Math 5615H, Fall 2020 **Solutions to Midterm Exam 2**
Posted: 12/16

Rules: **Unlike working on your homework, no study groups or cooperation when doing the exam, no asking questions on internet forums, etc.! You may use any textbooks and internet sources, but just copying arguments you might occasionally find will not gain any credit and will be regarded as plagiarism. You have to present all solutions in your own words.**

Regarding justifying your solutions: You may use any statement stated in class or in our textbook, baby Rudin, or stated in the homework, unless it makes your solution ridiculous, such as "Stated in class." You may also use one exam problem in your solution of another exam problem. You may use whatever theorems of algebra you wish to use.

You should also write on your paper the following *honor pledge*: "I pledge my honor that I have not violated the Honor Code during this examination" and sign your name under it.

Problem 1. Define a sequence $\{a_n\}_{n\geq 1}$ recursively by $a_1 = 1/2$ and $a_{n+1} = a_n + a_n^2$. Show that $\lim_{n\to\infty} a_n = +\infty$.

Solution. This is obviously $(a_{n+1} = a_n + a_n^2 \geq a_n)$ a monotone increasing sequence, and by a famous theorem, it has a finite limit L, if the sequence happens to be bounded above. In this case, $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$, because $\{a_{n+1}\}$ is a subsequence of a converging sequence, and $\lim_{n\to\infty} a_n + a_n^2 = L + L^2$ by theorems on adding and multiplying conerging sequences. Therefore, $L = L + L^2$, whence $L = 0$, but this contradicts $a_n \geq a_1 = 1/2$. Thus, it cannot happen that $\{a_n\}$ is bounded above.

Now, an unbounded above monotone increasing sequence must have an infinite limit, because if one element $a_N > M$, then $a_n > M$ for all $n \geq N$.

Problem 2. Let $\{a_n\}$ be a sequence of positive real numbers. Suppose that the series $\sum_{n=1}^{\infty} a_n^2$ converges. Show that the series $\sum_{n=1}^{\infty} a_n a_{n+1}$ also converges.

Solution. Note that for each n, $0 \geq 2a_n a_{n+1} \leq a_n^2 + a_{n+1}^2$, because $(a_n - a_{n+1})^2 \geq 0$. The series $\sum_{n=1}^{\infty} a_n^2 + a_{n+1}^2$ converges as the sum of two convergent series. This implies that the series $\sum_{n=1}^{\infty} 2a_n a_{n+1}$ converges by the comparison test. Therefeore, $\sum_{n=1}^{\infty} a_n a_{n+1}$ converges a convergent series multiplied by a constant.

Problem 3. Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 1 - x & \text{if } x \text{ is irrational.} \end{cases}$$

At which points $x \in \mathbb{R}$ is the function f continuous? Justify your answer using the ε-δ definition of continuity.

Solution. Since the rational and irrational numbers are dense among the reals, we feel that the function is continuous exactly when $x = 1-x$, *i.e.*, $x = 1/2$. Let us prove that with ε and δ, as required.

At $x = 1/2$, given an $\varepsilon > 0$, take $\delta = \varepsilon$. Then for each x such that $|x - 1/2| < \delta$, we have

$$\begin{aligned} |f(x) - f(1/2)| &= \begin{cases} |x - 1/2|, & \text{if } x \in \mathbb{Q}, \\ |1 - x - 1/2|, & \text{if } x \notin \mathbb{Q}, \end{cases} \\ &= |x - 1/2| < \delta = \varepsilon, \end{aligned}$$

and we are done with $x = 1/2$.

Suppose $a \neq 1/2$. Let us prove $f(x)$ is not continuous at a, that is to say, there exists an $\varepsilon > 0$ such that for any $\delta > 0$, there is an x such that $|x - a| < \delta$ but $|f(x) - f(a)| \geq \varepsilon$.

