

## LECTURE 6: OPERADS VIA GENERATORS AND RELATIONS

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### 1. OPERADS VIA GENERATORS AND RELATIONS

The tree operads that we looked at in Lecture 4, such as the associative and the Lie operads, are actually operads defined by generators and relations. Here is a way to define such operads in general. To fix notation, assume throughout this section that we work with operads  $\mathcal{O}(n)$ ,  $n \geq 1$ , of vector spaces.

**Definition 1.1.** An *ideal* in an operad  $\mathcal{O}$  is a collection  $\mathcal{I}$  of  $S_n$ -invariant subspaces  $\mathcal{I}(n) \subset \mathcal{O}(n)$ , for each  $n \geq 1$ , such that whenever  $i \in \mathcal{I}$ ,  $\gamma(\dots, i, \dots) \in \mathcal{I}$ .

The intersection of an arbitrary number of ideals in an operad is also an ideal, and one can define the ideal generated by a subset in  $\mathcal{O}$  as the minimal ideal containing the subset.

**Definition 1.2.** The *quotient operad*  $\mathcal{O}/\mathcal{I}$  is the collection  $\mathcal{O}(n)/\mathcal{I}(n)$ ,  $n \geq 1$ , with the structure of operad induced by that on  $\mathcal{O}$ .

The *free operad*  $F(V)$  generated by a collection  $V = \{V(n) \mid n \geq 1\}$  of vector spaces, is defined as follows.

$$F(V)(n) = \bigoplus_{n\text{-trees } T} V(T),$$

where the summation runs over all planar rooted trees  $T$  with  $n$  labeled leaves and

$$V(T) = \bigotimes_{v \in T} V(\text{in}(v)),$$

the summation running over all vertices  $v$  of the tree  $T$ ,  $\text{in}(v)$  being the number of incoming edges for the vertex  $v$  (the edges are directed toward the root). In other words, an element of  $F(V)(n)$  is an planar  $n$ -tree whose vertices decorated with elements of  $V$ . There is a special tree with no vertices:



The component  $F(V)(1)$  contains the one-dimensional subspace spanned by this tree.

The following data defines an operad structure on  $F(V)$ .

1. The identity element is the special tree in  $F(V)(1)$  with no vertices.
2. The symmetric group  $S_n$  acts on  $F(V)(n)$  by relabeling the inputs.
3. The operad composition is given by grafting the roots of trees to the leaves of another tree. No new vertices are created.

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**Definition 1.3.** Now let  $R$  be a subset of  $F(V)$ , i.e., a collection of subsets  $R(n) \subset F(V)(n)$ . Let  $(R)$  be the ideal in  $F(V)$  generated by  $R$ . The quotient operad  $F(V)/(R)$  is called the *operad generated by  $V$  with defining relations  $R$* .

**Exercise 1.** The associative operad of Lecture 4 is the operad generated by  $V = V(2) = k$  with defining relation given by the associativity condition, see Section 1.2.2 of Lecture 4.

**Exercise 2.** The Lie operad of Lecture 4 is the operad generated by  $V = V(2) = k$  with defining relations given by the skew symmetry condition and the Jacobi identity, see Section 1.2.3 of Lecture 4.