

LECTURE 4: THE LIE BIALGEBRA PROP

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1. THE LIE BIALGEBRA PROP

1.1. The Lie bialgebra PROP. This is an example, due to D. Sullivan, of a nicely defined PROP, the definition being intrinsic to the theory of graph homology. This PROP governs the class of Lie bialgebras.

Definition 1. A *Lie bialgebra* is a Lie algebra \mathfrak{g} with the structure of a Lie coalgebra given by a one-cocycle $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ on \mathfrak{g} with values in the \mathfrak{g} -module $\mathfrak{g} \wedge \mathfrak{g}$, *i.e.*, the linear map $\delta(g)$ must satisfy the following cocycle condition:

$$\delta([g_1, g_2]) = g_1\delta(g_2) - g_2\delta(g_1) \quad \text{for all } g_1, g_2 \in \mathfrak{g}.$$

Lie bialgebras are so-called quasi-classical limits of quantum groups (more precisely, quantum universal enveloping algebras). The structure of a Lie bialgebra also arises on the tangent space to the unit element of a *Poisson-Lie group*, a Lie group having a Poisson manifold structure, so that the group law defines a morphism of Poisson manifolds, *i.e.*, respects the Poisson bracket.

Exercise 1. The structure of a Lie coalgebra is dual to that of a Lie algebra: it is a good exercise on abstract nonsense (here: linear algebra) to write down the co-Jacobi identity in terms of δ . Make sure that if \mathfrak{g} is finite-dimensional, a Lie coalgebra structure is equivalent to a Lie algebra structure on \mathfrak{g} .

The following PROP will be called the *Lie bialgebra PROP*. In this PROP $\text{Mor}(m, n)$ will be vector spaces (thus, it will be a PROP of vector spaces) defined as quotient spaces of vector spaces spanned by graphs of a certain type. Whenever m or $n = 0$, we define $\text{Mor}(m, n) := 0$. For $m, n \geq 1$, the space $\text{Mor}(m, n)$ may be defined as follows.

Consider the vector space spanned freely by the (isomorphism classes of) directed oriented trivalent graphs Γ with $m + n$ legs labeled as inputs $1, \dots, m$ and outputs $1, \dots, n$. The graphs need not be connected, but must be finite. A *leg* is either an edge whose one end is free, that is, not a vertex, while the other end is a vertex, or a half-edge of an edge with two free ends. The adjective *directed* refers to the choice of directions on each edge, so that the legs are directed from the inputs and toward the outputs and the directions define a partial order on the set of vertices. *Trivalent* here means that all vertices must have one incoming and two outgoing edges or two incoming and one outgoing edges. Graphs with no vertices, *i.e.*, disjoint unions of edges each of which connects an input with an output, are allowed. An *orientation* on a graph means the choice of an ordering on the set of edges, up to the sign of a permutation. We also impose a relation $\Gamma^{\text{op}} = -\Gamma$, where Γ^{op} is the graph with the opposite orientation. The space $\text{Mor}(m, n)$ is a quotient space of the space of such graphs, defined as follows.

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The quotient is done with respect to the subspace generated by the edge expansions of a four-valent vertex in a graph of a similar kind. In fact, this edge expansion is precisely the differential in graph cohomology. The differentials of single-vertex four-valent graphs are defined as follows:

