ELEMENTS OF SUPERGEOMETRY

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This paper is devoted to an exposition of the structure theory of supermanifolds and bundles on them, a description of Serre duality on supermanifolds, investigation of inverse sheaves, definition of characteristic classes and proof of the Grothendieck–Riemann–Roch theorem for supermanifolds.

This paper is devoted to certain questions of supergeometry not yet incorporated in the standard textbooks (see, e.g., [1, 3, 5]) or not treated therein in adequate detail. Some of these questions will be presented with proofs, for others we shall only provide references. Our main topics are: Serre duality on supermanifolds, the structure theory of supermanifolds and bundles on them, inverse sheaves, characteristic classes, and the Grothendieck–Riemannn–Roch theorem for supermanifolds. We are deeply indebted to I. A. Skornyakov for his permission to present his study of 1|1-dimensional bundles (Appendix to Sec. 4). The exposition in Sec. 3 is also based on his writings. We are grateful to A. Yu. Vaintrob for his useful comments.

1. Notation and Conventions

Our notations will largely coincide with those of Chaps. 3 and 4 in [5]. However, the evenness of a homogeneous element a in a $\mathbf{Z}_2$-graded Abelian group $A_0 \oplus A_1$ will be denoted by $\bar{a} \in \mathbf{Z}_2$ (rather than $\tilde{a} \in \mathbf{Z}_2$ as in [5]).

In contradistinction to [5], given modules M, N over a superring, we shall use the symbol Hom(M, N) for the group of even homomorphisms, i.e., those preserving the $\mathbf{Z}_2$-grading, and $\underline{\mathrm{Hom}}$(M, N) for the inner Hom, i.e., the ($\mathbf{Z}_2$-graded) module of all homomorphisms.

Let $X = (\mathfrak{X}, \mathcal{O}_X)$ be a superspace, i.e., a locally ringed space such that the structure sheaf $\mathcal{O}_X$ is a sheaf of supercommutative rings. For an ideal $\mathcal{N}_X := (\mathcal{O}_X)_1^2 \oplus (\mathcal{O}_X)_1$ in $\mathcal{O}_X$ we put $\mathrm{gr}_i \mathcal{O}_X := \mathcal{N}_X^i / \mathcal{N}_X^{i+1}$, $\mathrm{gr}\, X := (\mathfrak{X}, \mathrm{gr}\, \mathcal{O}_X := \oplus_i \mathrm{gr}_i \mathcal{O}_X)$, $\mathcal{O}_{\mathrm{red}} := \mathrm{gr}_0 \mathcal{O}_X$, $X_{\mathrm{red}} := (\mathfrak{X}, \mathcal{O}_{\mathrm{red}})$, $N_X^* := \mathrm{gr}_1 \mathcal{O}_X$ – the conormal sheaf of the natural embedding $X_{\mathrm{red}} \to X$. We shall say that X splits if there exists an isomorphism of superspaces $X \tilde{\to} \mathrm{gr}\, X$.

We shall work in one of three geometrical categories: 1) the category of smooth supermanifolds; 2) the category of analytic superspaces; 3) the category of superschemes. In the last two cases we shall also be interested in superspaces with smoothness conditions on the local rings – analytic and algebraic supermanifolds.

2. Serre Duality

1. Relative Berezinian (Dualizing Sheaf). Let $f: X \to \mathcal{M}$ be a proper (relative to the underlying space or scheme) smooth morphism of complex superspaces or superschemes of finite type over $\mathbf{C}$ of relative dimension M|N. We define the Berezinian Ber_f of f by

$$\mathrm{Ber}_f := \mathrm{Ber}\, \Omega^1_{X/\mathcal{M}, ev},$$

where $\Omega^1_{X/\mathcal{M}, ev}$ is the sheaf of relative even differentials, and the symbol Ber on the right denotes a simple generalization of the concept of Berezinian of a free module (the superanalog of maximal outer degree) to the case of a locally free sheaf.

