LECTURE 13: THE FULTON-MACPHERSON COMPACTIFICATION

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1. The Fulton-MacPherson compactification

The compactification we will be talking may be regarded as a higher dimensional generalization of the Deligne-Knudsen-Mumford (DKM) compactification $\overline{\mathcal{M}}_{0,n}$ of the genus zero moduli space $\mathcal{M}_{0,n}$. Start with the *configuration space* $C(X;n)$ of points in a complex compact manifold X of dimension d:

$$C(X;n) := \{\text{the space of configurations of } n \text{ distinct labeled points in } X\} = X^n \setminus \Delta,$$

where Δ is the large diagonal. To mention immediate connection with the moduli spaces, note that $\mathcal{M}_{0,n} = C(\mathbb{CP}^1;n)/\operatorname{PGL}_2(\mathbb{C}) = C(\mathbb{C};n-1)/\operatorname{Aff}(\mathbb{C})$, where $\operatorname{Aff}(\mathbb{C}) = \mathbb{C} \rtimes \mathbb{C}^*$ is the affine group of the complex line $\mathbb{C}$, which is at the same time the stabilizer in $\operatorname{PGL}_2(\mathbb{C})$ of the ∞ point in $\mathbb{CP}^1$.

The configuration space $C(X;n)$ is not compact, even if X is compact, unless we deal with such uninteresting cases as $n = 0$ or 1, or $d = \dim X = 0$. Whence we have the following "big" question.

Question 1. How to compactify $C(X;n)$?

The answer to this question becomes more and more complicated depending on how much you expect of it. There is always a topological resort, the one-point compactification: $(X^n \setminus \Delta)^\bullet$, which is singular. There is a common-sense compactification X^n, which is smooth, but not what is accepted to be smooth *as a compactification*. A *smooth compactification of a complex manifold* U is a compact complex manifold V containing U as an open subset, so that the complement $D := V \setminus U$ is a *normal-crossing divisor*, which in its turn means that each irreducible component of D is smooth and any number of components of D intersect transversally. These strong conditions are needed for many reasons: it is easier to describe the rational homotopy type of U, compute the mixed Hodge structure on it, study the Picard group and the Chow ring of U, if a smooth compactification is known.

When $d = \dim X = 1$, X^2 would serve the purpose: the diagonal Δ is a complex curve isomorphic to X, and it is always a normal-crossing divisor, since $\dim X^2 = 2$ and X is irreducible, which we assumed by saying that X is a complex manifold. It turns out that the problem of a smooth compactification in general was not resolved until the early nineties, when W. Fulton and R. MacPherson took care of it in [FM94]. Fulton-MacPherson's answer to this question is a clever sequence of blowups of the diagonals in X^n. The construction is so canonical and has such a nice combinatorial description that it makes sense in applying it blindly to a noncompact

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manifold X to get a number of nice things, including a cofibrant module over the configuration space operad, see M. Markl [Mar99].

1.1. **Stable degenerations.** The complex points of the Fulton-MacPherson (FM) compactification $\overline{C}(X;n)$ may be described as the set of equivalence classes of stable degenerations of X with n labeled punctures, similar to the case of the DKM compactification. A *stable degeneration* here is a copy of the space X along with a number of labeled punctures and a number of "double points", all distinct. At each double point x, a copy of the projective space $\mathbb{P}(\mathcal{T}_x X \oplus \mathbb{C}) \cong \mathbb{CP}^d$, where $\mathcal{T}_x X$ is the holomorphic tangent space, is attached by blowing up the double point x on X and identifying the resulting exceptional divisor $\mathbb{P}(\mathcal{T}_x X)$ with the same hyperplane in $\mathbb{P}(\mathcal{T}_x X \oplus \mathbb{C})$. These copies of $\mathbb{CP}^d$ may be thought of as bubbles attached to X, see the figure. This is not the whole story yet. On each of these bubbles, there is at least two distinct marked points, either labeled punctures, or new double points, all away from that hyperplane $\mathbb{P}(\mathcal{T}_x X)$. Note that the complement to this hyperplane is naturally identified with $\mathcal{T}_x X$. At each of these double points, if any, another copy of $\mathbb{P}(\mathcal{T}_x X \oplus \mathbb{C}) \cong \mathbb{CP}^d$ is attached through the blowup of the double point. Each of the new bubbles may have more punctures and bubbles attached, so that the total number of double points and punctures on each bubble (but not necessarily X itself) is at least three, and the dual graph of this configuration is a tree, which has a distinguished vertex, the one corresponding to X. (Fulton and MacPherson prefer to cut this vertex along with the adjacent edges off this tree and speak about a forest, which is a disjoint union of a finite number of trees.) This is not it, though. These stable degenerations are *equivalent*, so that the FM compactification becomes the set of equivalence classes of stable degenerations, under the action of the affine group $\mathrm{Aff}(\mathbb{C}^d) = \mathbb{C}^d \rtimes \mathbb{C}^*$ on each bubble. This group acts as the stabilizer of the hyperplane $\mathbb{P}(\mathcal{T}_x X)$ in the group $\mathrm{PGL}_{d+1}(\mathbb{C})$.

