OPERADS AND CONFORMAL FIELD THEORY
LECTURE 1: CONFORMAL FIELD THEORY

ALEXANDER A. VORONOV

Introduction

From a mathematician's point of view, physics is a science which studies too particular things. For example, quantum field theory is basically the science about computing integrals over $\mathbb{R}^n$, not even general integrals, but integrals of the following very specific form

$$\int_{\mathbb{R}^n} f_1(x) \dots f_N(x) e^{-S(x)} dx.$$

Physicists have developed very advanced methods to compute such integrals. For example, the famous Feynman diagram technique allows us to compute the above integral for a rather particular function $S(x)$, a small perturbation of a positive definite quadratic form in $\mathbb{R}^n$. Sometimes, elaborate techniques of physicists to compute simple things find an amazing application to very general things in mathematics, such as invariants of smooth four-manifolds.

The purpose of this course is to give a mathematical introduction to two-dimensional quantum field theory (2d QFT), which includes such theories as conformal field theory (CFT), topological quantum field theory (TQFT), string theory, quantum gravity, and cohomological field theory. 2d QFT studies some particular objects related to Riemann surfaces (that is why 2d) or complex algebraic curves. These objects are close counterparts of the mathematical theory of operads, whose study is another purpose of the course. 2d QFT supplies ideas and motivation for such fields of mathematics as enumerative algebraic geometry, theory of moduli spaces, representation theory, low-dimensional topology, theory of knots, and a few others. That is why I believe it should be standard part of education of every serious mathematician.

Theory of operads was created by algebraic topologists (J. P. May, J. M. Boardman, R. M. Vogt) in the seventies, but the very structure, later dubbed an "operad" by Peter May, had made its appearance in the sixties in the works of Jim Stasheff on loop spaces and Murray Gerstenhaber on algebraic structures arising in deformation theory. Theory of operads is basically the theory of operations, as the name suggests, and no wonder, operations, such as the cup product, the Steenrod squares, the Dyer-Lashof operations, are an essential part of algebraic topology, whereas any product or bracket is what algebra is all about. Operads seemed to have played their role in the seventies, in the problem of recognizing a loop space in a given topological space, and since then has become in topology something like a steam engine. But in the nineties, the operads unexpectedly came back (Let me remind you that steam engines are not so harmful to the environment), allowing

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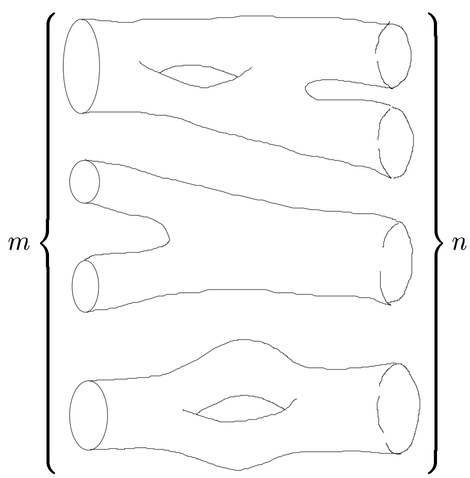
a break-through (V. Ginzburg, M. M. Kapranov) in understanding such complicated algebraic structures as A_∞- and L_∞-algebras, as well as the discovery of the relevance of operads to 2d QFT (J. Stasheff, M. Kontsevich, E. Getzler), which provided an effective tool of studying the complicated algebraic structure of the physical theories. Learning how it works will be our end-of-term goal.

1. Basic notions

1.1. **Conformal field theory.** Conformal field theory (CFT) provides a geometric background of string theory, which is considered as one of the steps toward *grand unification*, the unification of all forces of nature in a single theory. Whereas standard physics treats a particles as an ideal point, string theory thinks of a particle as a tiny loop or string. Respectively, a particle propagating in space along a path, a world line, becomes a string propagating along a world sheet, which is an orientable surface. The standard theory is quantized by using Feynman integral over the path space, whereas Feynman integral in string theory is an integral over the space of orientable surfaces. This integral can ultimately be integrated out to an integral over the moduli space of Riemann surfaces. To write down such an integral, one usually has to begin with certain data associated to Riemann surfaces. This data, called a CFT, should satisfy certain axioms. We will now describe G. Segal's approach to CFT, see [5, 3, 6].

