MATH 5615H: HONORS ANALYSIS I
HOMEWORK 1
DUE THURSDAY, SEPTEMBER 12, 2024, 11:59 P.M.

INSTRUCTOR: SASHA VORONOV

1. I encourage you to collaborate with each other when you do homework, but write the solutions in your own words. Please write on your work with whom you collaborated.

2. I do not encourage you, but do not mind if you use literature and internet sources to help you with solutions. This includes using Artificial Intelligence (AI), such as ChatGPT. Ask it to give you a hint, when you run out of ideas. If you use a source, please quote it briefly.

Read: The Syllabus. Your course notes. Chapters 1 and 2 of the textbook (Terence Tao, Analysis I, fourth edition, 2022). Read also some general notes from Terence Tao about proofs on the next page.

Exercises: 2.2.1, 2.2.3, 2.2.4, 2.3.1, 2.3.2, 2.3.3, 2.3.4, 2.3.5.

Problem A (Suggested by Mike)**.** Show that Axiom 5 (Axiom 2.5 in the text)

> **Axiom 5** (Principle of mathematical induction). Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n++)$ is also true. Then $P(n)$ is true for every natural number n.

is equivalent to

> **Axiom 5$'$**. If S is a set such that: 0 is in S, and for every natural number n, n being in S implies that $n++$ is in S, then S contains every natural number.

Date: September 6, 2024.

Some general notes on proofs: One purpose of this course is to help you write proofs in a correct and professional manner. To this end, you are encouraged to make your proofs as detailed as possible. You are certainly encouraged to write several English sentences in your proofs, and to use logical connectives ("since", "if... then", "because", "we have", "let", "therefore", "thus", "by hypothesis", etc.) to clarify the logical structure of your proofs; merely laying out a long sequence of mathematical equations without supplying any explanation may (barely) be acceptable, but don't count on it. Also, the following types of proofs may result in partial credit at best:

- **Circular reasoning.** This can occur if you use a result from later in the notes to prove a result from earlier in the notes, or if you use more advanced theory to prove simpler concepts. Of course, using results from earlier in the notes to prove results later in the notes is OK; so is using earlier HW questions to prove later HW questions. Because of the need to avoid circular reasoning, some of the most elementary results may, paradoxically, be the hardest for you to prove (because so many results that you know are not permitted to be used).

- **Proof by example.** This occurs when one is asked to prove a *universal* statement (e.g. "Property P is true for all integers n") and instead one just supplies a single example ("Property P is true for a single integer n"). However, proofs by example are valid for *disproving* a universal statement (i.e. finding a counterexample), or in proving *existence* questions (e.g. "Show that there exists an integer n which satisfies property P").

- **Proof by appeal to intuition.** You have plenty of intuition already about the natural numbers and related number systems, and this will be very helpful for you in constructing proofs. However, even when claiming an intuitively obvious statement in a proof, you still have to supply rigorous justification (though you are certainly encouraged to also *add* your own intuitive remarks; just don't rely on them by themselves!). Also, your intuition may use more advanced concepts than what is needed for the question at hand, and so by relying on intuition too much you may fall into circular reasoning (see above). One of the purposes of this course is to allow you to analyze your own intuition and see what assumptions it is really based on.