LECTURE 3: ALGEBRAS OVER A PROP AND CFT'S

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1. Algebras over a PROP and CFT's

Definition 1. We say that a vector space V is an *algebra over a PROP P*, if a morphism of PROP's from P to the endomorphism PROP of V is given. A morphism of PROP's should be a functor respecting the symmetric monoidal structure and the symmetric group actions, and equal to the identity on the objects.

An algebra over a PROP could have been called a *representation*, but since algebras over operads, which are similar objects, are nothing but familiar types of algebras, it is more common to use the term "algebra".

1.1. **Conformal field theory.**

Example 2. An example of an algebra over a PROP is a *Conformal Field Theory* (*CFT*), which may be defined as an algebra over the Segal PROP, see the previous lecture. The fact that the functor respects compositions of morphisms translates into the sewing axiom of CFT in the sense of G. Segal. Usually, one also asks for the functor to depend smoothly on the point in the moduli space $\mathcal{P}_{m,n}$.

Remark 3. This definition of a CFT describes only theories with a vanishing central charge. One needs to extend the Segal PROP by a line bundle to cover the case of an arbitrary charge, see Huang's book on vertex operator algebras.

Conformal field theory (CFT) provides a geometric background of string theory, which is considered as one of the steps toward *grand unification*, the unification of all forces of nature in a single theory. Whereas standard physics treats a particle as an ideal point, string theory thinks of a particle as a tiny loop or string. Respectively, a particle propagating in space along a path, a world line, becomes a string propagating along a world sheet, which is an orientable surface. The standard theory is quantized by using Feynman integral over the path space, whereas Feynman integral in string theory is an integral over the space of orientable surfaces. This integral can ultimately be integrated out to an integral over the moduli space of Riemann surfaces. To write down such an integral, one usually has to begin with certain data associated to Riemann surfaces. This data, called a CFT, should satisfy certain axioms. These axioms were singled out by G. Segal, see [**?**, **?**, **?**]. The definition above compresses Segal's definition using the notion of a PROP. Below, we will write down the data and axioms implied by our definition of a CFT, which at the same time show that it is equivalent to Segal's definition.

The definition of a CFT, see Example 2, may be unraveled as certain data satisfying certain axioms. Part of the data is a complex vector space V, called the

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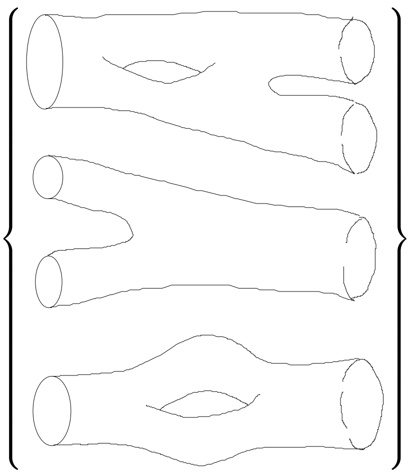
state space, together with a correspondence

m  n

$$\longmapsto \qquad |\Sigma\rangle : V^{\otimes m} \to V^{\otimes n}$$

A Riemann surface Σ bounding $m+n$ circles

A linear operator $|\Sigma\rangle$

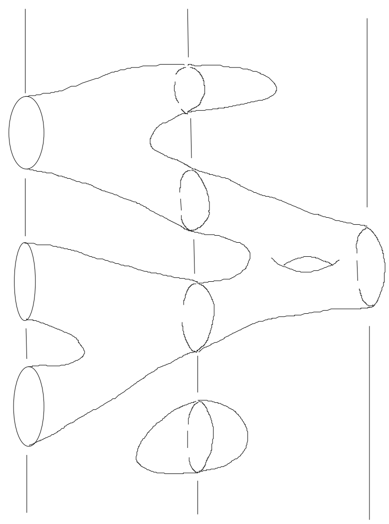
Here a surface Σ is a (not necessarily connected) compact Riemann surface, a complex manifold of dimension one. It has labeled (enumerated), nonoverlapping holomorphic holes, which are nothing but biholomorphic embeddings $\phi : D^2 \to \Sigma$, where $D^2 = \{z \in \mathbb{C} \mid |z| \le 1\}$ is the standard unit disk. One can think of ϕ as the choice of a holomorphic coordinate z at the hole. The first $m \ge 0$ circles are called *inputs* and the remaining $n \ge 0$ circles are called *outputs*.

