LECTURE 4: OPERADS

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1. Operads

1.1. Definitions and examples. A commutative algebra is a Frobenius algebra without an invariant inner product. Similarly, an operad is a PROP without things that can possibly create something like an inner product in a representation. More specifically, an operad is the part $\mathrm{Mor}(n,1)$, $n \geq 0$, of a PROP. Of course, given only the collection of morphisms $\mathrm{Mor}(n,1)$, it is not clear how to compose them. The idea is to take the union of a m elements from $\mathrm{Mor}(n,1)$ to be able to compose them with an element of $\mathrm{Mor}(m,1)$. This leads to cumbersome notation and ugly axioms, compared to those of PROP's. However operads are in a sense more basic than the corresponding PROP's: as Victor Kac put it on this lecture, the difference is similar to the difference between Lie algebras and the universal enveloping algebras.

Definition 1.1. Let k be a ground field. An *operad* $\mathcal{O}$ is a collection of sets (vector spaces, complexes, topological spaces, manifolds, ... , objects of a tensor category) $\mathcal{O}(n)$, $n \geq 0$, with

1. A composition law:
$$\gamma : \mathcal{O}(m) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_m) \to \mathcal{O}(n_1 + \cdots + n_m).$$
2. A right action of the symmetric group S_n on $\mathcal{O}(n)$.
3. A unit $e \in \mathcal{O}(1)$.

such that the following properties are satisfied:

1. The composition is associative, i.e., the following diagram is commutative:
$$\begin{array}{ccc} \left\{ \begin{array}{l} \mathcal{O}(l) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_l) \\ \qquad \otimes \mathcal{O}(n_{11}) \otimes \cdots \otimes \mathcal{O}(n_{l,n_l}) \end{array} \right\} & \xrightarrow{\mathrm{id} \otimes \gamma^l} & \mathcal{O}(l) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_l) \\ \Big\downarrow {\scriptstyle \gamma \otimes \mathrm{id}} & & \Big\downarrow {\scriptstyle \gamma} \\ \mathcal{O}(m) \otimes \mathcal{O}(n_{11}) \otimes \cdots \otimes \mathcal{O}(n_{m,n_m}) & \xrightarrow{\gamma} & \mathcal{O}(n) \end{array},$$
where $m = \sum_i m_i$, $n_i = \sum_j n_{ij}$, and $n = \sum_i n_i$.
2. The composition is equivariant with respect to the symmetric group actions: $S_m \times S_{n_1} \times \cdots \times S_{n_m}$ acts on the left-hand side and maps naturally to $S_{n_1+\cdots+n_m}$, acting on the right-hand side.
3. The unit e satisfies natural properties with respect to the composition: $\gamma(e; f) = f$ and $\gamma(f; e, \dots, e) = f$ for each $f \in \mathcal{O}(k)$.

The notion of a *morphism of operads* is introduced naturally.

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Remark 1.2. One can consider *nonsymmetric operads*, not assuming the action of the symmetric groups. Not requiring the existence of a unit e, we arrive to *nonunital operads*. Do not mix this up with operads with no $\mathcal{O}(0)$, algebras over which (see next section) have no unit. There are also good examples of operads having only $n \geq 2$ components $\mathcal{O}(n)$.

An equivalent definition of an operad may be given in terms of operations $f \circ_i g = \gamma(f; , \mathrm{id}, \dots, \mathrm{id}, g, \mathrm{id}, \dots, \mathrm{id})$, $i = 1, \dots, m$, for $f \in \mathcal{O}(m), g \in \mathcal{O}(n)$. Then the associativity condition translates as $f \circ_i (g \circ_j h) = (f \circ_i g) \circ_{i+j-1} h$.

Example 1.3 (The Riemann surface and the endomorphism operad)**.** $\mathcal{O}(n)$ is the space of all connected Riemann surfaces (with or without complex structure) with n inputs and 1 output. Another example is the *endomorphism operad of a vector space* V: $\mathcal{E}nd_V(n) = \mathrm{Hom}(V^{\otimes n}, V)$, the space of n-linear mappings from V to V.

