Math 5615H: Honors: Introduction to Analysis
The Complete Ordered Field of Real Numbers

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Zoom Session Rules of Conduct

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- You are encouraged to answer my questions, ask questions and interrupt me. You may ask questions out loud or by via Zoom chat. If you do not want the whole class to see your question, you may choose the option of private chat message to me on Zoom. If I do not pay attention to Chat, please interrupt me and say: "Sasha, would you, please check your Chat window?"
- **Reward system**: You may get a maximum of 1 point per class, basically for any math related activity.

Supremum and Infimum

Definition

Let P be a partially ordered set, $S \subset P$.
1. $b \in P$ is an *upper bound of* S if $b \geq x \quad \forall x \in S$. Say *$S$ is bounded above*.
2. $b \in P$ is the *least upper bound* or *supremum of* S, $b = \sup S$, if b is an upper bound and $b \leq u$ for all upper bounds u of S.
3. If S has no upper bound, say $\sup S = \infty$.
4. *Lower bnd, bdd below, and grtst lower bnd or infimum...*

Example

$S = \{x \in \mathbb{Q} \mid x^2 < 2\} \subset \mathbb{Q}$ bdd above, e.g., by $b = 2$, but no least upper bnd. (You have read in 1.1 that the set $B = \{x \in \mathbb{Q} \mid x > 0, x^2 > 2\}$ contains no least element. B is the set of upper bnds of S: if $x \in \mathbb{Q}$ is such that $x^2 \leq 2$, it cannot be an upper bnd for S.) Expect: $A = \{x \in \mathbb{R} \mid x^2 < 2\}$ has $\sup A$.

otherwise, $\exists x \in \mathbb{Q}$ $x > 2 \Rightarrow x^2 > 4$

~~Antisym~~

(PO set)

Dichotomy

Total order

$\forall x, y$ such that

$x \le y$ and $y \le x \Rightarrow x = y$

$\mathbb{R}^2$ $(x_1, y_1) \le (x_2, y_2)$ if and $x_1 \le x_2$, $y_1 \le y_2$

$\forall x, y \in P$ $\quad x \le y$ or $y \le x$

$(1, -1) \;\not\le\; (-1, 1)$

incomparable

$1 \ge -1$

and

$-1 \ge 1$

The least upper bound property and complete ordered fields

Definition

Let P be a partially ordered set. If every nonempty subset $S \subset P$ that is bounded above has a least upper bound in P; that is, there is $b \in P$ such that $b = \sup S$, we say that P has the *least upper bound property* or *P is complete*.

Definition

A *complete ordered field* is an ordered field which is complete.

Example

$\mathbb{Q}$ is an ordered field, but not complete.

$\mathbb{Z}$ is a PO set which is complete.

The complete ordered field of real numbers

Theorem

1. A complete ordered field always contains $\mathbb{Q}$ as an ordered subfield. (E.g., $n = 1 + 1 + \cdots + 1 \in F$ gives $\mathbb{Z} \subset F$.)
2. A complete ordered field exists and is unique up to isomorphism (of ordered fields).

$F_1 \xrightarrow{\cong} F_2$ bijection respect $+, \cdot, 0, 1, P$

F_1, F_2 complete ord. flds $\Rightarrow \exists\, F_1 \to F_2$

Definition

Once and for all, fix a complete ordered field and call it $\mathbb{R}$. A *real number* is an element of the complete ordered field $\mathbb{R}$.

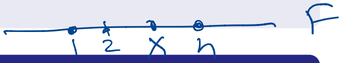
Comments...

To prove $\exists$ in 2, you present a construction of the reals; can use Dedekind cuts of $\mathbb{Q}$ (see appendix in the text) or decimal fractions; or Cauchy sequences in $\mathbb{Q}$

Archimedian Fields

Definition

An ordered field F is called *Archimedean* if for every $x \in F$ there is an $n \in \mathbb{N}$ such that $n > x$.



Theorem

$\mathbb{R}$ is Archimedian.

Proof. Next slide.

Corollary

$\mathbb{Q}$ is Archimedian.

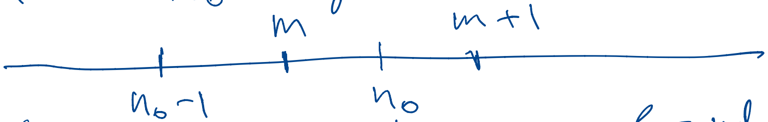
Proof that $\mathbb{R}$ is Archimedian

Proof. Suppose not

$\exists\, x \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$ $n \leq x$

$\Rightarrow \mathbb{N} \subset \mathbb{R}$ is bdd above

Take $n_0 = \sup \mathbb{N} \in \mathbb{R}$

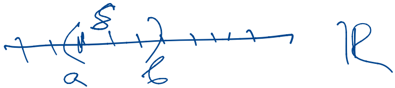


Claim: $n_0 - 1$ is not an upper bound of $\mathbb{N}$

B/c $n_0 - 1 < n_0$ ($\Leftarrow 1 > 0 \Leftarrow 1^2 > 0$) Thus, n_0

$\Rightarrow \exists m \in \mathbb{N}$ $m > n_0 - 1$. But then $n_0 < m + 1$. is not upper bdd!

The Density of Rationals in $\mathbb{R}$



Definition

A subset S of the real numbers is *dense* in $\mathbb{R}$ if for any two real numbers $a < b$, there is an $s \in S$ such that $a < s < b$.

Theorem

1 (The Density of Rationals). For any two real numbers $a < b$, there is a rational number q such that $a < q < b$.
2 (The Density of Irrationals). For any two real numbers $a < b$, there is an irrational number x such that $a < x < b$.

Proof of the Density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$

Proof.

2. Density $\mathbb{Q}$ $\Rightarrow$ density of $\mathbb{R} \setminus \mathbb{Q}$

$\cancel{a<b}$ $a < b$ $\qquad a\sqrt{2} < b\sqrt{2}$

(1) $\Rightarrow$ $\exists q \in \mathbb{Q}$: $\quad a\sqrt{2} < q < b\sqrt{2}$

$a < \frac{q}{\sqrt{2}} < b \qquad \frac{q}{\sqrt{2}} \notin \mathbb{Q}$

The existence of the square root $\sqrt{2}$

Theorem

There exists a unique positive real number r such that $r^2 = 2$.

Proof. Take $A := \{s \in \mathbb{R} \mid s \geq 0 \text{ and } s^2 \leq 2\}$ and $r = \sup A \ldots$

$\sqrt{2} := r$ (this r)

Continuation of proving that $r^2 = 2$

Proof.

The existence of the nth root $\sqrt[n]{a}$

Theorem

There exists a unique positive real number r such that $r^n = a$ for any given $n \in \mathbb{N}$ and $a \geq 0 \in \mathbb{R}$.

Proof. ($\sqrt[n]{a} :=$ this r.)

$r = \sup \{ s \in \mathbb{R} \mid s \geq 0, s^n \leq a \}$

Exclude $r^n < a$, $r^n > a$.

Leaves $r^n = a$.

Archimedean

Read how in the text.