LECTURE 7: MORE OPERADS AND ALGEBRAS

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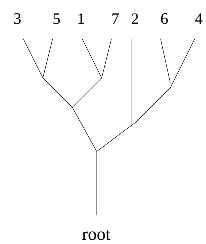
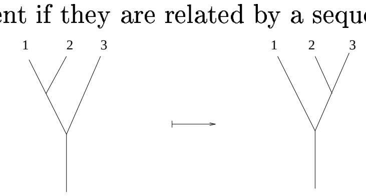
1. More Operads and Algebras

1.1. **The associative operad.** The *associative operad* $\mathcal{A}ssoc$ can be considered as a one-dimensional analogue of the commutative operad $\mathcal{T}op$. $\mathcal{A}ssoc(n)$ is the set of equivalence classes of connected planar binary (each vertex being of valence 3) trees that have a root edge and n leaves labeled by integers 1 through n:



If $n = 1$, there is only one tree — it has no vertices and only one edge connecting a leaf and a root. If $n = 0$, the only tree is the one with no vertices and no leaves — it only has a root. Unfortunately, I have a problem sketching it: it probably exists only in the quantum world.

Two trees are equivalent if they are related by a sequence of moves of the kind



performed over pairs of two adjacent vertices of a tree. The symmetric group acts by relabeling the leaves, as usual. The composition is obtained by grafting the roots of m trees to the leaves of an m-tree, no new vertices being created at the grafting points. Note that this is similar to sewing Riemann surfaces and erasing the seam, just as we did to define operad composition in that case. By definition, grafting a 0-tree to a leaf just removes the leaf and, if this operation creates a vertex of valence 2, we should erase the vertex.

Exercise 1. Prove that the structure of an algebra over the associative operad $\mathcal{A}ssoc$ on a vector space is equivalent to the structure of an associative algebra with a unit.

1.2. **The Lie operad.** The *Lie operad* $\mathcal{L}ie$ is another variation on the theme of a tree operad. Consider the same planar binary trees as for the associative operad, except that we do not include a 0-tree, *i.e.*, the operad has only positive components $\mathcal{L}ie(n)$, $n \geq 1$, and there are now two kinds of equivalence relations:

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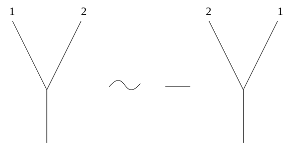
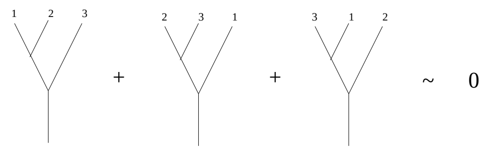


Skew Symmetry

and



Jacobi Identity

Now that we have arithmetic operations in the equivalence relations, we consider the Lie operad as an operad of vector spaces. We also assume that the ground field is of a characteristic other than 2, because otherwise we will arrive at the wrong definition of a Lie algebra.

Exercise 2. Prove that the structure of an algebra over the Lie operad $\mathcal{L}ie$ on a vector space over a field of a characteristic other than 2 is equivalent to the structure of a Lie algebra.

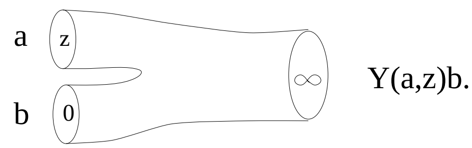
Exercise 3. Describe algebraically an algebra over the operad $\mathcal{L}ie$, if we modify it by including a 0-tree, whose composition with any other tree is defined as (a) zero, (b) the one for the associative operad.

1.3. **The Poisson operad.** Recall that a *Poisson algebra* is a vector space V (over a field of characteristic zero) with a unit element e, a dot product ab, and a bracket $[a, b]$ defined, so that the dot product defines the structure of commutative associative unital algebra, the bracket defines the structure of a Lie algebra, and the bracket is a derivation of the dot product:

$$[a, bc] = [a, b]c + b[a, c] \qquad \text{for all } a,\ b, \text{ and } c \in V.$$

Exercise 4. Define the *Poisson operad*, using a tree model similar to the previous examples. Show that an algebra over it is nothing but a Poisson algebra. [*Hint*: Use two kinds of vertices, one for the dot product and the other one for the bracket.]

1.4. **The Riemann surface operad and vertex operator algebras.** Just for a change, let us return to the operad $\mathcal{P}$ of Riemann surfaces, more exactly, isomorphism classes of Riemann spheres with holomorphic holes. What is an algebra over it? Since there are infinitely many nonisomorphic pairs of pants, there are infinitely many (at least) binary operations. In fact, we have an infinite dimensional family of binary operations parameterized by classes of pairs of pants. However modulo the unary operations, those which correspond to cylinders, we have only one fundamental binary operation corresponding to a fixed pair of pants. An algebra over this operad $\mathcal{P}$ is part of a CFT data at the tree level, the central charge $c = 0$. If we consider a holomorphic algebra over this operad, that is, require that the defining mappings $\mathcal{P}(n) \to \mathcal{E}nd_V(n)$, where V is a complex vector space, be holomorphic, then we get part of a chiral CFT, or an object which may be called a *vertex operator algebra* (*VOA*). This kind of object is not equivalent to what people used to call a VOA, but according to Huang's Theorem, a true VOA is a holomorphic algebra over a "partial pseudo-operad of Riemann spheres with rescaling", which is a version of $\mathcal{P}$, where the disks are allowed to overlap. The fundamental binary operation $Y(a,z)b$ for $a, b \in V$ of a VOA is commonly chosen to be the one corresponding to a pair of pants which is the Riemann sphere with a standard holomorphic coordinate and three unit disks around the points 0, z, and ∞ (No doubt, these disks overlap badly, but we shrink them on the figure to look better):



The famous associativity identity

$$Y(a, z-w)Y(b,-w)c = Y(Y(a,z)b,-w)c$$

for vertex operator algebras comes from the following natural isomorphism of the Riemann surfaces: