Math 5615H                Homework 3
Posted: 12:30 a.m., 09/20, modified on 09/24; due: Friday, 09/26/2014

The problem set is due at the beginning of the class on Friday.

Reading: Chapter 2: pages 24-37

Problems:
1. Suppose a set $A$ is infinite and $B$ is countable. Show that $A \cup B \sim A$.
   
   Hint: Start with selecting a countable subset in $A$.

2. Reproduce and complete the argument we had in class (see also a very
   sketchy note in the text after Theorem 2.14) to show that $|[0, 1)| = 2^\aleph_0$,
   where by definition $|A|$ is the cardinality of $A$, $\aleph_0 := |\mathbb{N}|$, and
   $2^{|A|} := |P(A)|$, the cardinality of the set $P(A)$ of all subsets of $A$,
   for any set $A$.

3. Show that $c = 2^{\aleph_0}$, where by definition $c := |\mathbb{R}|$.

4. If $|A_n| = c$ for all $n \geq 1$ under the notation of the previous problem,
   then the countable disjoint union $\bigcup_{n \geq 1} A_n$ has the same cardinality $c$.

5. What is the cardinality of the irrationals?

6. Prove that the following sets are open:
   
   (1) the first quadrant $\{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 0\}$;
   
   (2) any subset of the discrete metric space.

7. Find an infinite collection of distinct open sets in $\mathbb{R}$ whose
   intersection is a nonempty open set. (Thus infinite intersections of open sets
   may or may not be open.)

8. Show that $\mathbb{Q}$ as a subset of $\mathbb{R}$ is neither open, nor closed.

9. Show that the closure of set $A$ in a metric space is the intersection of all
   the closed sets which contain $A$. 