Here is a solution of Problem 6(2) on Homework 9.

**Solution:** We define \( g(q) := \lim_{p\to q} f(p) \) for each \( q \in \overline{A} \). Since \( f(p) \) is continuous on \( A \), \( g(p) = f(p) \) for each \( p \in A \). The limit exists, because for any sequence \( \{p_n\} \subset A \) converging to \( q \in \overline{A} \), the sequence \( \{f(p_n)\} \) will converge in \( \mathbb{R} \), as \( \{p_n\} \) is a Cauchy sequence, and so is \( \{f(p_n)\} \) by Part (1).

We need to show that \( g \) is continuous on \( \overline{A} \). Let us do it using the definition. For each point \( q \in \overline{A} \), given \( \epsilon > 0 \), we need to find a \( \delta > 0 \) such that \( |g(p) - g(q)| < \epsilon \) whenever \( |p - q| < \delta \) and \( p \in \overline{A} \). Start with a \( \delta_1 > 0 \) such that \( |f(p') - g(q)| < \epsilon/2 \) whenever \( 0 < |p' - q| < \delta_1 \) and \( p' \in A \). Such \( \delta_1 \) exists, because \( g(q) = \lim_{p' \to q} f(p') \). Note also that if \( |p' - q| = 0 \), which may happen only when \( q \in A \), we have \( |f(p') - g(q)| = |f(q) - g(q)| = 0 < \epsilon/2 \).

Define \( \delta := \delta_1/2 \). Now for any \( p \in \overline{A} \) such that \( |p - q| < \delta \), we can find \( \delta_2 > 0 \) such that \( |f(p') - g(p)| < \epsilon/2 \) whenever \( 0 < |p' - p| < \delta_2 \) and \( p' \in A \). Such \( \delta_2 \) exists, because \( g(p) = \lim_{p' \to p} f(p') \). Note also that if \( |p' - p| = 0 \), which may happen only when \( p \in A \), we have \( |f(p') - g(p)| = |f(p) - g(p)| = 0 < \epsilon/2 \).

Take any point \( p' \in A \) such that \( |p' - p| < \min(\delta, \delta_2) \). Such \( p' \) exists, because if \( p \in A \), we can take \( p' = p \). Otherwise, \( p \) is a limit point of \( A \) and there are points of \( A \) arbitrarily close to \( p \). Then \( |p' - q| \leq |p' - p| + |p - q| < \delta + \delta = \delta_1 \), and we have

\[
|g(p) - g(q)| \leq |g(p) - f(p')| + |f(p') - g(q)| < \epsilon/2 + \epsilon/2 = \epsilon.
\]

Thus, \( g(q) \) is continuous on \( \overline{A} \).

Now, let us show the uniqueness of \( g \). Indeed, if we have another continuous function \( g_1 \) on \( \overline{A} \) extending \( f \) from \( A \), then we have \( g_1(p) = f(p) = g(p) \) for all \( p \in A \). If \( q \in \overline{A}' \), then take a sequence \( \{p_n\} \subset A \) such that \( p_n \to q \). Then, since \( g \) and \( g_1 \) are continuous on \( \overline{A} \), we have \( \lim_{n \to \infty} g_1(p_n) = g_1(q) \) and \( \lim_{n \to \infty} g(p_n) = g(q) \). However, since \( p_n \in A \), we have \( g_1(p_n) = f(p_n) = g(p_n) \) and thereby \( g_1(q) = g(q) \).

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