2. Duality Theorems. It is natural to assume that in the situation of subsection 1 Ber_f is a dualizing sheaf in Grothendieck's sense. This means that for any bounded complex of $\mathcal{O}_X$-modules $\mathcal{F}^{\cdot}$ with coherent cohomology, considered as an object of the derived category $D^b_X(\mathrm{Coh})$ of bounded complexes of $\mathcal{O}_X$-modules with coherent cohomology, one has a canonical isomorphism into $D^b_{\mathcal{M}}(Coh)$:

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$$\theta_f^{\mathcal{F}}: Rf_*R\underline{Hom}_X(\mathcal{F}^\cdot, \mathrm{Ber}_f[-M]) \xrightarrow{\sim} R\underline{Hom}_{\mathcal{M}}(Rf_*\mathcal{F}^\cdot, \mathcal{O}_{\mathcal{M}}),$$

where $\mathrm{Ber}_f[-M]$ is the complex whose M-th term is the sheaf Ber_f and the other terms vanish.

This assertion may apparently be proved along the lines of the proof in the usual case. As communicated to us by Vaintrob (see also his paper [2] in this volume), the existence of a <u>dualizing complex</u> $K^\cdot$, i.e., an object $K^\cdot$ of the category $D_X^b(\mathrm{Coh})$ such that (1) is true with $\mathrm{Ber}_f[-M]$ replaced by $K^\cdot$, is readily proven even for any proper (relative to the underlying space or scheme) morphism of complex superspaces or superschemes. The uniqueness of $K^\cdot$ is verified by standard means. Thus, it remains only to show that for smooth f, $K^\cdot \simeq \mathrm{Ber}_f[-M]$. Nevertheless, up to the present only special cases have been treated in the literature. The first important special case is when $\mathcal{M}$ is a point and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Regarding this case see [15], where a more general assertion is proved, concerning non-proper analytic supermanifolds as well. One of the papers in this volume [8] will also use another special case of duality, when $\mathcal{M}$ is arbitrary and the morphism f is smooth and projective.

This means that there exists a commutative diagram

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & P_{\mathcal{M}}(\mathcal{E}) \\ & \searrow^{f} & \downarrow \\ & & \mathcal{M} \end{array},$$

in which $P_{\mathcal{M}}(\mathcal{E}) \to \mathcal{M}$ is a projectivization of a locally free $\mathcal{O}_{\mathcal{M}}$-module $\mathcal{E}$ and j is a closed embedding. In this situation the methods of [13] admit a direct supergeneralization, which leads to the duality theorem (see [6]). The following proposition sums up the facts that will be used in this volume.

<u>3. Proposition.</u> Let $f: X \to \mathcal{M}$ be a proper morphism of complex superspaces or superschemes of finite type over $\mathbf{C}$. Then:

a) There exists a dualizing complex $K^\cdot$.

b) If $\mathcal{M} = \mathrm{Spec}\,\mathbf{C}$, f is smooth of relative dimension M|N (i.e., X is an analytic or algebraic supermanifold of dimension M|N) and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then the dualizing complex of part (a) is exactly $\mathrm{Ber}_f[-M]$. In particular, one has a canonical isomorphism of $\mathbf{Z}_2$-graded vector spaces over $\mathbf{C}$

$$\underline{\mathrm{Ext}}_X^{M-i}(\mathcal{F}, \mathrm{Ber}_f = \mathrm{Ber}_X) = H^i(\mathcal{F})^*$$

for all $i \in \mathbf{Z}_+$.

c) If f is smooth and projective and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then the dualizing complex of part (a) is exactly $\mathrm{Ber}_f[-M]$. In particular, for example, if $\mathcal{F}$ and all $R^k f_* \mathcal{F}$, $k \in \mathbf{Z}_+$ are locally free, then one has a canonical isomorphism of $\mathcal{O}_{\mathcal{M}}$-modules

$$R^{M-i} f_*(\mathcal{F}^* \otimes_{\mathcal{O}_X} \mathrm{Ber}_f) \simeq (R^i f_* \mathcal{F})^*$$

for all $i \in \mathbf{Z}_+$. □

<u>3. Structure Theory of Supermanifolds and Locally Free $\mathcal{O}$-Modules</u>

