Math 5615 Honors: Higher-Derivative Test for Relative Extrema and The Contraction Mapping Theorem

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Higher-Derivative Test for Relative Extrema

Let $n \geq 0$,

Theorem

Let I be an open interval and assume that $f : I \to \mathbb{R}$ has $n+1$ derivatives, $f', f'', \ldots, f^{(n+1)}$ all defined on I, and $f^{(n+1)}$ is continuous on I. Let x_0 be a point in I such that

notation: $f \in C^{n+1}(I)$

$$f'(x_0) = f''(x_0) = \cdots = f^{(n)}(x_0) = 0 \text{ and } f^{(n+1)}(x_0) \neq 0.$$

The following statements are true:
1. If n is even, then x_0 is not an extreme point for f.
2. If n is odd, then x_0 is an extreme point, which is a local minimum point if $f^{(n+1)}(x_0) > 0$ and a local maximum point if $f^{(n+1)}(x_0) < 0$.

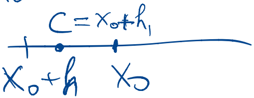
Proof

By Taylor's thm, for $h := x - x_0$, we have

(*) $$f(x_0+h) - f(x_0) = \frac{f^{(n+1)}(c)}{(n+1)!} h^{n+1}$$

for some c between x_0 and x_0+h.

Thus, $c = x_0 + h_1$ for some h_1: $|h_1| < |h|$.



Since $f^{(n+1)}$ is conts, $f^{(n+1)}(x_0) \neq 0$, have $f^{(n+1)}(x_0+h_1) \neq 0$ for $|h_1| < \delta$ and does not change sign. Now look at $h: |h| < \delta$.

1. n even. h^{n+1} takes both >0 and <0 values near 0.

$\Rightarrow$ $f(x_0+h) - f(x_0)$ changes sign when h passes thru 0. No extremum.

Proof

2. n odd. $h^{n+1} \geq 0 \ \forall h \in \mathbb{R}$

If $f^{(n+1)}(x_0) > 0 \Rightarrow$ RHS of $(*)$ is ≥ 0 for $|h| < \delta$

$\Rightarrow f(x_0+h) - f(x_0) \geq 0$ for such h

$\Rightarrow$ get a local minimum

If $f^{(n+1)}(x_0) < 0 \Rightarrow$ RHS of $(*)$ is ≤ 0 for $|h| < \delta$. $\Rightarrow f(x_0+h) - f(x_0) \leq 0$

$\Rightarrow x_0$ is a local maximum. $\Box$

Example

$$f'(x) = e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3}$$

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

From HW: $f^{(n)}(0) = 0 \ \forall n \geq 1$. This test does not apply, as no $f^{(n+1)}(0) \neq 0$.

Nevertheless, $f(x)$ has a local minimum at 0, b/c $f'(x) < 0 \ \forall x < 0$ and therefore $f(x)$ is decreasing on $(-\infty, 0)$, and $f'(x) > 0 \ \forall x > 0$, and $f(x)$ is increasing on $(0, \infty)$.

Actually as Xiang Liu suggested in chat, $f(x) \geq 0 \ \forall x \in \mathbb{R}$ and therefore it has a minimum at $x=0$.

Contraction Mapping

Definition

A mapping $T : X \to X$ of a metric space X with metric d is a *contraction mapping* if there is a number $0 < r < 1$ such that $d(T(x), T(y)) \leq r\, d(x, y)$ for all $x, y \in X$. Such a constant r is called a *contraction constant for* T.

Note that a contraction mapping is always continuous: indeed, if for a given y, $x \to y$, *i.e.*, $d(x, y) \to 0$, then $d(T(x), T(y)) \to 0$ by the contraction condition. Thus, $\lim_{x \to y} T(x) = T(y)$.

Contraction Mapping Theorem

Def. x^* is a fixed pt of $T: X \to X$, if $T(x^*) = x^*$.

Theorem (Contraction Mapping Theorem)

A contraction mapping $T : X \to X$ of a complete metric space X has a unique fixed point x^*. Moreover, if r is a contraction constant for T, then given any $x_0 \in X$, the iteration $x_{k+1} = T(x_k)$, $k = 0, 1, 2, 3, \ldots$ defines a sequence $\{x_k\}$ that converges to x^*, and for each k, we have

$$d(x_k, x^*) \leq \frac{r^k}{1-r} d(x_1, x_0).$$

Proof

$x_0 \in X$

$$d(x_{k+1}, x_k) \le r^k d(x_1, x_0) \quad \forall k \ge 0$$

(by induction on k: $d(x_{k+1}, x_k) = d(T(x_k), T(x_{k-1})) \le r\, d(x_k, x_{k-1})$).

$m > n > 0$

$$d(x_m, x_n) \le d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \dots + d(x_{n+1}, x_n)$$

$$\le (r^{m-1} + r^{m-2} + \dots + r^n)\, d(x_1, x_0)$$

$$(**) \quad \le r^n(1 + r + \dots + r^{m-1-n})\, d(x_1, x_0)$$

$$< r^n(1 + r + r^2 + \dots)\, d(x_1, x_0)$$

$$= \frac{r^n}{1-r} \cdot d(x_1, x_0) \Rightarrow \text{the sequence is Cauchy.}$$

Proof

X complete $\Longrightarrow \{x_k\}$ has a limit $x^* \in X$: $x^* = \lim_{k\to\infty} x_k$.

Then $T(x^*) \overset{T \text{ conts.}}{=} \lim_{k\to\infty} T(x_k) = \lim_{k\to\infty} x_{k+1} = x^*$

Thus, x^* is a fixed pt of T.

To prove $d(x^*, x_n) \le \frac{r^n}{1-r} d(x_1, x_0)$,

pass to $m \to \infty$ in $(**)$.

Suppose x^{**} another fixed of T

$d(x^*, x^{**}) = d(T(x^*), T(x^{**})) \le r\, d(x^*, x^{**})$

$r<1 \Rightarrow d(x^*, x^{**}) = 0 \Rightarrow x^* = x^{**}$ $\square$

Contraction Mapping Theorem for a Closed Subset

Corollary

Let Y be a closed, nonempty subset of a complete metric space X. A contraction mapping $T : Y \to Y$ has a unique fixed point.

Proof.

Exercise: recall that Y will be complete as a metric space in its own right.

□