This correspondence should satisfy the following axioms.

(1) **Conformal invariance**: The linear mapping $|\Sigma\rangle$ is invariant under isomorphisms of the surface Σ taking holes to the corresponding holes and preserving the holomorphic coordinates there. This axiom is implicit in our approach, because we considered only complex Riemann surfaces, which is equivalent to considering conformal classes thereof.
(2) **Permutation equivariance**: The correspondence $\Sigma \mapsto |\Sigma\rangle$ commutes with the action of the symmetric groups S_m and S_n on surfaces and linear mappings $V^{\otimes m} \to V^{\otimes n}$ by permutations of inputs and outputs.
(3) **Superposition property**:

$$|\Sigma_1 \coprod \Sigma_2\rangle = |\Sigma_1\rangle \otimes |\Sigma_2\rangle,$$

where $\Sigma_1 \coprod \Sigma_2$ is the disjoint union of two Riemann surfaces.
(4) **Factorization (sewing) property**: Sewing along the boundaries of the holes corresponds to composing of the corresponding operators:



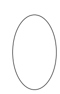
$$\longmapsto \qquad V^{\otimes m} \to V^{\otimes n} \to V^{\otimes k}$$

The sewing of the outputs of a surface with the inputs of another surface

The composition of the corresponding linear operators

Here *sewing along the boundaries* of two holomorphic holes with coordinates z and w means identifying two tubular neighborhoods $1/r < |z| < r$ and $1/r < |w| < r$, $r > 1$, of the boundaries via $z = 1/w$.

(5) **Normalization**:



$$\longmapsto \qquad \mathrm{id} : V \to V$$

The unit circle

The identity operator

Here the unit circle is understood as the "cylinder of zero width", the Riemann sphere $S^2 = \overline{\mathbb{R}}$ with the standard coordinate z and two holomorphic holes of radius one around 0 and ∞.

(6) **Smoothness**: Sometimes one requires that the operator $|\Sigma\rangle$ depends smoothly (or continuously) on the Riemann surface Σ. To make this assumption, one needs to assume that V is at least a topological space, introduce the structure of a smooth compex manifold on the space of linear mappings, and think of Σ as a point of the infinite dimensional moduli space of Riemann surfaces with holes. This axiom will not be essential to our considerations for the time being, and we will omit it. If instead of smoothness, we assumed *holomorphicity* in a certain sense, then we would be talking about a *chiral CFT* or a *vertex operator algebra* (*VOA*). This axiom follows from the PROP definition, if we assume that both the Segal PROP and the endomorphism PROP are enriched over the category of Fréchet orbifolds, that is, the morphisms $\mathrm{Mor}(m, n)$ are Fréchet orbifolds for our PROP's and the algebra structure on V respects this enrichment.

Remark 4. This definition describes in fact a CFT of *central charge* $c = 0$. An arbitrary central charge CFT relaxes Axiom 4: the operator $|\Sigma_2 \cup \Sigma_1\rangle$ corresponding to the result of sewing of two Riemann surfaces Σ_1 and Σ_2 is equal to the composition of two operators $|\Sigma_2\rangle \circ |\Sigma_1\rangle$ up to a nonzero factor λ:

$$|\Sigma_2 \cup \Sigma_1\rangle = \lambda |\Sigma_2\rangle \circ |\Sigma_1\rangle.$$

Throughout this course we will be mostly dealing with $c = 0$ theories.

Exercise 1. The constant λ generalizes the notion of a two-cocycle on $\mathrm{Diff}(S^1)$. Find out an equation of this type on λ. Is this the only condition λ must satisfy?