<u>(Component Analysis)</u>

<u>1. Component Analysis of Supermanifolds.</u> Let $\mathfrak{X}$ be a manifold and E a locally free $\mathcal{O}_{\mathfrak{X}}$-module of rank 0|N, $N \in \mathbf{Z}_+$. To each such pair $(\mathfrak{X}, E)$ we associate a supermanifold $X^0 := (\mathfrak{X}, S^\cdot(E))$. Obviously, $X^0_{\mathrm{red}} = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, $N^*_{X^0} = E$, and $\mathrm{gr}\, X^0 = X^0$. It is natural to ask: How many supermanifolds X are there such that $\mathrm{gr}\, X = X^0$? Essentially, this is a question about a certain noncommutative cohomology series, whose structure we shall investigate below.

We introduce a point set

$$\sigma_{X^0} := \left\{ (X, \varphi) \,\middle|\, \begin{array}{l} X \text{ is a supermanifold and} \\ \varphi: \mathrm{gr}\, X \xrightarrow{\sim} X^0 \text{ is an isomorphism} \end{array} \right\} \Big/ \left\{ \begin{array}{c} \text{isomorphisms} \\ \text{of pairs} \end{array} \right\}$$

with distinguished point (X^0, id). By definition, two pairs (X_1, φ_1) and (X_2, φ_2) are isomorphic if there exists an isomorphism $\eta: X_1 \to X_2$ such that $\varphi_2 \circ \mathrm{gr}\, \eta = \varphi_1$.

Let $\mathcal{H}$ be the sheaf of (unipotent) groups on $\mathfrak{X}$ associated with the presheaf

$$U \mapsto \mathcal{H}(U) := \{ g \in \mathrm{Aut}(\mathcal{O}_{X^0}|_U) \mid \mathrm{gr}(g) = \mathrm{id} \},$$

where $U \subset \mathfrak{X}$ is an open set. Then a standard construction leads to an identification of pointed sets

$$\sigma_{X^0} = H^1(\mathfrak{X}, \mathcal{H}).$$

Note that the $\mathbf{Z}$-grading on $\mathcal{O}_{X^0}$ induces a natural filtration on $\mathcal{H}$. Indeed, the equality gr(g) = id for all U and all $g \in \mathrm{Aut}(\mathcal{O}_{X^0}|_U)$ is equivalent to the inclusion

$$(g-\mathrm{id})(S^i(E)) \subset S^{>i}(E) := \bigoplus_{k=i+1}^{N} S^k(E) \text{ for all } i, \quad 0 \leqslant i \leqslant N.$$

Thus, we define $\mathcal{H}^{2j} \subset \mathcal{H}$ as the sheaf associated with the presheaf

$$U \mapsto \mathcal{H}^{2j}(U) := \{g \in \mathrm{Aut}(\mathcal{O}_{X^0}|_U) \mid (g-\mathrm{id})(S^i(E)) \subset S^{\geqslant i+2j}(E) \text{ for } i \in \mathbf{Z}_+\}.$$

It is easy to see that $\mathcal{H}^2 = \mathcal{H}$, $\mathcal{H}^{2j} = 1$ for 2j > N and

$$[\mathcal{H}^{2k}, \mathcal{H}^{2l}] \subset \mathcal{H}^{2(k+l)}.$$

A filtration on $\mathcal{H}$ also induces a filtration on the sheaf of Lie algebras Lie($\mathcal{H}$) of the sheaf of groups $\mathcal{H}$.

Next, it is not hard to establish an isomorphism of sheaves of $\mathbf{Z}$-graded Lie algebras

$$\mathrm{gr}\,\mathrm{Lie}(\mathcal{H}) \xrightarrow{\sim} \mathcal{T}_n,$$

where gr Lie($\mathcal{H}$) is the sheaf of $\mathbf{Z}$-graded Lie algebras associated with the above described filtration on Lie($\mathcal{H}$) and $\mathcal{T}_n$ is the sheaf of Lie algebras of nilpotent vector fields on X^0, defined as

$$\mathcal{T}_n := \bigoplus_{j \geqslant 1} \mathcal{T}_n^{2j},$$

$$\mathcal{T}_n^{2j} := \{\psi \in \mathcal{T}_{X^0} \mid \psi(S^i(E)) \subset S^{i+2j}(E) \text{ for } i \in \mathbf{Z}_+\},$$

$\mathcal{T}_{X^0}$ is the tangent sheaf to X^0.