1.2. **Algebras over operads.**

Definition 1.4. An *algebra over an operad* $\mathcal{O}$ (in other terminology, a *representation of an operad*) is a morphism of operads $\mathcal{O} \to \mathcal{E}nd_V$, that is, a collection of maps

$$\mathcal{O}(n) \to \mathcal{E}nd_V(n) \qquad \text{for } n \geq 0$$

compatible with the symmetric group action, the unit elements, and the compositions. If the operad $\mathcal{O}$ is an operad of vector spaces, then we would usually require the morphism $\mathcal{O} \to \mathcal{E}nd_V$ to be a morphism of operads of vector spaces. Otherwise, we would think of this morphism as a morphism of operads of sets. Sometimes, we may also need a morphism to be continuous or respect differentials, or have other compatibility conditions.

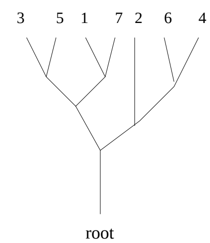
1.2.1. *The commutative operad.* The *commutative operad* is the operad of k-vector spaces with the nth component $\mathcal{C}omm(n) = k$ for all $n \geq 0$. We assume that the symmetric group acts trivially on k and the compositions are just the multiplication of elements in the ground field k. An algebra over the commutative operad is nothing but a commutative associative algebra with a unit, as we see from the following exercise.

Another version of the commutative operad is $\mathcal{C}omm(n) = \text{point}$ for all $n \geq 0$. This is an operad of sets. An algebra over it is also the same as a commutative associative unital algebra.

Exercise 1. Show that the operad $\mathcal{T}(n) = \{$the set of diffeomorphism classes of connected Riemann surfaces of genus zero with n input holes and 1 output hole$\}$ is isomorphic to the commutative operad of sets.

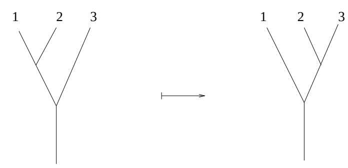
Exercise 2. Prove that the structure of an algebra over the commutative operad $\mathcal{C}omm$ on a vector space is equivalent to the structure of a commutative associative algebra with a unit.

1.2.2. *The associative operad.* The *associative operad* $\mathcal{A}ssoc$ can be considered as a 1d analogue of the commutative operad $\mathcal{T}(n)$. $\mathcal{A}ssoc(n)$ is the set of equivalence classes of connected planar binary (each vertex being of valence 3) trees that have a root edge and n leaves labeled by integers 1 through n:



If $n = 1$, there is only one tree — it has no vertices and only one edge connecting a leaf and a root. If $n = 0$, the only tree is the one with no vertices and no leaves — it only has a root. Unfortunately, I have a problem sketching it: it probably exists only in the quantum world.

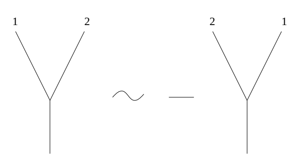
Two trees are equivalent if they are related by a sequence of moves of the kind



performed over pairs of two adjacent vertices of a tree. The symmetric group acts by relabeling the leaves, as usual. The composition is obtained by grafting the roots of m trees to the leaves of an m-tree; no new vertices being created at the grafting points. Note that this is similar to sewing Riemann surfaces and erasing the seam, just as we did to define operad composition in that case. By definition, grafting a 0-tree to a leaf just removes the leaf and, if this operation creates a vertex of valence 2, we should erase the vertex.

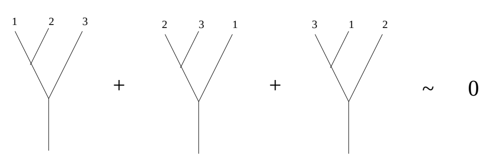
Exercise 3. Prove that the structure of an algebra over the associative operad $\mathcal{A}ssoc$ on a vector space is equivalent to the structure of an associative algebra.

