Math 5615H: Honors: The algebra of limits in $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{R}^n$
Subsequences and sequential compactness
[redacted]

Sasha Voronov

University of Minnesota

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The Definition of a Limit of a Sequence

Definition A sequence $\{a_n\}$ in a metric space X has limit $L \in X$ if $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} : \forall n > N$ we have $d(a_n, L) < \varepsilon$.

The Algebra of Sequences in $\mathbb{R}$ and $\mathbb{R}^n$ a and $\mathbb{C}$

Theorem

Suppose for two sequences in $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{R}^n$

$$\lim_{k\to\infty} a_k = a \text{ and } \lim_{k\to\infty} b_k = b.$$

$(\forall c \in \mathbb{R}$ (or $\mathbb{C}$) $\lim_{k\to\infty}(ca_k) = ca.$

1. *Scalar Multiplication* $\forall c \in \mathbb{R}$ (or $\mathbb{C}$) $\lim(ca_k)$ exists and equals $c \cdot a$.
2. *Sum* $\lim_{k\to\infty}(a_k + b_k) = a + b$
3. *Product (not for $\mathbb{R}^n$)* $\lim_{k\to\infty}(a_k \cdot b_k) = a \cdot b$
4. *Quotient (not for $\mathbb{R}^n$)* $\lim_{k\to\infty}(a_k / b_k) = \frac{a}{b}$, provided $b_k \neq 0\ \forall k$ and $b \neq 0$.

Proof of (3): Given $\varepsilon > 0$, let's choose N_1: $\forall n \geq N_1$ $|a_n - a| < \varepsilon/2M$, and N_2: $\forall n \geq N_2$ $|b_n - b| < \varepsilon/2|a|$ (unless $|a| \neq 0$, then drop $|a|$). Take $N = \max(N_1, N_2)$.
Then $\forall n > N$ $|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \leq$

$$\leq |b_n| \cdot |a_n - a| + |a| \cdot |b_n - b|$$

$$< |b_n| \cdot \varepsilon/2M + \varepsilon/2 \quad (\text{want} < \varepsilon)$$

$$\leq |b_n| \cdot \varepsilon/2M + \frac{\varepsilon}{2}$$

$$< M \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2} = \varepsilon .$$

Recall $\{b_n\}$ bdd, being convergent, i.e., $\exists M > 0 : |b_n| < M \ \forall n$

Adjust N_1

The rest: see text.

Subsequences

$\{a_n\}$ sequence in a metric space X, $n_1 < n_2 < \dots$ infinite sequence of naturals. Then $\{a_{n_k}\}$ is called a *subsequence* and its limit, ~~of~~ if exists, a *subsequential limit* of $\{a_n\}$. **Observe**: $n_k \geq k \quad \forall k \geq 1$.

Theorem

Every subsequence of a convergent sequence converges.

Proof.

$$\left.\begin{array}{l} \lim_{n\to\infty} a_n = L, \quad a_n \in X. \\ \{a_{n_k}\} \text{ subsequence} \end{array}\right\} \text{Given}$$

$\forall \varepsilon > 0$ take $N \in \mathbb{N}$: $\forall n > N$ $d(a_n, L) < \varepsilon$.

Then $\forall k > N$, we have $d(a_{n_k}, L) < \varepsilon$,

because $n_k \geq k > N$. $\square$

Subsequences in Compact Subsets

Theorem

If K is a compact subset of a metric space X, then every sequence in K has a subsequence that converges to a point in K.

Proof. (By contrapositive: If K has a sequence with no subsequence converging to a point in K, then K is not compact.)
Enough to consider sequences with infinite range, because every sequence with a finite range has a convergent subsequence. $\{a_n\} \subset K$

$\forall x \in K \ \exists \varepsilon > 0$: $B_\varepsilon(x)$ will contain $< \infty$ many terms of the sequence (b/c ~~x is not~~ otherwise, easy to find a subsequence converging to x). $\{B_\varepsilon(x) \mid x \in K\}$ open cover this cover has no finite subcover. Otherwise,

$$\exists \{B_{\varepsilon_1}(x_1), B_{\varepsilon_2}(x_2), \ldots, B_{\varepsilon_n}(x_n)\}, \quad \bigcup_{i=1}^{n} B_{\varepsilon_n}(x_n) \supseteq K \supseteq \{a_n\}$$

$\Rightarrow$ sequence has a finite range $\square$

Sequential Compactness

Definition

Let X be a metric space. A subset $K \subset X$ is *sequentiqally compact* if every sequence in K has a subsequence that converges to a point in K.

Compare to the *Bolzano-Weirestrass property*: every infinite subset of K has a limit (cluster) point in K.

Theorem

$K \subset X$ TFAE:

1. *K is compact;*
2. *K is sequentially compact;*
3. *K has the B-W property.*

$(1) \Rightarrow (2)$
$\Downarrow$?
(3)

$(1) \Rightarrow (3)$ is redundant

Proof. $(1) \Rightarrow (2)$: Previous theorem.
$(1) \Rightarrow (3)$: A theorem proven last Friday, 10/02/2020.
$(3) \Rightarrow (1)$: A problem on the Midterm.

Proof of Compactness Criterion, continued

To complete the proof: (2) (3)

The simplest thing now: Show (3) $\Rightarrow$ (2).

Next time ...