figure of a degenerate configuration

figure of the corresponding tree

As in the case of the DKM compactification, the FM compactification admits a "topological" stratification, the one in which the strata correspond to the trees, which are the dual graphs of stable degenerations. The trees that occur are the trees with n labeled legs (which one may or may not separate into $n-1$ legs and a root) with a distinguished vertex, so that the valence of each vertex, but the dsitinguished one, is at least three. The valence of the distinguished vertex may be arbitrary.

The moral difference from the DKM compactification is that we do not consider isomorphism classes of stable degenerations. This simple difference turns out to be quite subtle, should we look for an explicit relation. In fact, for $X = \mathbb{CP}^1$, consider only the FM strata corresponding to the trees all of whose vertices have valence at least three. Now we have to take a quotient of this space by identifying those stable genus zero curves which are isomorphic. This may be achieved by fixing a distinguished vertex for each isomorphism class of trees, and modding out the union of the corresponding strata by the action of $\mathrm{PGL}_2(\mathbb{C})$ which moves double points and punctures around on the distinguished component.

Another relation with DKM is that for $X = \mathbb{C}$ the union of the strata corresponding to all trees with a distinguished vertex of valence one is a direct product

$\mathcal{M}_{0,n+1} \times \mathbb{C}$. Also, if we mod out these strata by the free action of the group $\mathbb{R}$ of translations by moving the attachment point on $\mathbb{C}$ around, we will just get $\mathcal{M}_{0,n+1}$.

Still other relation: for $X = \mathbb{C}$ the union of the strata corresponding to the trees with a distinguished vertex of valence at least two, modded out by the affine group $\mathbb{C} \rtimes \mathbb{C}^*$, is again $\mathcal{M}_{0,n+1}$.

Remark 1. In fact, one can formally apply the FM compactification to any complex manifold X, whether it is compact or not. We will use the same notation then and abuse the terminology by still calling $\overline{C}(X; n)$ the FM *compactification*.

1.2. **Construction of the FM compactification.** The FM compactification is constructed as a sequence of blowups of the diagonals in X^n and their proper transoforms. These diagonals correspond to subsets $S \subset \{1, 2, \dots, n\}$ having at least two elements: the corresponding diagonal $\Delta_S = \{(x_1, \dots, x_n) \in X^n \mid x_i = x_j \text{ for all } i, j \in S\}$.

The construction is inductive with respect to n. The FM compactification $\overline{C}(X; n)$ of $C(X; n)$ will be denoted $X[n]$ here and will come with a universal family $X[n]^+ \to X[n]$. For $n = 1$, $X[1] := X$ — there is nothing to compactify. The universal family will be $X[1]^+ := X^2 \to X$ with the natural projection onto the first factor. For $n = 2$, $X[2]$ is the blowup of X^2 along the only diagonal $\Delta = \Delta_{1,2}$. The universal family $X[2]^+ \to X[2]$ and the next step $X[3]$ are constructed as follows.

