Math 5615 Honors: Applications of the Derivative to Analysis

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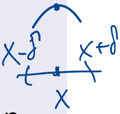
Relative extrema

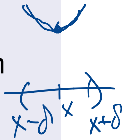
Definition

Let $f : (a, b) \to \mathbb{R}$.

1. The function f has a *relative (local) maximum* at a point $x \in (a, b)$ if there is a $\delta > 0$ such that $f(s) \leq f(x)$ for all $s \in (x - \delta, x + \delta)$. The *relative (local) maximum value* is then $f(x)$.

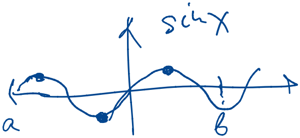


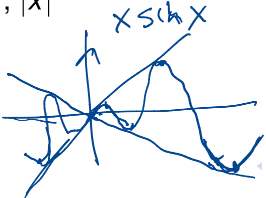
2. The function f has a *relative (local) minimum* at a point $x \in (a, b)$ if there is a $\delta > 0$ such that $f(s) \geq f(x)$ for all $s \in (x - \delta, x + \delta)$. The *relative (local) minimum value* is then $f(x)$.

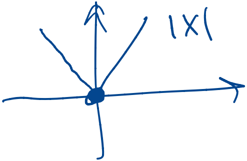


3. A *relative extremum* is a relative maximum or minimum.

Examples: $\sin x$, $x \sin x$, $|x|$







Relative extrema and critical points

The derivative is a great tool to find extrema:

Theorem-Definition

If $f : (a, b) \to \mathbb{R}$ and f has either a relative extremum at $c \in (a, b)$, and if $f'(c)$ exists, then c is a *critical point* of f, *i.e.*, $f'(c) = 0$.

Proof. f has a rel. max. at $c \in (a,b)$ (WLOG):

$\exists \delta > 0 : |x - c| < \delta \Rightarrow f(x) - f(c) \leq 0$

For $x > c$, $0 < x - c < \delta$, we have

$\dfrac{f(x) - f(c)}{x - c} \leq 0 \Rightarrow \lim_{x \to c+} \dfrac{f(x) - f(c)}{x - c} \leq 0$

The $\lim_{x \to c+}$ exists, b/c $\lim_{x \to c} \dfrac{f(x)-f(c)}{x-c} = f'(c)$. Thus $\lim_{x \to c+} \dfrac{f(x)-f(c)}{x-c} = \lim_{x \to c} \dfrac{f(x)-f(c)}{x-c} = f'(c)$

For $x < c$, $0 < c - x < \delta$, $\dfrac{f(x) - f(c)}{x - c} \geq 0 \Rightarrow \lim_{x \to c-} \dfrac{f(x)-f(c)}{x-c} \geq 0$

Thus, $0 \leq f'(c) \leq 0 \Rightarrow f'(c) = 0$ $\square$

Rolle's Theorem

The following particular case of the mean value theorem easily follows from the previous theorem on extrema and critical pts.

Theorem (Rolle)

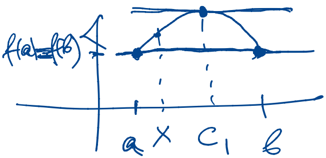
Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, then there is a point $c \in (a, b)$ such that $f'(c) = 0$.

Proof.

If $f =$const, done, as $f'(c) = 0$ at each point $c \in (a, b)$.

If not constant, then there must be $x \in (a, b)$ such that $f(x) \neq f(a)$. Then $f(x) > f(a) = f(b)$ or $f(x) < f(a) = f(b)$. Since f is continuos, it attains an absolute maximum at some $c_1 \in [a, b]$ and an absolute minimum at $c_2 \in [a, b]$. If $f(x) > f(a)$, then $c_1 \in (a, b)$. Then $f'(c_1) = 0$ by previous theorem. If $f(x) < f(a)$, then $c_2 \in (a, b)$ and $f'(c_2) = 0$ by previous theorem. $\square$

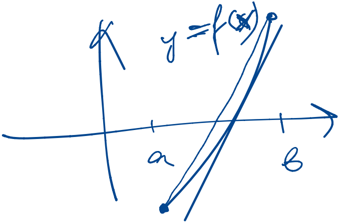
Illustration and Example



tangent's slope = inst. rate of change of $f(x)$ at $x = c$. Slope $= f'(c_1) = 0$

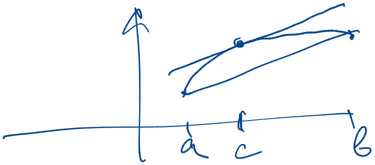
chord, slope = average rate of change of $f(x)$ on $[a, b] = \dfrac{f(b) - f(a)}{b - a} = 0$

What if



?

Mean Value Theorem



Theorem

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on the open interval (a,b). Then there is a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof of MVT

Proof. Geometric intuition helps: the equation of the line through $(a, f(a))$ and $(b, f(b))$ is

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

Consider

$$h(x) := f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then $h(a) = h(b)$, conts on $[a, b]$, diffble on (a, b) and Rolle's theorem applies: $\exists\, c \in (a, b) : h'(c) = 0$

But $h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \cdot 1 = 0$ $\square$

Monotone Functions and Derivative

Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on the open interval (a, b). Then
1. If $|f'(x)| \leq M \; \forall x \in (a, b)$ then

$$|f(x) - f(a)| \leq M(x - a) \leq M(b - a);$$

2. If $f'(x) = 0 \; \forall x \in (a, b)$, then f is constant;

3. If $f'(x) \geq 0 \; \forall x \in (a, b)$, then f is increasing. If $f'(x) > 0 \; \forall x \in (a, b)$, then f is strictly increasing;

4. If $f'(x) \leq 0 \; \forall x \in (a, b)$, then f is decreasing. If $f'(x) < 0 \; \forall x \in (a, b)$, then f is strictly decreasing.

Proof

1. Apply MVT: $\exists\, c \in (a, x)$: $f'(c) = \frac{f(x) - f(a)}{x - a}$

Then $|f(x) - f(a)| = |f'(c)|\,(x-a) \le M\,(x-a)$

2. $\forall\, s, t \in [a,b]$, $s < t$, want $f(s) = f(t)$.

MVT $\Rightarrow$ $f(t) - f(s) = f'(c)\,(t-s) = 0$

for some c: $s < c < t$.

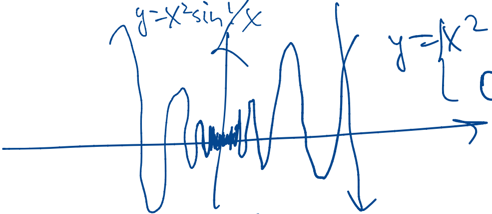
3. $\forall\, s, t \in [a,b]$, $s < t$, want $f(s) \underset{(\text{or} <)}{\le} f(t)$

MVT $f(t) - f(s) = f'(c)\,(t-s) \ge 0$ $(\text{or} > 0)$

4. Similar to 3. $\square$

Illustration and Example

An example of diffble $f(x)$ such that $f'(c) > 0$ at some $c \in (a, b)$ does not imply that $f(x)$ is increasing on an interval about c will be on the homework: $\cancel{f(x) = x^2 \sin 1/x}$ for $c = 0$.



$$y = \begin{cases} x^2 \sin\frac{1}{x} + \frac{x}{2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

$$f(x) = x^2 \sin\frac{1}{x} + \frac{x}{2}$$

$$\left(\text{Sorry, } \left(x^2 \sin\frac{1}{x}\right)' \Big|_{x=0} = 0. \right)$$