1.2.3. *The Lie operad.* The *Lie operad* $\mathcal{L}ie$ is another variation on the theme of a tree operad. Consider the same binary trees as for the associative operad, except that we do not include the zero-tree, *i.e.*, the operad has only positive components $\mathcal{L}ie(n)$, $n \geq 1$, and there are now two kinds of equivalence relations:



Skew Symmetry

and



Jacobi Identity

Now that we have arithmetic operations in the equivalence relations, we consider the Lie operad as an operad of vector spaces. We also assume that the ground field is of characteristic zero, because otherwise we will arrive to the wrong definition of a Lie algebra.

Exercise 4. Prove that the structure of an algebra over the Lie operad $\mathcal{L}ie$ on a vector space over a field of characteristic zero is equivalent to the structure of a Lie algebra.

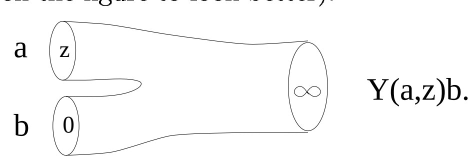
Exercise 5. Describe algebraically an algebra over the operad $\mathcal{L}ie$, if we modify it by including the zero-tree, whose composition with any other tree is defined as (a) zero, (b) for the associative operad.

1.2.4. *The Poisson operad.* Recall that a *Poisson algebra* is a vector space V (over a field of characteristic zero) with a unit element e, a dot product ab, and a bracket $[a, b]$ defined, so that the dot product defines the structure of commutative associative unital algebra, the bracket defines the structure of a Lie algebra, and the bracket is a derivation of the dot product:

$$[a, bc] = [a, b]c + b[a, c] \qquad \text{for all } a,\ b, \text{ and } c \in V.$$

Exercise 6. Define the *Poisson operad*, using a tree model similar to the previous examples. Show that an algebra over it is nothing but a Poisson algebra. [*Hint*: Use two kinds of vertices, one for the dot product and the other one for the bracket.]

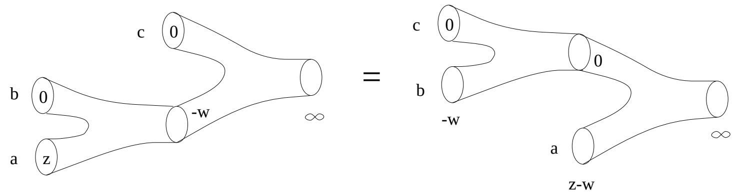
1.2.5. *The Riemann surface operad and vertex operator algebras.* Just for a change, let us return to the operad of Riemann surfaces $\mathcal{P}$ — this time, of isomorphism classes of genus zero Riemann surfaces with holomorphic holes. What is an algebra over it? Since there are infinitely many nonisomorphic pairs of pants, there are infinitely many (at least) binary operations. In fact, we have an infinite dimensional family of binary operations parameterized by classes of pairs of pants. However modulo the unary operations, those which correspond to cylinders, we have only one fundamental binary operation corresponding to a fixed pair of pants. An algebra over this operad $\mathcal{P}$ is part of a CFT data at the tree level, $c = 0$. If we consider a holomorphic algebra over this operad, that is, require that the defining mappings $\mathcal{P}(n) \to \mathcal{E}nd_V(n)$, where V is a complex vector space, be holomorphic, then we get part of a chiral CFT, or an object which may be called a *vertex operator algebra* (*VOA*). This kind of object is not equivalent to what people used to call a VOA, but according to Huang's Theorem, a true VOA is a holomorphic algebra over a "partial pseudo-operad of Riemann surfaces with rescaling", which is a version of $\mathcal{P}$, where the disks are allowed to overlap. The fundamental binary operation $Y(a, z)b$ for $a, b \in V$ of a VOA is commonly chosen to be the one corresponding to a pair of pants which is the Riemann sphere with a standard holomorphic coordinate and three unit disks around the points 0, z, and ∞ (No doubt, these disks overlap badly, but we shrink them on the figure to look better):



The famous associativity identity

$$Y(a, z - w)Y(b, -w)c = Y(Y(a, z)b, -w)c$$

for vertex operator algebras comes from the following natural isomorphism of the Riemann surfaces: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