LECTURE 14: A REAL VERSION OF FULTON-MACPHERSON COMPACTIFICATION

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1. A real version of Fulton-MacPherson compactification

Although Fulton-MacPherson's paper [FM94] is done in principle over an arbitrary field, applying it to the field $\mathbb{R}$ of reals leaves the meticulous reader with a sudden sense of disorientation. Indeed, about half of the projective spaces encountered by the reader will be not orientable. Do not be discouraged, the meticulous reader! The reals always offer you the shelter of orientation.

1.1. **The real blowup.** A real blowup is not quite out of reach for a good old mathematician, [Kac66]. The construction goes along the complex one, except one speaks of the sphere of normal directions rather than the space of complex normal lines.

The construction is done locally, using coordinates, and then one shows that it is coordinate independent. Let D be the unit disk $\{|x| = 1 \mid x = (x_1, \dots, x_n) \in \mathbb{R}^n\}$ in $\mathbb{R}^n$. Let V be the submanifold defined by equations $x_{k+1} = \dots = x_n = 0$. Let $d_{k+1}, \dots, d_n$ be coordinates in a copy of $\mathbb{R}^{n-k}$, where we want to consider the unit sphere S^{n-k-1}. The *real blowup of D along V* will then be

$$\tilde{D} = \{(x, d) \in D \times S^{n-k-1} \mid x_i d_j = x_j d_i \text{ for } k+1 \le i, j \le n\}.$$

There is a projection $\pi : \tilde{D} \to D$ on the first factor of $D \times S^{n-k-1}$, so that $\pi^{-1}(p) \cong S^{n-k-1}$ for $p \in V$, whereas $E := p^{-1}(V)$ is a real codimension one subspace of $\tilde{D}$, the real analogue of the exceptional divisor.

This local construction fortunately behave well with respect to coordinate changes. If $x' = f(x)$ is a diffeomorphism of D with itself so that V is still defined as $x'_{k+1} = \dots = x'_n = 0$, then for the blowup $\tilde{D}'$ of D along V in the new coordinates , we have a diffeomorphism:

$$\begin{aligned} \tilde{f} : \tilde{D} \setminus E &\to \tilde{D}' \setminus E', \\ z &\mapsto f(z), \end{aligned}$$

extends to E as

$$(x, d) \mapsto (f(x), d'),$$

where $x_{k+1} = \dots = x_n = 0$ and $d'_j := \sum_{i=k+1}^n \frac{\partial f_j}{\partial x_i}(x) d_i$.

Topologically speaking, a blowup along V is homeomorphic to the complement of an open tubular neighborhood of V in D, as on can see from the following figure. Therefore, a real blowup is homotopy equivalent to $D \setminus V$.

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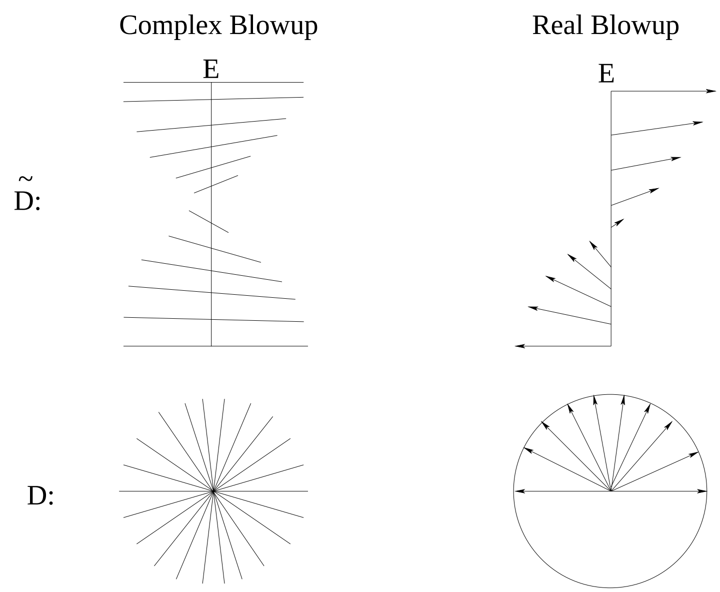

The *real FM compactification* is obtained by repeating Fulton-MacPherson's construction in the previous lecture by substituting real blowups everywhere in place of their complex counterparts.

