LECTURE 2: MODULAR FUNCTOR

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1. Modular Functor

In physical examples, especially when one deals with the structure of chiral, holomorphic CFT, it is more natural to consider a more general theory, where a Riemann surface Σ with holomorphic holes is assigned a finite-dimensional space V_Σ of operators rather than a unique operator $|\Sigma\rangle$. This situation is formalized by the notion of a modular functor, which starts with a finite set I of indices, with an involution $\alpha \mapsto \bar{\alpha}$ for $\alpha \in I$ defined and a unit element $\mathbf{1} \in I$, such that $\bar{\mathbf{1}} = \mathbf{1}$, fixed. The physical meaning of this set I is the space of quantized momenta of the string. A *modular functor* assignes a Riemann surface Σ with n holomorphic holes labeled not only by the numbers 1, ... , n, but also by a vector $\alpha = (\alpha_1, \dots, \alpha_n)$ of elements of the index set I, a vector space $V_{\Sigma,\alpha}$ depending holomorphically on Σ:

$$(\Sigma, \alpha) \mapsto V_{\Sigma,\alpha}, \qquad \dim V_{\Sigma,\alpha} < \infty.$$

The holomorphic dependance is understood as follows. Every Riemann surface gives rise to a vector space holomorphically depending on the class of a Riemann surface means that the moduli space of Riemann surfaces with holes (or just the base S any holomorphic family $X \to S$) is provided with a holomorphic vector bundle $\mathcal{V}$ whose fiber over the Riemann surface Σ is V_Σ.

This assignment must satisfy the following axioms.

1. **Disjoint union**: $V_{\Sigma_1 \coprod \Sigma_2, \alpha_1 \coprod \alpha_2} = V_{\Sigma_1,\alpha_1} \otimes V_{\Sigma_2,\alpha_2}$.
2. **Sewing**: If Σ is a Riemann surface with $n+2$ holes, and $\widehat{\Sigma}$ is the result of sewing the $n+1$st hole to the $n+2$nd hole, and α is an indexing of the remaining n holes on $\widehat{\Sigma}$, then
$$V_{\widehat{\Sigma},\alpha} = \bigoplus_{\beta \in I} V_{\Sigma,(\alpha,\beta,\bar{\beta})}.$$
3. **Normalization 1**: If D is the unit disk, understood as the Riemann sphere with one hole around infinity: $D = S^2 \setminus \{|z| > 1\}$, then
$$V_{D,\alpha} = \begin{cases} \mathbb{C} & \text{if } \alpha = \mathbf{1}, \\ 0 & \text{otherwise.} \end{cases}$$
4. **Normalization 2**: If Σ is an annulus, then
$$\dim V_{\Sigma,(\alpha_1,\alpha_2)} = \begin{cases} 1 & \text{if } \alpha_2 = \bar{\alpha}_1, \\ 0 & \text{otherwise.} \end{cases}$$

Here is a number of consequences of the above axioms, which make the story more similar to the set of axioms of a CFT.

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Exercise 1. If Σ_1 and Σ_2 are sewn end-to-end, then there exists a composition map

$$V_{\Sigma_1,(\alpha,\beta)} \otimes V_{\Sigma_2,(\bar{\beta},\gamma)} \to V_{\Sigma_1 \cup \Sigma_2,(\alpha,\gamma)}.$$

Exercise 2. If Σ is a torus, then $V_\Sigma = \langle I \rangle_{\mathbb{C}}$, the linear span of the index set I.

Theorem 1.1 (Physics folklore and G. Segal)**.** *If $X \to S$ is a holomorphic family of surfaces, there exists a canonical flat projective connection in the holomorphic vector bundle $\mathcal{V}$ defined by the modular functor.*

Remark 1.2. Here a *projective connection* means an isomorphism $p_* : V_{X_s,\alpha} \to V_{X'_s,\alpha}$, defined for any path p conneting points s and $s' \in S$ up to a nonzero scalar factor. A projective connection is *flat*, if p_* does not change if the path p is deformed smoothly leaving its ends fixed.

Proof. The proof is based on the following exercise on differential geometry.

Exercise 3.
1. If a Lie algebra $\mathfrak{g}$ acts locally transitively on a manifold M, *i.e.*, there is a morphism $\mathfrak{g} \to \operatorname{Vect}(M)$, such that the evaluation map $\mathfrak{g} \to T_m M$ is surjective for any point $m \in M$, and the action of $\mathfrak{g}$ on M lifts to an action on a vector bundle V over M, then there exists a natural flat connection on V
2. Same, assuming the action of $\mathfrak{g}$ on V is projective: show that there is a flat projective connection on V.

Assume for simplicity that the Riemann surfaces in question have only one holomorphic hole.

Lemma 1.3. *The tangent space to the space of Riemann surfaces X with one holomorphic hole is naturally isomorphic to* $\operatorname{Vect}_{\mathbb{C}}(S^1)/\operatorname{Vect}(X)$*, where* $\operatorname{Vect}_{\mathbb{C}}(S^1)$ *is the space of smooth vector fields on the circle S^1 (in fact, those which come from smooth vector fields on an annulus containing S^1), and* $\operatorname{Vect}(X)$ *is the space of holomorphic vector fields on the complement of the disk in X identical on the boundary of the hole.*

Proof of Lemma. The classical theory of Beltrami differentials suggests that an infinitesimal deformation of the complex structure on a Riemann surface preserving the holomorphic hole is governed by the class of a smooth $(-1,1)$-form $\mu d\bar{z}/dz$, z being a local holomorphic coordinate on the Riemman surface, $\mu = 0$ on the hole, modulo the $(-1,1)$-forms of the type $\bar{\partial}\eta d\bar{z}/dz$, where η/dz is a smooth vector field on the surface vanishing on the hole, an infinitesimal diffeomorphism. Computing this cohomology group in Čech cohomology using the covering by two open sets, a neighborhood of the hole and a neighborhood of the complement, we see that the tangent space is given by $\operatorname{Vect}(S^1)/\operatorname{Vect}(X)$. □

□

Corollary. *Given a modular functor, the projective space $\mathbb{P}(V_\Sigma)$ is naturally associated to a smooth surface Σ without any choice of a complex structure.*

Proof. The fact is that the Teichmüller space, the space of conformal classes of complex structures on a Riemann surface (before taking the quotient by the diffeomorphism group of the Riemann surface) is contractible. Therefore, lifting the projective connection to the Teichmüller space, we get an identification of the fibers of the corresponding projective bundle. □

2. To be added later

Example: loop groups, coinvariants
Verlinde Algebra