omega} - \left(\frac{k_+}{k_-}\right)\right]^2} \ . \tag{19}$$

To model extrinsic noise, we would like the network defined by Eq. (5) to act as multiplicative noise given that the species r is coupled to another reactant. Note that Eqs. (3) and (19) are identical if

$$x = \frac{r}{\Omega} \qquad ; \qquad \mu = \frac{k_+}{k_-} \qquad ; \qquad \frac{\lambda}{D} = \Omega \frac{k_-}{k_+} \ . \tag{20}$$

In order to compare the theoretical predictions of Eqs. (3) and (19), note that the domain of the distribution in the discrete limit has to be redefine according to $x = r/\Omega$. The distribution has to be normalized accordingly.

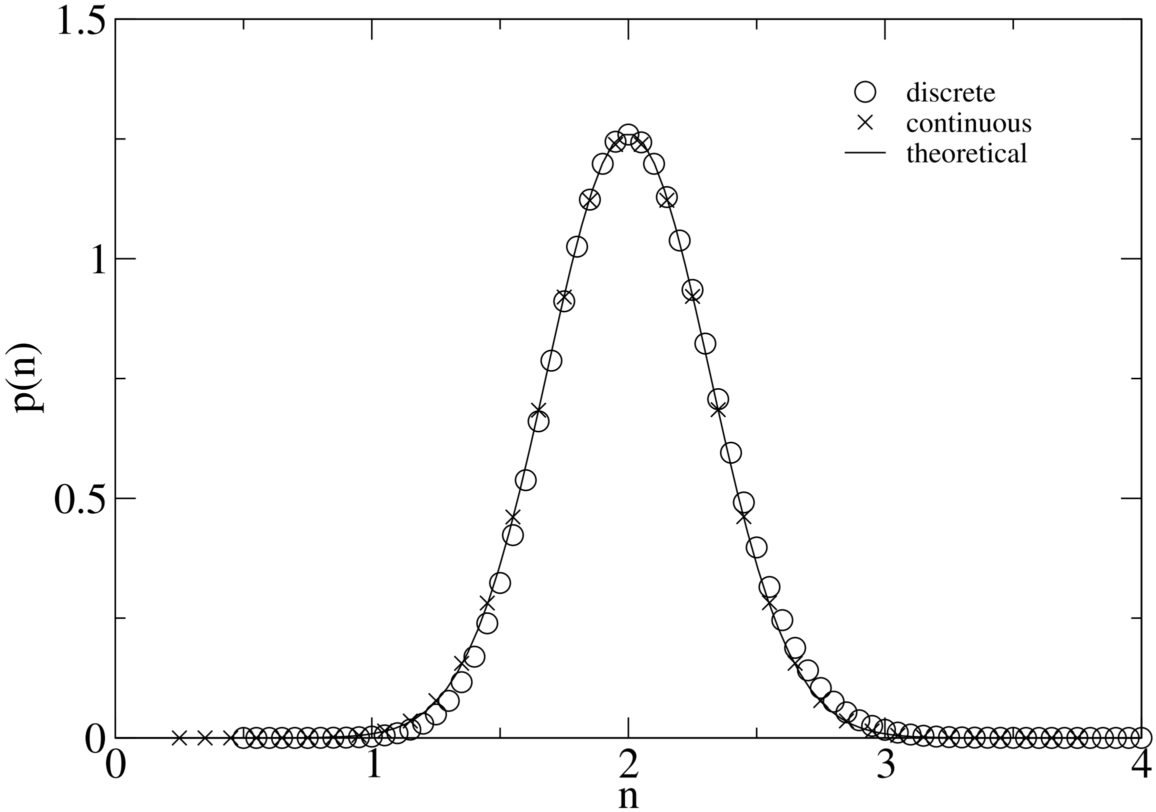

FIG. 1: Comparison of various probability densities. We show result from the simulation of the chemical reactions Eq. (5) with the Gillespie algorithm ($\circ$). The stationary probability distribution function $p_s(r)$ as a function of r is obtained with $\Omega = 20$, $k_+ = 2$, and $k_- = 1$. This distribution is compared to the stationary density $p_s(n)$ as a function of n calculated from a first order numerical algorithm ($\times$) with $\lambda = 1$ and $D = 0.1$. The two distributions are compared to the analytical prediction (solid curve), Eqs. (3) and (19).