Math 5615 Honors: Proof Riemann’s Theorem on Rearrangements
Limits of Functions

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Riemann’s Theorem

**Theorem**

Suppose \( \sum_{k=1}^{\infty} a_k \) is conditionally convergent. Given any real number \( S \), there is a rearrangement \( \sum_{k=1}^{\infty} a_{p_k} \) that converges to \( S \).

Given a series \( \sum_{k=1}^{\infty} a_k \), let

\[
a_k^+ := \max\{a_k, 0\} \quad \text{and} \quad a_k^- := \max\{-a_k, 0\}.
\]

Then \( a_k^+ = a_k \) if \( a_k > 0 \) and \( a_k^+ = 0 \) otherwise; \( a_k^- = |a_k| \) if \( a_k < 0 \) and \( a_k^- = 0 \) otherwise.

**Proposition (Proved last time)**

If \( \sum_{k=1}^{\infty} a_k \) is absolutely convergent, then the series \( \sum_{k=1}^{\infty} a_k^+ \) and \( \sum_{k=1}^{\infty} a_k^- \) are both convergent. If \( \sum_{k=1}^{\infty} a_k \) is conditionally convergent, then the series \( \sum_{k=1}^{\infty} a_k^+ \) and \( \sum_{k=1}^{\infty} a_k^- \) are both divergent.
Proof of Riemann’s Theorem

Know \( \sum a_k^+ \), \( \sum a_k^- \) diverge, \( \sum a_k \) converges \( \Rightarrow \sum a_k \to 0 \) as \( k \to \infty \). Suppose \( S > 0 \).

Construct a rearrangement of \( \sum a_k \) as per algorithm:

1. Add \( a_k^+ \)’s from \( \sum a_k \) (in their original order) up to the first \( a_k^+ \) term so that we exceed \( S \) (possible because \( \sum a_k^+ \) diverges to \( \infty \)).

2. Add negative terms of \( \sum a_k \) (in their original order) up to the first \( a_k^- \) term so that the resulting sum < \( S \) (possible b/c \( \sum a_k^- \) diverges to \( \infty \)).

3. Repeat steps 1 & 2. Never terminates b/c \( \sum a_k^- \), \( \sum a_k^+ \) diverge.
Given $\varepsilon > 0$, find $N$ so that $|a_k| < \varepsilon$. Choose $K; a_1, \ldots, a_K$ are among $a_1, a_2, \ldots$, and $K > N$. Thus, if $h > K$, then $|a_h| > \varepsilon$.

Claim: if $h > K$, then $\sum_{k=1}^{h} |a_k| > \varepsilon$. Otherwise, if $|S - \sum_{k=1}^{h} a_k| \geq \varepsilon$ for some $n > K$, then we've added too many terms of the same sign in the algorithm.

Thus, $\sum_{k=1}^{\infty} a_k$ converges to $S$. If $S < 0$, then start with step 2.
Example

Take the series

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots,$$

which converges and has sum $\log 2$ (the Taylor series of $\log x$). The rearrangement

$$\left(1 + \frac{1}{3}\right) - \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4} + \left(\frac{1}{9} + \frac{1}{11}\right) - \frac{1}{6} + \ldots$$

converges to $\frac{3}{2} \log 2$, being $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} + \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$. The parentheses reflect the algorithm used in the proof of Riemann’s theorem to achieve $S = \frac{3}{2} \log 2$.

$$\left(\frac{1}{5} + \frac{1}{7}\right) > \frac{1}{4} > \frac{1}{9} + \frac{1}{11} \text{ etc.}$$
Example, Continued

\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \ldots
\]

\[
= (1 + \frac{1}{3}) - \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4} + \left(\frac{1}{9} + \frac{1}{11}\right) - \ldots
\]
Limit of a Function

**Definition**

Let $X$ and $Y$ be metric spaces (important case $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$), $D \subset X$ and $f : D \rightarrow Y$. Let $a$ be a cluster point of $D$, and let $L \in X$. We say that $f$ has limit $L$ as $x$ approaches $a$, and write $\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$x \in D \text{ and } 0 < d(x, a) < \delta \Rightarrow d(f(x), L) < \varepsilon.$$

**Theorem**

*If, under the assumption of the definition above, $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} f(x) = L_2$, then $L_1 = L_2$.***

**Proof.**

For any $\varepsilon > 0$, let $d(L_1, L_2) = \frac{\varepsilon}{3}$. If we choose $\delta_1$ and $\delta_2$ such that

$$0 < d(f(x), L_1), d(f(x), L_2) < \frac{\varepsilon}{3},$$

we have

$$d(L_1, L_2) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

Then, we have

$$d(L_1, L_2) = 0.$$