LECTURE 3: TOPOLOGICAL QUANTUM FIELD THEORY AND FROBENIUS ALGEBRAS

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1. Topological quantum field theory and Frobenius algebras

We saw in Lecture 2 that given a modular functor, the vector space V_Σ associated to a Riemann surface depends, up to a scalar, only on the diffeomorphism class of the Riemann surface Σ. This defines what is called a topological quantum field theory. We will be mostly interested in the situation when the central charge $c = 0$ and there is a specific operator $|\Sigma\rangle$ (as opposed to a vector space of such operators) corresponding at least continuously to a surface Σ. This all translates as follows into our language.

Definition 1.1. A *Topological Quantum Field Theory* (*TQFT*) is a correspondence of the same type as a Conformal Field Theory, see Lecture 1, except that the conformal invariance axiom is replaced with diffeomorphism invariance: the operator $|\Sigma\rangle$ must be invariant under diffeomorphisms of the surface Σ taking holes to the corresponding holes and preserving the holomorphic coordinates there. The normalization axiom is convenient to change so as the operator corresponding to a cylinder (a sphere with two holes) is equal to identity.

Theorem 1.2 (Folklore)**.** *The structure of a TQFT based on vector space V is equivalent to the structure of a finite-dimensional Frobenius algebra on it. A* Frobenius algebra *structure on a vector space V means the structure of a commutative associative algebra with a unit and a nondegenerate symmetric bilinear form $\langle , \rangle : V \otimes V \to \mathbb{C}$ which is invariant with respect to the multiplication: $\langle ab, c\rangle = \langle a, bc\rangle$.*

Proof. Let V be the state space of a TQFT. We would like to construct the structure of a Frobenius algebra on V. Consider the following Riemann surfaces and the

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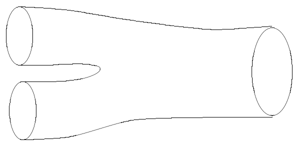
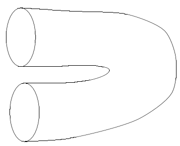
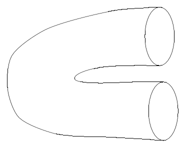
corresponding operators, which we denote ab, $\langle a, b\rangle$, *etc.*



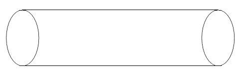
$\longmapsto \quad V \otimes V \to V, \qquad a \otimes b \mapsto ab$



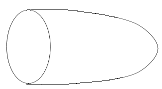
$\longmapsto \quad V \otimes V \to \mathbb{C}, \qquad a \otimes b \mapsto \langle a, b\rangle$



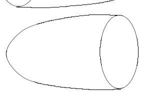
$\longmapsto \quad \psi : \mathbb{C} \to V \otimes V$



$\longmapsto \quad \mathrm{id} : V \to V$



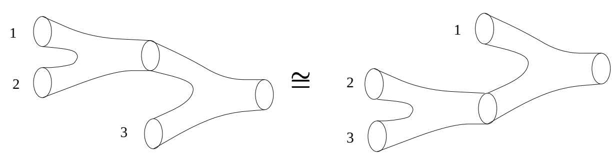
$\longmapsto \quad \mathrm{tr} : V \to \mathbb{C}$



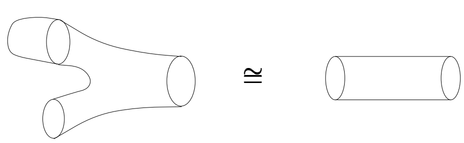
$\longmapsto \quad \mathbb{C} \to V, \qquad 1 \mapsto e$

We claim that these operators define the structure of a Frobenius algebra on V. Indeed, the multiplication is commutative, because if we interchange labels at the legs of a pair of pants we will get a diffeomorphic Riemann surface. Therefore, the corresponding operator $a \otimes b \mapsto ba$ will be equal to ab. Similarly, the associativity $(ab)c = a(bc)$ of multiplication is based on the fact that the following two surfaces are diffeomorphic:

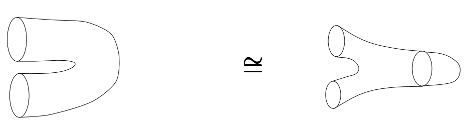
(1.1)



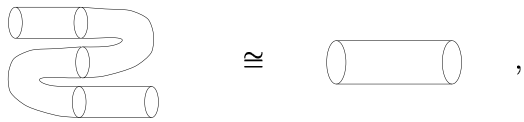
The property $ae = ea = a$ of the unit element comes from the diffeomorphism



Thus, we see that V is a commutative associative unital algebra. Now, the following diffeomorphism



proves the identity $\langle a, b\rangle = \mathrm{tr}(ab)$, which, along with the associativity, implies $\langle ab, c\rangle = \langle a, bc\rangle$. The fact that the inner product $\langle\ ,\ \rangle$ is nondegenerate follows from the diffeomorphism



which implies that the composite mapping

$$\begin{array}{ccccc} V & \xrightarrow{\mathrm{id}\otimes\psi(1)} & V\otimes V\otimes V & \xrightarrow{\langle,\rangle\otimes\mathrm{id}} & V, \\ v & \longmapsto & \sum_{i=1}^n v\otimes u_i\otimes v_i & \longmapsto & \sum_{i=1}^n \langle v, u_i\rangle v_i \end{array}$$

is equal to $\mathrm{id} : V \to V$. This also implies that $\dim V < \infty$, because the v_i's, $i = 1, \dots, n$, must span V. This completes the construction of the structure of a Frobenius algebra on the state space V of a TQFT.

Conversely, if we have a finite-dimensional Frobenius algebra V, we can define the structure of a TQFT on the vector space V by (1) cutting a Riemann surface down into pairs of pants, cylinders, and caps; (2) defining the operators corresponding to those basic objects using the multiplication (or its linear dual), the identity map, and the unit element $e \in V$ (or the dual of the map $\mathbb{C} \to V$, $1 \mapsto e$, as the trace functional), respectively; and (3) using the sewing axiom. The fact that the composite operator is independent of the way we cut down the surface follows from (1.1) and the associativity of multiplication. □

Exercise 1. Find a more elegant way (not using any elements of V) to show that $\dim V < \infty$.