$$(1.1) \quad \begin{array}{l} \begin{array}{c} 1 \\ \diagdown \\ \text{d} \quad \text{---} \rightarrow \\ \diagup \\ 2 \\ \text{---} \rightarrow \\ \diagdown \\ 3 \end{array} \quad := \quad \begin{array}{c} 1 \\ \diagdown \\ \text{---} \rightarrow \\ \diagup \\ 2 \\ \text{---} \rightarrow \\ \diagdown \\ 3 \end{array} + \begin{array}{c} 1 \\ \diagdown \\ \text{---} \rightarrow \\ \diagup \\ 2 \\ \text{---} \rightarrow \\ \diagdown \\ 3 \end{array} + \begin{array}{c} 1 \\ \diagdown \\ \text{---} \rightarrow \\ \diagup \\ 2 \\ \text{---} \rightarrow \\ \diagdown \\ 3 \end{array} \\ \\ \begin{array}{c} 1 \\ \diagup \\ \text{d} \quad \text{---} \rightarrow \\ \diagdown \\ 2 \\ \text{---} \rightarrow \\ \diagup \\ 3 \end{array} \quad := \quad \begin{array}{c} 1 \\ \diagup \\ \text{---} \rightarrow \\ \diagdown \\ 2 \\ \text{---} \rightarrow \\ \diagup \\ 3 \end{array} + \begin{array}{c} 1 \\ \diagup \\ \text{---} \rightarrow \\ \diagdown \\ 2 \\ \text{---} \rightarrow \\ \diagup \\ 3 \end{array} + \begin{array}{c} 1 \\ \diagup \\ \text{---} \rightarrow \\ \diagdown \\ 2 \\ \text{---} \rightarrow \\ \diagup \\ 3 \end{array} \\ \\ \begin{array}{c} 1 \\ \diagdown \\ \text{d} \quad \text{---} \rightarrow \\ \diagup \\ 2 \\ \text{---} \rightarrow \\ \diagdown \\ 2 \end{array} \quad := \quad \begin{array}{c} 1 \\ \diagdown \\ \text{---} \rightarrow \\ \diagup \\ 2 \\ \text{---} \rightarrow \\ \diagdown \\ 2 \end{array} + \begin{array}{c} 1 \\ \diagdown \\ \text{---} \rightarrow \\ \diagup \\ 2 \\ \text{---} \rightarrow \\ \diagdown \\ 2 \end{array} + \begin{array}{c} 1 \\ \diagdown \\ \text{---} \rightarrow \\ \diagup \\ 2 \\ \text{---} \rightarrow \\ \diagdown \\ 2 \end{array} \\ \\ \quad \quad \quad + \quad \begin{array}{c} 1 \\ \diagup \\ \text{---} \rightarrow \\ \diagdown \\ 2 \\ \text{---} \rightarrow \\ \diagup \\ 2 \end{array} + \begin{array}{c} 1 \\ \diagup \\ \text{---} \rightarrow \\ \diagdown \\ 2 \\ \text{---} \rightarrow \\ \diagup \\ 2 \end{array} + \begin{array}{c} 1 \\ \diagup \\ \text{---} \rightarrow \\ \diagdown \\ 2 \\ \text{---} \rightarrow \\ \diagup \\ 2 \end{array} \end{array}$$

Then the *defining relations* between graphs are given by setting each of these differentials to zero. These defining relations give rise to relations in our graph spaces as follows. Take a graph with m inputs and n outputs of the same kind as above, except that one vertex of it is of valence four. Expand that vertex, using one of the three figures above, depending on the type of the four-valent vertex. This gives a linear combination of graphs. The subspace of relations, that is the subspace with respect to which we take the quotient to obtain $\text{Mor}(m, n)$ is the subspace spanned by such linear combinations. The orientation on the edge expansion is given by putting the new edge to the end of the ordering.

The *PROP composition* $\text{Mor}(m, n) \times \text{Mor}(n, k) \rightarrow \text{Mor}(m, k)$ is given by grafting the outputs of a graph to the corresponding inputs of another graph, i.e., connecting the respective legs of the two graphs so that a pair of legs becomes an edge connecting two vertices of the new graph. An *orientation on the composition* of two graphs is given by (1) reordering the edges of the first graph in such a way that the output legs follow the remaining edges, (2) reordering the edges of the second graph in such a way that the input legs precede the remaining edges, and (3) after grafting, putting the edges of the second graph after the edges of the first graph. The resulting ordering should look like that: the newly grafted edges in the middle, preceded by the remaining edges of the first graph and followed by the remaining edges of the second graph. The *symmetric groups act* by relabeling the inputs and the outputs and not changing the orientation of the graph.

Theorem 2 (Sullivan). *A vector space V has the structure of an algebra over the Lie bialgebra PROP, if and only if V has the structure of a Lie bialgebra. Moreover, this correspondence is functorial, that is, defines an isomorphism between the category of algebras over the Lie bialgebra PROP and the category of Lie bialgebras.*

Proof. The proof is more or less tautological.