Finally, there are two obvious mutually inverse isomorphisms of sheaves of sets

$$\mathcal{H} \underset{\exp}{\overset{\log}{\rightleftarrows}} \mathcal{T}_n.$$

In particular, $H^0(\mathcal{H}) = H^0(\mathcal{T}_n) = \bigoplus_{j \geqslant 1} H^0(\mathcal{T}_n^{2j})$. Hence it follows that for any $j \geqslant 1$ the restriction map

$$H^0(\mathcal{H}) \to H^0(\mathcal{H}/\mathcal{H}^{2j})$$

is surjective, and so the following sequences of pointed sets are exact:

$$\begin{pmatrix}\text{distingu-}\\ \text{ished point}\end{pmatrix} \to H^1(\mathcal{H}^{2j}) \to H^1(\mathcal{H}) \to H^1(\mathcal{H}/\mathcal{H}^{2j}), \qquad (1)_j$$

$$\begin{pmatrix}\text{distingu-}\\ \text{ished point}\end{pmatrix} \to H^1(\mathcal{H}^{2j+2}) \to H^1(\mathcal{H}^{2j}) \to H^1(\mathcal{H}^{2j}/\mathcal{H}^{2j+2}), \qquad (2)_j$$

$j \geqslant 1$.

Sequences $(1)_j$ imply that the sets $H^1(\mathcal{H}^{2j})$, $j \geqslant 1$ define a (pointed) filtration on $\sigma_{X^0} = H^1(\mathcal{H})$. Sequences (2_j) furnish the characteristic maps $c_{2j}: H^1(\mathcal{H}^{2j}) \to H^1(\mathcal{H}^{2j}/\mathcal{H}^{2j+2})$ that measure the deviation of the terms of the filtration from one another.

These arguments constitute an outline of the proof of the following theorem.

2. THEOREM. There exist a canonical pointed filtration on σ_{X^0}:

$$\sigma_{X^0} = \sigma_2 \supset \sigma_4 \supset \ldots \supset \sigma_{2r} = \begin{pmatrix}\text{distingu-}\\ \text{ished point}\end{pmatrix} = (X^0, \mathrm{id})$$

and a sequence of maps $c_{2j}: \sigma_{2j} \to H^1(\mathcal{T}_n^{2j})$ such that for any $j \in \mathrm{N}$ $(X, \varphi) \in \sigma_{2j+2}$ if and only if $(X, \varphi) \in \sigma_{2j}$ and $c_2{}^j(X, \varphi) = 0$. □

The theorem has several important corollaries.

3. COROLLARY. A supermanifold X with gr X = X^0 is split if and only if for some (equivalently: for any) isomorphism $\varphi: \mathrm{gr}\, X \xrightarrow{\sim} X^0$ $c_{2j}(X, \varphi) = 0$ for all $j \in \mathrm{N}$.

Proof. This is simply a reformulation of Theorem 2. □

4. COROLLARY. a) If $H^1(\mathcal{T}^{2j}_{n})=0$ for all $j\in\mathbf{N}$, then any X with $\operatorname{gr} X \simeq X^0$ is split.

b) If $H^1(E^*\otimes S^{2j+1}(E))=0$ and $H^1(\mathcal{T}_{X^0_{\text{red}}}\otimes S^{2j}(E))=0$ for all $j\in\mathbf{N}$, then any X with $\operatorname{gr} X \simeq X^0$ is split.