The real compactification $\mathcal{M}(n)$ of the moduli space $\mathcal{M}_{0,n+1}$, which we discussed in Lecture 12, is related to the real FM compactification in a similar way the DM compactification $\overline{\mathcal{M}}_{0,n+1}$ is related to the complex FM compactification, see remarks at the end of Section 1.1 in Lecture 13.

1.2. **Operad structures on FM compactifications.** The FM compactification is not in general an operad. First of all, an experienced "operadchik" may recognize a module structure in the strata pattern. A (*right*) *module over an operad* $\mathcal{O}$ is a collection of S_n-spaces $\mathcal{M}(n)$, $n \geq 1$, along with the structure maps

$$\mathcal{M}(m) \times \mathcal{O}(n_1) \times \cdots \times \mathcal{O}(n_m) \to \mathcal{M}(n_1 + \cdots + n_m)$$

satisfying axioms similar to those of an operad. When X is parallelized, *i.e.*, the tangent bundle to X is trivialized, the collection $\{\overline{C}(X;n) \mid n \geq 1\}$ is indeed a module over the operad of compactified moduli spaces of configurations of points in $\mathbb{C}^d$ or $\mathbb{R}^d$, where $d = \dim X$, depending on whether X is complex or real. The word moduli refers to modding out by the group $\mathbb{C}^d \rtimes \mathbb{C}^*$ or $\mathbb{R}^d \rtimes \mathbb{R}^*_+$, respectively. These spaces may be again obtained by taking in $\overline{C}(X;n)$ for $X = \mathbb{C}^d$ or $\mathbb{R}^d$ the union of strata corresponding to the trees with a root vertex of valence one and modding out by the group of translations. The module structure is of course given by gluing along $\mathbb{P}^{d-1}$'s inserted by blowups at punctures.

What is really needed for an operad module structure on $\overline{C}(X;n)$ is a projective parallelizability of X, which is always the case when $d = 1$. The two examples below are of this kind.

In general, one can bypass this problem by considering *framed configurations*, see Markl [Mar99], similar to our spaces $\underline{\mathcal{N}}$ that will appear later, in the discussion of Topological Conformal Field Theories. The case $\mathcal{M}(n)$ of Lecture 12 is special, because the choice of a tangent direction (which gives a trivialization of the projectivized tangent space) at ∞ provides tangent directions at all punctures.

Example 1 (Stasheff Associahedra)**.** Consider the real FM compactification $\overline{C}(X;n)$ for $X = \mathbb{R}$. Take the strata corresponding to trees with the root vertex of valence at least two and mod out by the affine group $\mathbb{R} \rtimes \mathbb{R}_+^*$. The result is $S_n \times K^{n-2}$, where K^{n-2} is the *Stasheff associahedron.* The symmetric group shows up because of the linear ordering of points positioned on the line. This fact was noticed by Kontsevich [Kon94]. These spaces form an operad by the operation of attaching at punctures. The nice feature of this operad is that it is a cellular operad, i.e., it has a cell structure compatible with the operad structure. The reason is that each stratum of the topological stratification is a cell. Therefore, the corresponding chain complex is a DG operad. This DG operad is nothing but the A_∞ operad of Lecture 9.

Example 2 (Cyclohedra)**.** Cyclohedra is something which showed up in the work of Kapranov, if I am not mistaking. Consider the similar (real) compactified moduli space of $n+1$ points on S^1. This space may be obtained as the real FM compactification $\overline{C}(X;n)$ for $X = \mathbb{R}$ by taking the union of strata corresponding to the trees with a root vertex of valence one and modding out by the group $\mathbb{R}$ of translations. A connected component of this space corresponding to the cyclic order $1 < 2 < \cdots < n+1 < 1$ of the points is a *cyclohedron*, a cyclic counterpart of the associahedron. The cyclohedra (with all cyclic orders of punctures) form not an operad, but a right module over the associahedra (times S_n) operad of the previous example.

References

[FM94] W. Fulton and R. MacPherson, *A compactification of configuration spaces*, Ann. Math. **139** (1994), 183–225.

[Kac66] T. J. Kaczynski, *On a boundary property of continuous functions*, Michigan Math. J. **13** (1966), 313–320.

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[Mar99] M. Markl, *A compactification of the real configuration space as an operadic completion*, J. Algebra **215** (1999), no. 1, 185–204.