Math 4242
Spring 2017
Final, section 1
Time Limit: 120 minutes

Name (Print): ______________________

Student ID: ______________________

This exam contains 13 pages (including this cover page) and 7 problems. Check to see if any pages are missing.

You may not use your books or calculators in this exam, and you may not bring any notes other than **two letter-sized double sided cheat sheets**.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided.
- **Mysterious or unsupported answers will *not* receive full credit**. A correct answer, unsupported by calculations or explanations will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **To cite a result** from class or the textbook, you should paraphrase the result and note it as a prior result.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	16	
2	13	
3	7	
4	12	
5	13	
6	4	
7	15	
Total:	80	

1. (16 points) For each statement below, determine whether it is true or false and give a brief explanation.

   (a) The function
   $$\det : \mathcal{M}_{n\times n} \to \mathbb{R}$$
   that takes an $n \times n$ matrix to its determinant is linear.

   (b) The formula
   $$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\| = \min(|x|, |y|)$$
   defines a norm on $\mathbb{R}^2$.

   (c) The formula
   $$\left\langle \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\rangle = w_1\overline{z_1} + \overline{w_2}z_2$$
   defines an inner product on $\mathbb{C}^2$.

   (d) A square matrix whose diagonal entries are all negative may have a positive eigenvalue.

(e) If A is a nonsingular symmetric matrix, then A^{-1} is also symmetric.

(f) If A is any $n \times n$ matrix, then $|\det A|$ is the product of the singular values of A.

(g) If A is a symmetric matrix, then its singular values are equal to its eigenvalues.

(h) If a vector space V has an inner product, it must be finite dimensional.

2. (13 points) Let

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 2 & -1 \\ -1 & 5 & -2 \end{pmatrix}.$$

Find an invertible matrix S such that $S^{-1}AS$ is in Jordan normal form, and write down that Jordan normal form.

3. (7 points) Find a QR decomposition of

$$A = \begin{pmatrix} 3 & 7 \\ 4 & 1 \end{pmatrix}.$$

(Hint: if you can find Q, you can easily compute R via the formula

$$R = Q^T A.)$$

4. Let $(\mathbb{R}^n)^* = \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ be the dual space of $\mathbb{R}^n$, defined as the space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}$. Recall that $(\mathbb{R}^n)^*$ can be thought of as the space of length n row vectors.

   If $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is a basis of $\mathbb{R}^n$, the row vectors

$$\mathbf{v}_1^*, \mathbf{v}_2^*, \cdots, \mathbf{v}_n^* \in (\mathbb{R}^n)^*$$

   are defined by

$$\mathbf{v}_i^*(\mathbf{v}_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

   (a) (5 points) Let

$$A = (\mathbf{v}_1 \ \mathbf{v_2} \ \cdots \ \mathbf{v}_n)$$

   be the matrix formed by taking the $\mathbf{v}_i$ as columns. Express the row vectors $\mathbf{v}_1^*, \mathbf{v}_2^*, \cdots, \mathbf{v}_n^*$ in terms of A. Justify your answer.

   (b) (3 points) Show that $\mathbf{v}_1^*, \mathbf{v}_2^*, \cdots, \mathbf{v}_n^*$ is a basis of $(\mathbb{R}^n)^*$. (This is called the *dual basis* to $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$.)

(c) (4 points) Let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Find $\mathbf{v}_1^*$ and $\mathbf{v}_2^*$. (This may be done independently of parts a) and b)), but part a) might help.)

5. Let

$$K = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 5 & 3 \\ -2 & 3 & x \end{pmatrix}.$$

(a) (5 points) For which values of x is K positive definite?

(b) (8 points) Suppose $x = 3$. Find a basis of $\mathbb{R}^3$ which is orthogonal with respect to the inner product $\langle\, ,\, \rangle_K$, where

$$\langle \mathbf{v}, \mathbf{w} \rangle_K = \mathbf{v}^T K \mathbf{w}.$$

6. (4 points) Suppose A is a 3×3 matrix such that $\operatorname{tr}(A) = -4$, $\det(A) = -6$, and there is some vector $\mathbf{v} \in \mathbb{R}^3$ such that

$$A\mathbf{v} = \mathbf{v}.$$

What are the eigenvalues of A and their multiplicities?

7. (a) (3 points) Give a condition on a, b and c for $(a\ b\ c)^T$ to belong to the range of

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$$

Show that $(4\ 2\ \ -4)^T$ is not in the range of A.

(b) (9 points) Find the pseudoinverse of A.

Blank space for calculations.

(c) (3 points) Using part b), or otherwise, find the least squares approximate solution to the linear system

$$\begin{aligned} x &= 4 \\ y &= 2 \\ x - y &= -4. \end{aligned}$$