1. Take the exceptional divisor D in $X[2]$. This is the preimage of the diagonal $X = \Delta_{1,2} \subset X^2$. Embed D in $X[2] \times X$ as the graph in $D \times X \subset X[2] \times X$ of the map $D \to \Delta_{1,2} = X$. This graph D is of dimension $\dim D$, which means it will be of codimension $d + 1$ in $X[2] \times X$. Under the natural projection to X^3, D will obviously map onto $\Delta_{1,2,3}$. Blow up $X[2] \times X$ along D and let $X[2]^+$ denote the result. As a universal family, take the natural composition $X[2]^+ \to X[2] \times X \to X[2]$.
2. Embed $X[2]$ into $X[2] \times X$ as the graphs of the two projections $X[2] \to X$. When projected to X^3, these graphs will map onto the diagonals $\Delta_{1,3}$ and $\Delta_{2,3}$. These graphs are of codimension d in $X[2] \times X$. Take the proper transforms of these two submanifolds under the blowup $X[2]^+$. They will now become disjoint. Blow up $X[2]^+$ along these two proper transforms. This is $X[3]$.

In general, this inductive procedure works as follows, if we assume $X[n]$ has already been constructed.

1. Take the exceptional divisor $D_{1,\dots,n}$ in $X[n]$ which is the preimage of the diagonal $\Delta_{1,\dots,n} \subset X^n$. Embed this divisor $D_{1,\dots,n}$ into $X[n] \times X$ as the graph in $D_{1,\dots,n} \times X \subset X[n] \times X$ of the map $D_{1,\dots,n} \to \Delta_{1,\dots,n} = X$. This graph $D_{1,\dots,n}$ is of codimension $d+1$ in $X[n] \times X$. Under the natural projection to X^n, this graph will obviously map onto $\Delta_{1,\dots,n}$. Blow up $X[n] \times X$ along $D_{1,\dots,n}$.
2. For each $n-1$-element subset $S \subset \{1, \dots, n\}$, take the corresponding exceptional divisor D_S in $X[n]$ and embed it into $X[n] \times X$ as the graph in $D_S \times X \subset X[n] \times X$ of the map $D_S \to \Delta_S \to X$, the last arrow taking $(x_1, \dots, x_n) \in \Delta_S$ to $x_i \in X$ for all $i \in S$. Then take the proper transform of this graph in the previous blowup of $X[n] \times X$. These codimension $d+1$ submanifolds are now separated, because of the previous blowup, and therefore it does not matter in which order you blowup along them. Blow

up along each of these submanifolds then. The image of the corresponding exceptional divisor under the projection to X^{n+1} is the diagonal $\Delta_{S\cup\{n+1\}}$.

(3) Repeat this construction consecutively, for each $n-2$-element subset $S \subset \{1,\dots,n\}$, *etc.*, down to the two element subsets S. Let $X[n]^+$ denote the resulting blowup space.

(4) Embed $X[n]$ into $X[n] \times X$ as the graphs of the n projections $X[n] \to X$. When projected to X^{n+1}, these graphs will map onto the diagonals $\Delta_{k,n+1}$ for $k = 1,\dots,n$. These graphs are of codimension d in $X[2] \times X$. Take the proper transforms of these two submanifolds under the blowup $X[n]^+$. They will now become disjoint. Blow up $X[n]^+$ along these two proper transforms. This is $X[n+1]$.

1.3. **Universality properties.** The description of complex points of the FM compactification in Section 1.1 suggests that it may represent a certain "point" functor, that is, the functor that to each scheme S assignes the set of some kind of isomorphism classes of families of stable degenerations of X over the base S.

Question 2 (really, to Fulton)**.** Is this the case?

Another universality property would concern the questions of minimality of the FM compactification and uniqueness of a minimal model of it.

Question 3. Is the FM compactification minimal? By *minimality* here we mean that it cannot be blown down to another compactification of the configuration space $C(X;n)$ with a normal-crossing divisor.

Question 4. If it is minimal, is it unique, that is, if there is another compactification of $C(X;n)$ with a normal-crossing divisor, does it blow down to $\overline{C}(X;n)$?

Surprisingly enough, none of these questions seems to have been addressed by the founding fathers [FM94].

References

[FM94] W. Fulton and R. MacPherson, *A compactification of configuration spaces*, Ann. Math. **139** (1994), 183–225.

[Mar99] M. Markl, *A compactification of the real configuration space as an operadic completion*, J. Algebra **215** (1999), no. 1, 185–204.