Problem 1. Determine whether the series converges or diverges:

1. \[ \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots 2n}; \quad \text{Hint: “Cancel” terms > 1.} \]
2. \[ \sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}; \]
3. \[ \sum_{n=1}^{\infty} (1-a)(1-a/2)(1-a/3)\ldots(1-a/n), \quad a > 0; \quad \text{Hint: Use Raabe’s test, which is Problem 6 from the previous homework.} \]
4. \[ 1 + \frac{1}{3^2} - \frac{1}{2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{4} + \ldots + \frac{1}{(4n+1)^2} + \frac{1}{(4n+3)^2} - \frac{1}{2n+2} + \ldots \]

Problem 2. Find the sum of the series

\[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)}. \]

\textit{Hint:} If you are able to find the sum, it means you have an idea about “telescoping” series.

Problem 3. Determine the coefficients \( a_n \) of the power series whose sum is \((1-z)^{-2}\) for \(|z| < 1\) by squaring \((1-z)^{-1}\).

Problem 4. Define \( f : \mathbb{R} \to \mathbb{R} \) by

\[ f(x) = \begin{cases} 
1/q & \text{if } x = p/q \text{ (reduced fraction)}, \quad x \neq 0, \\
0 & \text{if } x = 0 \text{ or } x \notin \mathbb{Q}
\end{cases} \]

Show that \( f \) is continuous at 0 and any irrational point, but not continuous at any nonzero rational point.

Problem 5. Describe all continuous functions \( f : \mathbb{R} \to X \), where \( X \) is a discrete metric space.

Problem 6. Let \( S \) be a metric space and \( q \in S \). Show that the distance function \( d(p, q) \) is a continuous function of \( p \).

Problem 7. Let \( E \) be a nonempty subset of a metric space \( S \). Define the distance from a point \( p \in S \) to the set \( E \) to be

\[ d_E(p) = \inf\{d(p, q) \mid q \in E\}. \]

Prove that \( d_E(p) = 0 \) iff \( p \in \overline{E} \), the closure of \( E \). Prove that \( d_E \) is a continuous function on \( S \).
Problem 8. Suppose that $E$ is a subset of a metric space $S$ that is not closed. Show that there is a continuous real-valued function on $E$ that is not bounded.