A *conformal field theory* (*CFT*) is a complex vector space V, called the *state space*, together with a correspondence



$$\longmapsto \qquad |\Sigma\rangle : V^{\otimes m} \to V^{\otimes n}$$

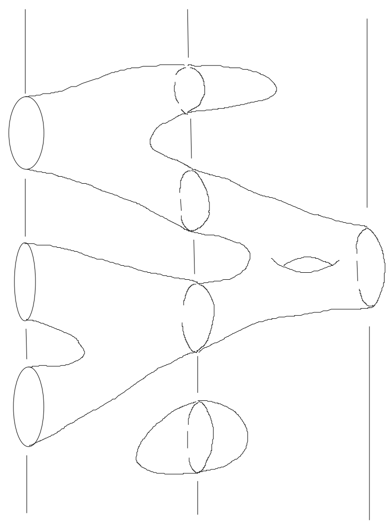
An orientable surface Σ bounding $m + n$ circles

A linear operator $|\Sigma\rangle$

Here a surface Σ is a (not necessarily connected) compact Riemann surface, a complex manifold of dimension one. It has labeled (enumerated), nonoverlapping holomorphic holes, which are nothing but biholomorphic embeddings $\phi : D^2 \to \Sigma$, where $D^2 = \{z \in \mathbb{C} \mid |z| < 1\}$ is the standard unit disk. One can think of ϕ as the choice of a holomorphic coordinate z at the hole. The first $m \geq 0$ circles are called *inputs* and the remaining $n \geq 0$ circles are called *outputs*. The notation $V^{\otimes n}$ means $V \otimes \cdots \otimes V$ n times.

This correspondence should satisfy the following axioms.

1. **Conformal invariance**: The linear mapping $|\Sigma\rangle$ is invariant under isomorphisms of the surface Σ taking holes to the corresponding holes and preserving the holomorphic coordinates there.
2. **Permutation equivariance**: The correspondence $\Sigma \mapsto |\Sigma\rangle$ commutes with the action of the symmetric groups S_m and S_n on surfaces and linear mappings by permutations of inputs and outputs.
3. **Factorization (sewing) property**: Sewing along the boundaries of the holes corresponds to composing of the corresponding operators:

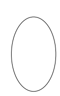


$$\longmapsto \qquad V^{\otimes m} \to V^{\otimes n} \to V^{\otimes k}$$

The sewing of the outputs of a surface with the inputs of another surface

The composition of the corresponding linear operators

Here *sewing along the boundaries* of two holomorphic holes with coordinates z_1 and z_2 means identifying two tubular neighborhoods $1/r < |z_1| < r$ and $1/r < |z_2| < r$, $r > 1$, of the boundaries via $z_1 = 1/z_2$.
4. **Normalization**:



$$\longmapsto \qquad \mathrm{id} : V \to V$$

The unit circle

The identity operator

Here the unit circle is understood as the "cylinder of zero width", the Riemann sphere $S^2 = \overline{\mathbb{R}}$ with the standard coordinate z and two holomorphic holes of radius one around 0 and ∞.
5. **Smoothness**: Sometimes one requires that the operator $|\Sigma\rangle$ depends smoothly (or continuously) on the Riemann surface Σ. To make this assumption, one needs to assume that V is at least a topological space, introduce a structure of smooth compex manifold on the space of linear mappings, and think of Σ as a point of the infinite dimensional moduli space of Riemann surfaces with holes. This axiom will not be essential to our considerations for the time being, and we will omit it. If instead of smoothness, we assumed *holomorphicity* in a certain sense, then we would be talking about a *chiral CFT* or a *vertex operator algebra* (*VOA*).