1. Suppose V is an algebra over the Lie bialgebra PROP. This means graphs in consideration give rise to operators between tensor powers of V . Define the bracket $[\cdot, \cdot]$ and the cobracket δ as the operators corresponding to the following *basic* graphs, respectively:

$$\begin{array}{c}
 \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 2 \end{array} \rightarrow 1 \mapsto [g_1, g_2] \\
 \\
 1 \leftarrow \begin{array}{c} \text{---} \\ \diagup \\ 2 \\ \diagdown \\ 3 \end{array} \mapsto \delta(g) := g_{(1)} \wedge g_{(2)} := \sum_i g_{i1} \wedge g_{i2}
 \end{array}$$

The relations (1.1) we have imposed on the space of graphs rewrite as the vanishing of linear combinations of the following operators, respectively:

$$\begin{aligned}
 & [[g_1, g_2], g_3] - [[g_1, g_3], g_2] + [[g_2, g_3], g_1] = 0, \\
 & -\delta(\delta(g)_{(1)}) \wedge \delta(g)_{(2)} - \delta(g)_{(1)}^2 \wedge \delta(\delta(g)_{(2)})^{13} + \delta(g)_{(1)} \wedge \delta(\delta(g)_{(2)}) = 0, \\
 & \delta([g_1, g_2]) - \delta(g_1)_{(1)} \wedge [\delta(g_1)_{(2)}, g_2] \\
 & \quad - [g_1, \delta(g_2)_{(1)}] \wedge \delta(g_2)_{(2)} - \delta(g_1)_{(2)} \wedge [g_2, \delta(g_1)_{(1)}] + [g_1, \delta(g_2)_{(2)}] \wedge \delta(g_2)_{(1)} = 0,
 \end{aligned}$$

where for elements $x \in V$ and $y = y_{(1)} \wedge y_{(2)} \in V \wedge V$, we used the following notation $x^2 \wedge y^{13} := y_{(1)} \wedge x \wedge y_{(2)}$. Thus V becomes a Lie bialgebra.

2. Conversely, let V be a Lie bialgebra. Start with assigning the bracket and the cobracket to the two basic graphs, as above. The skew symmetry of the bracket and cobracket follows from the equations

$$\begin{array}{c}
 \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 2 \end{array} \rightarrow 1 = - \begin{array}{c} 1 \\ \diagup \\ \text{---} \\ \diagdown \\ 2 \end{array} \rightarrow 1, \\
 \\
 1 \leftarrow \begin{array}{c} \text{---} \\ \diagup \\ 2 \\ \diagdown \\ 3 \end{array} = - \begin{array}{c} 1 \\ \diagdown \\ \text{---} \\ \diagup \\ 2 \end{array} \leftarrow 1,
 \end{array}$$

which are equations of the type $\Gamma^{\text{op}} = -\Gamma$.

For each element in our quotient space, take a representing linear combination of oriented graphs. Cut each graph in it into PROP compositions of disjoint unions of the two basic graphs and the “identity” graphs --- . It may be done in a unique way: just cut each edge which is not a leg into two halves. Then take a composition (in the sense of the endomorphism PROP of V) of the bracket and the cobracket prescribed by the way the basic graphs are grafted together into the big graph. This assigns an operator to each graph and a linear combination of graphs. This operator does not depend on the choice of an representative in the quotient graph space, because the Jacobi, the co-Jacobi, and the cocycle identities are satisfied in our Lie bialgebra. \square

1.2. The L_∞ -bialgebra PROP. The graph description above suggests that if we consider a more general space of graphs, allowing vertices of any valence higher than two, and use the usual graph-cohomology differential, we will get a certain dg PROP. This RPOP is freely generated by trees with one vertex and m inputs

and n outputs, one for each pair (m, n) . Perhaps, this PROP should be called the L_∞ -bialgebra PROP. To justify that name, one needs to prove this dg PROP is a cofibrant resolution of the Lie bialgebra PROP, described in the previous section. This seems to be a nontrivial computation:

Conjecture 3. *The homology of this dg PROP is exactly the Lie bialgebra PROP.*