

Math 5615H: Honors: The algebra of limits in
 \mathbb{R} , \mathbb{C} , and \mathbb{R}^n
Subsequences and sequential compactness

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The Definition of a Limit of a Sequence

Definition A sequence $\{a_n\}$ in a metric space X has limit $L \in X$ if $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} : \forall n > N$ we have $d(a_n, L) < \varepsilon$.

The Algebra of Sequences in \mathbb{R} and \mathbb{R}^n ^{a} and \mathbb{C}

Theorem

Suppose for two sequences in \mathbb{R} , \mathbb{C} , or \mathbb{R}^n

$$\lim_{k \rightarrow \infty} a_k = a \text{ and } \lim_{k \rightarrow \infty} b_k = b.$$

$$(\forall c \in \mathbb{R} \text{ (or } \mathbb{C})) \lim_{k \rightarrow \infty} (ca_k) = ca.$$

1 Scalar Multiplication

$\forall c \in \mathbb{R}$ (or \mathbb{C}) $\lim(ca_k)$ exists and equals $c \cdot a$.

2 Sum $\lim_{k \rightarrow \infty} (a_k + b_k) = a + b$

3 Product (not for \mathbb{R}^n) $\lim_{k \rightarrow \infty} (a_k \cdot b_k) = a \cdot b$

$b_k \neq 0 \forall k$

4 Quotient (not for \mathbb{R}^n) $\lim_{k \rightarrow \infty} (a_k / b_k) = \frac{a}{b}$, provided $b \neq 0$.

Proof of (3): Given $\epsilon > 0$, let's choose N_1 : $\forall n > N_1, |a_n - a| < \epsilon/2M$
and N_2 : $\forall n > N_2, |b_n - b| < \epsilon/2M$, unless $|a|$ is too large, then drop $|a|$.
Then $\forall n > N = \max(N_1, N_2)$,
 $|a_n b_n - ab| = |a_n b_n - a b_n + a b_n - ab| \leq$

$$\leq |b_n| \cdot |a_n - a| + \underbrace{|a| \cdot |b_n - b|}$$

$$< |b_n| \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2} \quad (\text{want } < \varepsilon)$$

$$\leq |b_n| \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2}$$

$$< M \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2} = \varepsilon.$$

Recall $\{b_n\}$ bdd, being
convergent, i.e.,
 $\exists M > 0: |b_n| < M \forall n$
Adjust $N_1 \subseteq$

The rest: see text.

Subsequences

$\{a_n\}$ sequence, $n_1 < n_2 < \dots$ infinite sequence of naturals. ^{is a metric space X}
Then $\{a_{n_k}\}$ is called a *subsequence* and its limit, if it exists, a *subsequential limit* of $\{a_n\}$. **Observe:** $n_k \geq k \quad \forall k \geq 1$.

Theorem

Every subsequence of a convergent sequence converges.

Proof. $\lim_{n \rightarrow \infty} a_n = L, a_n \in X.$ } Given
 $\{a_{n_k}\}$ subsequence

$\forall \varepsilon > 0$ take $N \in \mathbb{N}! \forall n > N \quad d(a_n, L) < \varepsilon.$
Then $\forall k > N$, we have $d(a_{n_k}, L) < \varepsilon$,
because $n_k \geq k > N$. \square

Subsequences in Compact Subsets

Theorem

If K is a compact subset of a metric space X , then every sequence in K has a subsequence that converges to a point in K .

Proof. (By contrapositive: If K has a sequence with no subsequence converging to a point in K , then K is not compact.)

Enough to consider sequences with infinite range, because every sequence with a finite range has a convergent subsequence.

$\{a_n\} \subset K$
 $\forall x \in K \exists \varepsilon > 0: B_\varepsilon(x)$ will contain $< \infty$ many terms of the sequence (b/c x is not otherwise, easy to find a subsequence converging to x). $\{B_\varepsilon(x) \mid x \in K\}$ open cover

this cover has no finite subcover. Otherwise,

$\exists \{B_{\varepsilon_1}(x_1), B_{\varepsilon_2}(x_2), \dots, B_{\varepsilon_n}(x_n)\}, \bigcup_{i=1}^n B_{\varepsilon_i}(x_i) \supseteq K$
 \Rightarrow sequence has a finite range $\{a_n\}$
□

Sequential Compactness

Definition

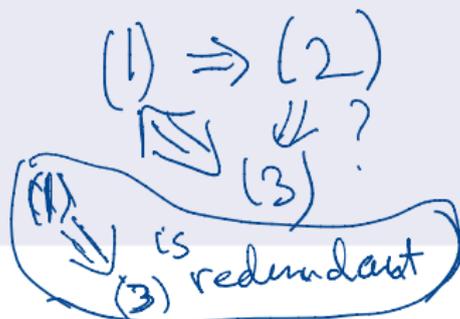
Let X be a metric space. A subset $K \subset X$ is *sequentially compact* if every sequence in K has a subsequence that converges to a point in K .

Compare to the *Bolzano-Weierstrass property*: every infinite subset of K has a limit (cluster) point in K .

Theorem

$K \subset X$ TFAE:

- 1 K is compact;
- 2 K is sequentially compact;
- 3 K has the B-W property.



Proof. (1) \Rightarrow (2): Previous theorem.

(1) \Rightarrow (3): A theorem proven last Friday, 10/02/2020.

(3) \Rightarrow (1): A problem on the Midterm.

Proof of Compactness Criterion, continued

To complete the proof: (2) (3)

The simplest thing now: Show $(3) \Rightarrow (2)$.

Next time ...