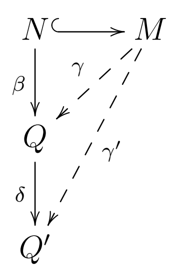
WHY $\gamma|_N = \beta$ IN THE PROOF OF LEMMA A3.8

Here is how one can check the property $\gamma|_N = \beta$ in the proof of Lemma A3.8. Let me repeat the setup: R is an S-algebra, Q' is an injective S-module, $Q := \mathrm{Hom}_S(R, Q')$ has the structure of an R-module defined by $(r\phi)(r') := \phi(rr')$, $N \hookrightarrow M$ is an R-submodule of M, $\beta : N \to Q$ is a given R-module map. We have added an S-module map $\delta : Q \to Q'$ defined by $\phi \mapsto \phi(1)$. We have constructed an S-module map $\gamma' : M \to Q'$ such that $\gamma'|_N = \delta\beta$, using the injectivity of Q'. And we have defined $\gamma : M \to Q$ by assigning to $m \in M$ a homomorphism $\gamma(m)$ defined by its values on $r \in R$: $(\gamma(m))(r) = \gamma'(rm)$. All we need is to check $\gamma|_N = \beta$ in the resulting diagram



This is not obvious, but can be done carefully as follows.

For each $n \in N$, we need to see that $\gamma(n) = \beta(n)$ in Q. Since elements of Q are S-module homomorphisms $R \to Q'$, we need to see if the values of $\gamma(n)$ and $\beta(n)$ on each $r \in R$ agree. Indeed,

$$(\gamma(n))(r) = \gamma'(rn) = \delta\beta(rn) = (\beta(rn))(1) = (r(\beta(n))(1) = (\beta(n))(r).$$

Here we used the definition of γ, the property $\gamma'_N = \delta\beta$, the definition of δ, the fact that β is an R-module map, and the definition of the R-module structure on Q.

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