Remark 1.1. This definition describes in fact a CFT of *central charge* $c = 0$. An arbitrary central charge CFT relaxes Axiom 3: the operator $|\Sigma_2 \cup \Sigma_1\rangle$ corresponding to the result of sewing of two Riemann surfaces Σ_1 and Σ_2 is equal to the composition of two operators $|\Sigma_2\rangle \circ |\Sigma_1\rangle$ up to a nonzero factor λ:

$$|\Sigma_2 \cup \Sigma_1\rangle = \lambda|\Sigma_2\rangle \circ |\Sigma_1\rangle.$$

Throughout this course we will be mostly dealing with $c = 0$ theories.

Exercise 1. The constant λ generalizes the notion of a two-cocycle on $\mathrm{Diff}(S^1)$. Find out an equation of this type on λ. Is this the only condition λ must satisfy?

1.2. **Categorical Terms.** These data and axioms can be formulated equivalently using tensor categories and functors, see [4] or [2] regarding basics of tensor (symmetric monoidal) categories. Within this approach, a CFT is a multiplicative functor from a "topological" tensor category "Segal" to a "linear" tensor category "Vect". An object of the category Segal is an integer $n \geq 0$. A morphism between two numbers m and n is an isomorphism class of Riemann surfaces with $m + n$ holomorphic holes. The identity morphism of an object n is the union of n zero-width cylinders. The addition of numbers and the disjoint union of Riemann spheres introduce tensor products on the objects and the morphisms, i.e., the structure of a tensor category on Segal.

The other category Vect is the category of complex vector spaces, not necessarily finite-dimensional, with the usual tensor product.

Exercise 2. Show that a CFT is a tensor functor from category Segal to category Vect. Ignore Axiom 5, which translates as a certain smoothness property of the functor.

1.3. **PROP's.** At this point, we have come very closely to the notion of a PROP, see *e.g.*, J. Adams [1], the big brother of an operad, so that it would be unwise not to mention it here.

Definition 1.2. A *PROP* is a tensor category whose objects are the nonnegative integers with the tensor product given by

$$m \otimes n = m + n.$$

PROP stands for PROducts, the compositions of morphisms in the category, and Permutations.

Note that we have kept the freedom of choice regarding the morphisms: the above category Segal with morphisms given by complex cobordisms, *i.e.*, Riemann surfaces bounding holomorphic circles, is the first example of a PROP. Another example is the *endomorphism PROP of a vector space* V: the set of morphisms from m to n is defined as $\mathrm{Mor}(m, n) = \mathrm{Hom}(V^{\otimes m}, V^{\otimes n})$. Exercise 2 can be reformulated as the following proposition.

Proposition 1.3. *A CFT is a morphism from the PROP* Segal *to the endomorphism PROP of a vector space* V.

A PROP is indeed the big brother of an operad: an *operad* is basically the part $\mathrm{Mor}(n, 1)$ of a PROP. What this means will be the topic of the lecture after the next lecture.

References

[1] J. F. Adams, *Infinite loop spaces*, Annals of Mathematics Studies, vol. 90, Princeton University Press; University of Tokyo Press, Princeton, N.J.; Tokyo, 1978.

[2] P. Deligne and J. S. Milne, *Tannakian categories*, Hodge cycles, motives, and Shimura varieties, Springer-Verlag, Berlin-New York, 1982.

[3] R. H. Dijkgraaf, *A geometrical approach to two-dimensional conformal field theory*, Ph.d. thesis, University of Utrecht, 1989.

[4] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics, vol. 155, Springer-Verlag, New York, 1995.

[5] G. Segal, *Two-dimensional conformal field theories and modular functors*, IXth International Congress on Mathematical Physics (Swansea, 1988), Hilger, Bristol, 1989, pp. 22–37.

[6] A. A. Voronov, *Topological field theories, string backgrounds and homotopy algebras*, Adv. Appl. Clifford Algebras **4** (1994), no. Suppl. 1, 167–178, Differential geometric methods in theoretical physics (Ixtapa-Zihuatanejo, 1993).