Math 5615 Honors: Limits of Functions and Continuity

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Limit of a Function

Definition

Let X and Y be metric spaces (important case $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$), $D \subset X$ and $f : D \to Y$. Let a be a cluster point of D, and let $L \in Y$. We say that f *has limit L as x approaches a*, and write $\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

($f(x) \to L$ as $x \to a$)

$$\left(x \in D \text{ and } 0 < d(x, a) < \delta\right) \Rightarrow d(f(x), L) < \varepsilon.$$

Theorem

If, under the assumption of the definition above, $\lim_{x \to a} f(x) = L_1$ *and* $\lim_{x \to a} f(x) = L_2$, *then* $L_1 = L_2$.

Proved last time.

Continuity

<u>Def.</u> Let $f: D \to Y$, $D \subset X$, $a \in D$. f is <u>continuous at</u> a, if $\forall \varepsilon > 0\ \exists \delta > 0$: $(x \in D \text{ and } d(x,a) < \delta) \Rightarrow d(f(x), f(a)) < \varepsilon$.

~~Definition~~ Thm.

Let $D \subset X$ and $f : D \to Y$. If $a \in D$ and a is a cluster point of D, then f is *continuous at* a if

$$\lim_{x \to a} f(x) = f(a).$$

Theorem (Sequential Characterization of Limits)

Let $D \subset X$ and $f : D \to Y$. Then the following are true:

1. Let a be a cluster point of D. Then $\lim_{x \to a} f(x) = L$ if and only if for every sequence $\{x_n\}$ in D such that $\forall n, x_n \neq a$ $\lim_{n \to \infty} x_n = a$, we have $\lim_{n \to \infty} f(x_n) = L$.

2. Function f is continuous at $a \in D$ if and only if for every sequence $\{x_n\}$ in D such that $\lim_{n \to \infty} x_n = a$, we have $\lim_{n \to \infty} f(x_n) = f(a)$.

Only 1 (2 is similar). Suppose $\{x_n\} \to a$, $x_n \in D \setminus \{a\}$.

Proof. $\Rightarrow$ Given $\varepsilon > 0$ $\exists\, \delta > 0$:

$\forall x \in D$ and $0 < d(x,a) < \delta$, we have $d(f(x), L) < \varepsilon$

Since $\lim\limits_{n\to\infty} x_n = a$, $\exists N$: $\forall n \geq N$ $d(x_n, a) < \delta$.

Note that $\forall n \geq N$ $x_n \in D$ and $0 < d(x_n, a) < \delta$.

Then $d(f(x_n), L) < \varepsilon$ by three lines above.

$\Leftarrow$ By contradiction: Suppose not true that $\lim\limits_{x\to a} f(x) = L$. Then $\exists\, \varepsilon > 0$: $\forall \delta_n = \frac{1}{n}$, $n \in \mathbb{N}$, $\exists x_n \in D$: $0 < d(x_n, a) < \delta_n$ but $d(f(x_n), L) \geq \varepsilon$. Then $\forall n$ $x_n \neq a$ and $\lim\limits_{n\to\infty} x_n = a$ but it's not true that $\lim\limits_{n\to\infty} f(x_n) = L$. $\square$

Further simple properties of limits

1. $\lim_{x \to a} f(x) = L \in Y$ if and only if $\lim_{x \to a} d(f(x), L) = 0$. (Note: $d(f(x), L)$ is a function $X \to \mathbb{R}$, whereas f is a function $X \to Y$.)

2. If $\lim_{x \to a} f(x) = L \in \mathbb{R}^n$, then for any scalar $c \in \mathbb{R}$, $\lim_{x \to a} cf(x) = cL$. Similar property for sums and inner products of $\mathbb{R}^n$-valued functions and quotients of real-valued functions. Same for continuous at a functions. (Use sequential charact. of limits)

3. If $g(x) \leq f(x) \leq h(x) \in \mathbb{R}$ for all x in a common domain having a as a cluster point, and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$ ~~exists~~, then $\lim_{x \to a} f(x) = L$. (Use sequential characterization of limits)

or $|f(x) - L| \leq |f(x) - g(x)| + |g(x) - L|$

$$\leq |h(x)-g(x)|+|g(x)-L|$$

$$\leq |h(x)-L|+|L-g(x)|+|g(x)-L|$$

$$< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$$

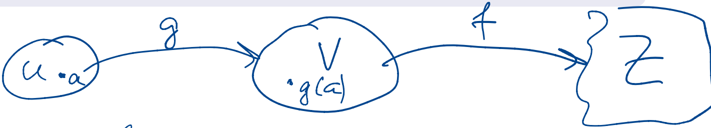
with a suitable choice of a δ-ball about a,

Continuity of Composition

Theorem

Let $U \subset X$ and $V \subset Y$, and suppose that $g : U \to \cancel{Y}\, V$ and $f : V \to Z$. If g is continuous at $a \in U$ and f is continuous at $g(a) \in V$, then $f \circ g$ is continuous at a.

Proof.



Sequential characterization!

$\forall \{x_n\} \subset U$, $\lim_{n\to\infty} x_n = a$;

we have $\lim_{n\to\infty} g(x_n) = g(a)$. Then $f(g(x_n))$

$\to f(g(a))$ b/c f is cts at $g(a)$.

Thus, $f \circ g$ is cnts at a. $\square$