LECTURE 1: TENSOR CATEGORIES

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An easy way to define a PROP uses the notion of a symmetric monoidal (or tensor) category, which is an interesting and useful notion on its own. We will give a brief introduction to tensor categories here. You can find more detail in the new book "Tensor Categories and Modular Functors" by Bakalov and Kirillov, Jr., or any textbook on quantum groups, such as Chari-Pressley, the original paper of MacLane or Saavedra Rivano's monograph "Catégories Tannakiennes".

Definition 0.1. A *symmetric monoidal* (*tensor*) category is a category $\mathcal{C}$ with a functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

and functorial isomorphisms

$$\begin{aligned} \alpha_{X,Y,Z} : &\quad X \otimes (Y \otimes Z) &\to (X \otimes Y) \otimes Z, \\ \tau_{X,Y} : &\quad X \otimes Y &\to Y \otimes X, \end{aligned}$$

satisfying the following identities:

$$\begin{aligned} \tau^2 &= \mathrm{id}, \\ \alpha_{X \otimes Y,Z,W} \alpha_{X,Y,Z \otimes W} &= (\alpha_{X,Y,Z} \otimes \mathrm{id}_W) \alpha_{X,Y \otimes Z,W} (\mathrm{id}_X \otimes \alpha_{Y,Z,W}), \\ (\tau_{X,Z} \otimes \mathrm{id}_Y) \alpha_{X,Z,Y} (\mathrm{id}_X \otimes \tau_{Y,Z}) &= \alpha_{Z,X,Y} \tau_{X \otimes Y,Z} \alpha_{X,Y,Z}. \end{aligned}$$

The second and the third identities express the commutativity of the famous pentagon and hexagon diagrams.

Theorem 0.2 (MacLane's Coherence Theorem)**.** *In a symmetric monoidal category, two compositions of α's, α^{-1}'s, and τ's between objects $X_1 \otimes X_2 \otimes \dots \otimes X_n$ with arbitrary positioning of parentheses and $X_{i_1} \otimes X_{i_2} \otimes \dots \otimes X_{i_n}$ with arbitrary positioning of parentheses for any permutation $i_1, \dots, i_n$ of $1, \dots, n$ are equal.*

This is a quite technical theorem, important for it asserts that in a tensor category, it makes sense to *identify* any possible tensor products of the same collection of objects, no matter in which order or succession it is made. This identification is done via a morphism whose uniqueness is guaranteed by the Coherence Theorem.

Examples 0.3.
(1) The category of vector spaces over a fixed ground field with the tensor product of vector spaces. The morphisms τ and α are defined element-wise: $\tau(x \otimes y) := y \otimes x$ and $\alpha(x \otimes (y \otimes z)) := (x \otimes y) \otimes z$.
(2) The category of complexes of vector spaces. The tensor product and α are defined the same way as for vector spaces, but $\tau(x \otimes y) = (-1)^{pq} y \otimes x$, where p and q are the degrees of elements x and y. The full subcategory of graded vector spaces gets the induced structure of a tensor category.
(3) The category of sets with respect to Cartesian product.

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(4) The "geometric" categories, those of topological spaces, manifolds (topological, smooth, or complex), and schemes — all with respect to the direct product. Moreover, the category of schemes over a fixed scheme is also a tensor category with respect to the fibered product.
(5) The category of representations of a fixed Lie group: the tensor product is given by the tensor product of the underlying vector spaces with the group action defined as $g(x \otimes y) := gx \otimes gy$. The same for the category of representations of a fixed Lie algebra, except that the Lie algebra action is defined as $g(x \otimes y) := gx \otimes y + x \otimes gy$.
(6) The category $A-\operatorname{mod}$ of left modules over an (associative) algebra A does not in general have the structure of a tensor category. The reason is that for two A-modules X and Y, the tensor product $X \otimes Y$ does not have a natural A-module structure — for instance, see why the formulas from the previous paragraph do not work. However, $X \otimes Y$ is naturally a module over the associative algebra $A \otimes A$. To have a universal A-module structure on $X \otimes Y$, it would be enough to have an algebra homomorphism $\Delta : A \to A \otimes A$, satisfying certain properties providing the existence of the morphisms α and τ satisfying the axioms of a symmetric monoidal category. It turns out that this all may be achieved if one requires that the homomorphism Δ defines the structure of a cocommutative coassociative bialgebra on A. The cocommutativity means $\Delta^{\mathrm{op}} = \Delta$, where $\Delta^{\mathrm{op}} := \Delta P$, where $P : A \otimes A \to A \otimes A$, $P(a \otimes b) := b \otimes a$. The coassociativity means $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.
(7) One can relax the cocommutativity condition in the previous example, which opens way to a whole field of mathematics called quantum group theory. Suppose we have a coassociative associative bialgebra A and an ivertible element $R \in A \otimes A$, such that $\Delta^{\mathrm{op}} = R \Delta R^{-1}$. Defining τ as PR, we get the structure of a symmetric monoidal category on $A-\operatorname{mod}$, provided $R_{12}R_{21} = 1$, which yields $\tau^2 = \mathrm{id}$, $(\mathrm{id} \otimes \Delta)R = R_{13}R_{12}$, and $(\Delta \otimes \mathrm{id})R = R_{13}R_{23}$, which imply the hexagon axiom. Here we used the standard notation: let $R_{12} := R = \sum R_{(1)} \otimes R_{(2)}$, then $R_{21} := \sum R_{(2)} \otimes R_{(1)}$, $R_{13} := \sum R_{(1)} \otimes 1 \otimes R_{(2)}$ and so on.

    Such a bialgebra is called a *quasi-triangular bialgebra*, although usually one does not require $R_{12}R_{21} = 1$, in which case we arrive to a more general notion of a *braided category*. These equations also imply the famous *quantum Yang-Baxter equation*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$