Take $\varepsilon := |a-(1-a)|/2 = |a-1/2|$, which is greater than 0, because $a \neq 1/2$. (The idea is to take ε to be small enough as compared to the difference of the values of the functions x and $1 - x$ out of which $f(x)$ is built. You might need to experiment with what should be small enough before arriving at a formula, as the one above.) Given a $\delta > 0$, take an irrational x closer to a than $\min(\varepsilon, \delta)$, if a is rational, or a rational x closer to a than $\min(\varepsilon, \delta)$, if a is irrational. This is possible because of the density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$. (The motivation is that we want to take x close to a but such that the value of $f(x)$ is further away from $f(a)$. If x and a belong to the same subset, be it $\mathbb{Q}$ or $\mathbb{R} \setminus \mathbb{Q}$, of $\mathbb{R}$, then $f(x)$ and $f(a)$ will be given by the same polynomial formula and be too close to each other.) Then, either way,

$$\begin{aligned} &|f(x) - f(a)| = |1 - x - a| = |1 - 2a - (x - a)| \\ &\geq ||1 - 2a| - |x - a|| = |2\varepsilon - |x - a|| = 2\varepsilon - |x - a| > 2\varepsilon - \varepsilon = \varepsilon. \end{aligned}$$

Problem 4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $(f(x) - f(y))^2 \leq |x - y|^3$ for all x and y. Prove that f is constant.

Solution. We feel like the inequality should force the derivative of f to exist and be equal to 0 everywhere, which is what we are going to show. For $x \neq y$, we have

$$0 \leq \frac{(f(x) - f(y))^2}{(x - y)^2} \leq |x - y|$$

and therefore, after taking the square root,

$$0 \leq \left| \frac{f(x) - f(y)}{x - y} \right| \leq \sqrt{|x - y|}.$$

Since $\lim_{x \to y} \sqrt{|x - y|} = 0$ (from $\sqrt{x}$ being continuous on the right at 0), the Squeeze thereom implies that the following limit exists and is equal to zero:

$$\lim_{x \to y} \left| \frac{f(x) - f(y)}{x - y} \right| = 0.$$

For any function $g(x)$, $\lim_{x \to y} g(x)$ exists and equals 0 if and only if $\lim_{x \to y} |g(x)|$ exists and equals 0, because $||g(x)| - 0| = |g(x) - 0|$. Therefore, $\lim_{x \to y} \frac{f(x)-f(y)}{x-y}$ exists and equals 0, whence $f'(y)$ exists and equals 0 for every y, and the function f is constant by a corollary of the Mean Value Theorem (MVT).

Problem 5. Let $f : [a, b] \to \mathbb{R}$ be a function. We say f is *convex* on $[a, b]$ if $l(x) > f(x)$ for all $x \in (a, b)$, where $l(x) := m(x - a) + f(a)$, with $m = (f(b) - f(a))/(b - a)$, is the line from $(a, f(a))$ to $(b, f(b))$. Prove that if f is continuous on $[a, b]$, is differentiable on (a, b), and f' is strictly increasing on (a, b), then f is convex on $[a, b]$.

Solution. (This is a somewhat different solution, as compared to the one given in class on 12/16, just for the fun of it.) Form $g(x) := l(x) - f(x)$, motivated by the proof of the MVT. We have $g(a) = 0 = g(b)$, g is continuous on $[a, b]$, differentiable on (a, b), and $g'(x) = m - f'(x)$ must be strictly decreasing. All we need is to show that $g(x) > 0$ on (a, b).

It looks like g is set for applying Rolle's theorem, which says there is a $c \in (a, b)$ such that $g'(c) = 0$. Then for each $x \in (a, c)$, we have $g'(x) > 0$ and for each $x \in (c, b)$, we have $g'(x) < 0$, because of the monotonicity condition. By the MVT, for each $y \in (a, c]$, $g(y) = g(y) - g(a) = g'(c_1)(y - a) > 0$ and for each $y \in [c, b)$, $g(y) = g(y) - g(b) = g'(c_2)(y - b) > 0$. Thus, $g(y) > 0$ for all $y \in (a, b)$.