Math 5615H Homework 10
Posted: 11/14; Updated 11/19; due: Friday, 11/21/2014

The problem set is due at the beginning of the class on Friday.

Reading: Chapter 5: pages 103-109.
For this homework, you may assume that \((\sqrt{x})' = 1/(2\sqrt{x})\), \(\sin' x = \cos x\) and similar computations known to you from calculus.

Problem 1. Let \(f : \mathbb{R} \to \mathbb{R}\) be defined by
\[
f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational}, \\ 0, & \text{if } x \text{ is irrational}. \end{cases}
\]
At which points is the function differentiable?

Problem 2. Give the critique of (i.e., find a gap in) the following supposed “proof” of the chain rule:
\[
\lim_{t \to x} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} = \left( \lim_{t \to x} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \right) \left( \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \right) = g'(f(x))f'(x).
\]

Problem 3. Consider the function
\[
f(x) = \begin{cases} 2x^2 \sin(1/x) + x, & x \neq 0, \\ 0, & x = 0. \end{cases}
\]
Show that \(f\) has a positive derivative at \(x = 0\) but is not monotonically increasing in any neighborhood of \(x = 0\).

Problem 4. Suppose \(f : \mathbb{R} \to \mathbb{R}\) is a function whose derivative exists at each point and is bounded. Show that \(f\) is uniformly continuous.

Problem 5. (1) Suppose that \(f : \mathbb{R} \to \mathbb{R}\) is continuous and \(\lim_{|x| \to +\infty} f(x) = 0\). Prove that \(f\) is uniformly continuous.
(2) Find a bounded function \(f : \mathbb{R} \to \mathbb{R}\) such that \(f\) is differentiable at every point and uniformly continuous, but \(f'\) is not bounded. Hint: Make use of Part (1). Note that the derivative must oscillate between large positive and negative values as \(|x| \to \infty\), because if, say, \(f'\) just grows unboundedly with \(x \to +\infty\), then it should force \(f\) to do the same. For instance, try to see why \(\sin x/x\) does not work and cook something based on that.

Problem 6. (1) Suppose that \(f : \mathbb{R} \to \mathbb{R}\) and \(|f(x)| \leq x^2\) for all \(x\). Prove that \(f\) is differentiable at \(x = 0\).
(2) Find a function \(f : \mathbb{R} \to \mathbb{R}\) that is differentiable at one point and not continuous at any other point.

Problem 7. Suppose that \(f\) is differentiable at each point of \((a, b)\) and its derivative is never 0. Prove that \(f\) is strictly increasing or strictly decreasing on the interval. (Note that \(f'\) is not assumed to be continuous.)

Problem 8. Suppose \(f\) is a real-valued function on \((0, +\infty)\) with the properties:
(1) \(f(xy) = f(x) + f(y)\) for all positive \(x\) and \(y\);
(2) \(f'(1)\) exists and equals \(1\).
Prove that \(f(1) = 0\) and \(f'(x)\) exists and equals \(1/x\) for all \(x > 0\). Hint: For the second statement, do \(x + h = x(1 + h/x)\).