Math 5615H: Honors: Introduction to Analysis
Useful Lemma
Decimal Expansions
The Euclidean Space $\mathbb{R}^n$

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Useful Lemma

Lemma

Let z be a real or complex number. If $|z| \leq \varepsilon$ for every $\varepsilon > 0$, then $z = 0$.

Proof. Clearly, $|z| \geq 0$. If $|z| = 0$, then $z = 0$. Done.

If $|z| > 0$. Take $\varepsilon = \frac{|z|}{2} > 0$.

Then $|z| \leq \frac{|z|}{2} \Rightarrow 2|z| \leq |z| \Rightarrow |z| \leq 0$.

$(\Rightarrow 2 \leq 1)$

contradictions

$\square$

Decimals

Let $x \geq 0$ be a real number. Define a set E of ~~real~~ rational numbers

$$y_0 := n_0 \le x, \qquad \text{largest } n_0 \in \mathbb{N} \cup \{0\}$$

$$y_1 := n_0 + n_1/10 \le x, \quad \text{largest } n_1 \in \mathbb{N} \cup \{0\}$$

$$\ldots$$

$$y_k := \underline{n_0 + \frac{n_1}{10} + \cdots + \frac{n_k}{10^k}} \le x, \quad \text{largest } n_k \in \mathbb{N} \cup \{0\} \tag{1}$$

$$\ldots$$

$$E := \{y_0, y_1, y_2, \ldots, y_k, \ldots\} \subset \mathbb{Q}$$

Then $x = \sup E$. (Why exists?) The *decimal expansion* of x is

If $x_1 < x$ is another upper bound for E

$y_k \quad x_1 \quad x$

$\frac{1}{10^k} < x - x_1 \qquad y_k + \frac{1}{10^k}$

$$x = n_0.n_1 n_2 n_3 \ldots \tag{2}$$

We know enough to prove: $x = \sup E$; $0 \le n_i \le 9$ for $i \ge 1$.

Bijection Statement

Would need to know that

$$\sum_{k \geq n+1} \frac{9}{10^k} = \frac{1}{10^n}$$

to show that $n_i < 9$ for infinitely many i. This would imply that n_n was not the largest. $n_0 . n_1 n_2 \ldots n_n 999 \ldots = n_0 . n_1 \ldots n_n \overline{9}$
$= n_0 . n_1 \ldots (n_n + 1) 0000 \ldots$

Theorem

There is a bijection between $\mathbb{R}$ and decimal expansions (2) *with $n_0 \in \mathbb{Z}$, $0 \leq n_i \leq 9$ for $i \geq 1$, and $n_i < 9$ for infinitely many i.*

Idea of Proof.

Given a decimal expansion (2), the set E of numbers (1) is bounded above, and $x = \sup E$ has (2) as decimal expnsn. □

Binary Expansions

Theorem

There is a bijection between $\mathbb{R}$ and expansions (2) with $n_0 \in \mathbb{Z}$, $n_i = 0$ or 1 for $i \geq 1$, and $n_i = 0$ for infinitely many i.

This expansion is called the *binary expansion* of x.

Idea of Proof.

Use (for $x \geq 0$)

$$n_0 + \frac{n_1}{2} + \cdots + \frac{n_k}{2^k} \leq x.$$

Everything else is the same as in previous theorem. □

$\mathbb{R}$ is uncountable

Theorem

The complete ordered field $\mathbb{R}$ is uncountable.

$S = \bigcup_{n \geq 1} S_n$

Proof. Add to the set of binary expansions the set S of those expansions for which $n_i = 0$ for finitely many i. This is a countable set as a countable union of ~~finite~~ countable sets. We want to prove that the set $\mathbb{R} \cup S$ of binary sequences like (2), starting with $n_0 \in \mathbb{Z}$ and $n_i = 0$ or 1 for $i > 0$, is uncountable. This will imply $\mathbb{R}$ is uncountable, because if $\mathbb{R}$ were countable, then $\mathbb{R} \cup S$ would also be countable.

$S_n = \mathbb{Z} \times \{0,1\}^n$ countable

$S_n = \{ n_0 . n_1 n_2 n_3 \dots n_n \bar{1} \mid n_i = 0 \text{ or } 1,\ i \geq 0,\ n_i = 0 \text{ for } < \infty \text{ many } i\text{'s} \}$

$\mathbb{R} \cup S$ is uncountable

Suppose it's countable

$r_0 = n_{00}.n_{01}n_{02}n_{03}\dots$

$r_1 = n_{10}.n_{11}n_{12}n_{13}\dots$

$r_2 = n_{20}.n_{21}n_{22}n_{23}\dots$

$\vdots$

Cantor's diagonal argument
(Cf. $|\mathcal{P}(A)| \neq |A|$)

Claim: I can find a binary expansion not on this list. Indeed, take
$r = n_0'.n_1'n_2'\dots$, where $n_0' \in \mathbb{Z}$, $n_0' \neq n_{00}$
$n_1' = 0$ or 1, $n_1' \neq n_{11}$
$n_2' = 0$ or 1, $n_2' \neq n_{22}$
etc.

Then $r \neq r_i$ for any i (they will differ in the ith binary place) $\square$

$\mathbb{R}^n$: Definition and vector-space structure

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}}_{n \text{ times}}, \quad n \geq 0$$

$$= \{ (x_1, \dots, x_n) \mid x_i \in \mathbb{R} \}$$

$$(x_1, x_2, \dots, x_n) + (y_1, \dots, y_n) := (x_1 + y_1, \dots, x_n + y_n)$$

$\alpha \in \mathbb{R}$ $\quad \alpha (x_1, \dots, x_n) := (\alpha x_1, \dots, \alpha x_n)$

Scalar multiplication

The Euclidean Inner Product and Properties

$\vec{x} = (x_1, \ldots, x_n)$ $\quad$ $\vec{y} = (y_1, \ldots, y_n)$

$(\vec{x}, \vec{y}) := \sum_{i=1}^{n} x_i y_i = x_1 \cdot y_1 + \ldots + x_n \cdot y_n \in \mathbb{R}$

$$\begin{cases} (\vec{x}, \vec{x}) \geq 0 \text{ and } = 0 \text{ iff } \vec{x} = 0 \\ (\vec{x} + \vec{y}, \vec{z}) = (\vec{x}, \vec{z}) + (\vec{y}, \vec{z}) \\ (\vec{x}, \vec{y}) = (\vec{y}, \vec{x}) \end{cases}$$

(properties)

The Euclidean Norm

$$|\vec{x}| = \sqrt{(\vec{x}, \vec{x})}$$

norm properties derived from those of $(-,-)$

E.g., $|(\vec{x}, \vec{y})| \le |\vec{x}| \cdot |\vec{y}|$ Schwarz inequalities

...

$$d(\vec{x}, \vec{y}) := |\vec{x} - \vec{y}|$$

distance betw $\vec{x}$ & $\vec{y}$