LECTURE 9: HOMOTOPY ALGEBRA

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1. Homotopy Algebra

The idea of a *homotopy something algebra* is to relax the axioms of the *something algebra*, so that the usual identities are satisfied up to homotopy. For example in a homotopy Lie algebra, the Jacobi identity looks like

$$[[a,b],c] \pm [[b,c],a] \pm [[c,a],b] \text{ is homotopic to zero.}$$

Or in a homotopy Gerstenhaber (G-) algebra, the Leibniz rule is

$$[a,bc] - [a,b]c \mp b[a,c] \text{ is homotopic to zero.}$$

Usually, a homotopy something algebra arises when one wants to lift the structure of a something algebra *a priori* defined on cohomology to the level of cochains.

Exercise 1. Try to lift the BV structure on the cohomology of a TVOA as defined by Lian and Zuckerman, see Lecture 8 or [**?**], to the level of cochains.

This kind of relaxation seems to be too much for many, practical and categorical, purposes, and one usually requires that the null-homotopies, regarded as new operations, satisfy their own identities, up to their own homotopy. These homotopies should also satisfy certain identities up to homotopy and so on. This resembles Hilbert's chains of syzygies in early homological algebra.

Operads are especially helpful when one needs to work with homotopy something algebras. We already know that defining the class of something algebras is equivalent to defining the something operad. Thus, if we have an operad $\mathcal{O}$, what is *the* homotopy $\mathcal{O}$ operad? In such generality, there is no completely satisfactory answer to this question. Here is one possible answer: *a homotopy $\mathcal{O}$ operad* is a resolution $h\mathcal{O}$ of $\mathcal{O}$ in the category of operads of complexes, *i.e.*, an operad of complexes whose cohomology is $\mathcal{O}[0]$, the operad $\mathcal{O}$ sitting in degree zero, if it was an operad of vector spaces, and the operad $\mathcal{O}$ sitting in the original degrees, if it was already an operad of graded vector spaces.

Example 1.1. The singular chain operad $\{C_\bullet(D(n)) | n \geq 1\}$ of the little disks operad D is obviously a homotopy G-operad: its cohomology is the operad $H_\bullet(D)$, which we know is the G-operad. However, this homotopy G-operad is too big. In an algebra over it, there are even infinitely many dot products, corresponding to points in $D(2)$, which are all homotopic to each other (because $D(2)$ is connected). Also, the operad structure on the singular chains of a topological operad is rather complicated: the map $C_\bullet(X) \otimes C_\bullet(Y) \to C_\bullet(X \times Y)$ needed to define the operad composition is defined using a cumbersome shuffle formula.

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However, one manages to define *the* homotopy something operad for certain specific classes of operads. For example, Ginzburg and Kapranov [?] do it for so-called quadratic operads. Markl [?] defines the homotopy something operad as the minimal model, a notion he introduces, of the something operad. Since homotopy something algebras are usually defined by concrete examples, we will just describe a few examples of homotopy something operads.

1.1. **Homotopy Lie operad and algebras.**

Definition 1.2. A *homotopy Lie algebra* is a complex $V = \sum_{i \in \mathbb{Z}} V^i$ with a differential d, $d^2 = 0$, of degree 1 and a collection of n-ary brackets:

$$[v_1, \dots, v_n] \in V, \qquad v_1, \dots, v_n \in V, \ n \geq 2,$$

which are homogeneous of degree $2 - n$ and graded skew:

$$[v_1, \dots, v_i, v_{i+1}, \dots, v_n] = -(-1)^{|v_i||v_{i+1}|}[v_1, \dots, v_{i+1}, v_i, \dots, v_n],$$

$|v|$ denoting the degree of $v \in V$, and satisfy the relations

$$d[v_1, \dots, v_n] - (-1)^n \sum_{i=1}^{n} \epsilon(i)[v_1, \dots, dv_i, \dots, v_n]$$
$$= \sum_{\substack{k+l=n \\ k \geq 2, l \geq 1}} \sum_{\substack{\text{unshuffles } \sigma: \\ \{1,2,\dots,n\} = I_1 \cup I_2, \\ I_1 = \{i_1, \dots, i_k\}, \ I_2 = \{j_1, \dots, j_l\}}} \epsilon(\sigma) \operatorname{sign}(\sigma)(-1)^{kl}[[v_{i_1}, \dots, v_{i_k}], v_{j_1}, \dots, v_{j_l}],$$

where $\epsilon(i) = (-1)^{|v_1| + \dots + |v_{i-1}|}$ is the sign picked up by taking d through $v_1, \dots,$ v_{i-1}, $\operatorname{sign}(\sigma)$ is the sign of the permutation σ, and $\epsilon(\sigma)$ is the sign picked up by the elements v_i passing through the v_j's during the unshuffle of $v_1, \dots, v_n$, as usual in superalgebra.

According to a Hinich-Schechtman theorem [?], homotopy Lie algebras can be described as algebras over a certain tree operad, which is encoded in the topology of the moduli spaces due to Beilinson and Ginzburg [?]. We will recall these results briefly.

To be finished ...