Proof. a) $H^1(\mathcal{T}^{2j}_{n})=0$ implies that $c_{2j}(X,\varphi)=0$ for all φ and all $j\in\mathbf{N}$.

b) The restriction $\eta\mapsto\eta|_{X^0_{\text{red}}}$, where $\eta\in\mathcal{T}^{2j}_{n}$ defines an exact sequence of $\mathcal{O}_{X^0_{\text{red}}}$-modules

$$0\to E^*\otimes S^{2j+1}(E)\to\mathcal{T}^{2j}_{n}\to\mathcal{T}_{X^0_{\text{red}}}\otimes S^{2j}(E)\to 0.$$

Consequently, since all $H^1(E^*\otimes S^{2j+1}(E))$ and $H^1(\mathcal{T}_{X^0_{\text{red}}}\otimes S^{2j}(E))$, $j\in\mathbf{N}$ vanish, the same is true of $H^1(\mathcal{T}^{2j}_{n})$, $j\in\mathbf{N}$, and our assertion reduces to (a). □

5. COROLLARY. a) Every C^∞-smooth supermanifold is split.

b) If Y is a complex analytic supermanifold and Y_{red} is a Stein manifold, then Y is split.

c) If Y is an algebraic supervariety and Y_{red} is affine, then Y is split.

Proof. If Y_{red} is C^∞-smooth, Stein or affine, then all higher cohomology groups with coefficients in locally free $\mathcal{O}_{Y_{\text{red}}}$-modules vanish. □

6. We now show that there exist nonsplit analytic and algebraic supermanifolds. Since all supermanifolds of dimension M|1 are split by definition, nonsplit supermanifolds must be sought in dimensions M|N, $N\geqslant 2$. In fact, it suffices to consider dimension 1|2. Putting $\mathfrak{X}=X_{\text{red}}:=\mathbf{C}P^1$, $E:=\Pi\mathcal{O}_{\text{red}}(-2)\oplus\Pi\mathcal{O}_{\text{red}}(-2)$, where $\mathcal{O}_{\text{red}}(k)$ denotes the $-k$-th power of the tautological bundle on CP^1, one can show by direct calculation that for $X^0=(\mathfrak{X}, S^\cdot(E))$, considered as an analytic supermanifold, the canonical filtration on σ_{X^0} is of length 2. In particular, $\sigma_{X^0}\neq\left(\begin{smallmatrix}\text{distinguished}\\ \text{point}\end{smallmatrix}\right)$. Consequently, $c_2\neq 0$ and, by Corollary 3, there exists a nonsplit analytic supermanifold X such that $\operatorname{gr} X\simeq X^0$. Moreover, it is clear that in that case all X with $\operatorname{gr} X\neq X^0$ are automatically algebraic, and the condition of algebraic splitting is equivalent to that of analytic splitting.

We would like to emphasize that the case of a nonsplit analytic or algebraic supermanifold is not pathological, but in fact represents the general position. Thus, the super-Grassmannians $\operatorname{Gr}(a|b, S^{m|n})$ with $a,b\neq 0$, $a\neq m$, $b\neq n$ are nonsplit – see [9].

7. If a supermanifold $X=(\mathfrak{X},\mathcal{O}_X)$ is split, then there exists a projection $X\to X_{\text{red}}$. The converse is false, but one can establish a criterion, analogous to Theorem 2, for the existence of such a projection.

We first note that, since any projection $X\to X_{\text{red}}$ factors through the natural projection $X\to X_0:=(\mathfrak{X},(\mathcal{O}_X)_0)$, it will suffice to consider the existence of a projection $X_0\to X_{\text{red}}$. We now again fix $\mathfrak{X}=X^0_{\text{red}}$ and E and let $\mathcal{H}^0$ denote the sheaf of (unipotent) groups on $\mathfrak{X}$ associated with the presheaf

$$U\mapsto\mathcal{H}^0(U):=\left\{g_0\in\operatorname{Aut}\left(\mathcal{O}_{X^0_0}|_U\right)\,\middle|\,\operatorname{gr}(g_0)=\operatorname{id}\right\},$$

where $X^0_0:=(\mathfrak{X}, S^\cdot(E)_0)$ and $\operatorname{gr}(g_0)$ are considered relative to the standard filtration $\mathcal{O}_{X^0_0}$ by powers of $\mathcal{N}^2_{X^0}$. Let $\sigma_{X^0_0}:=H^1(\mathcal{H}^0)$. There exists a canonical identification of pointed sets:

$$H^1(\mathcal{H}^0)=\left\{(X_0,\psi_0)\;\middle|\;\begin{array}{l}X_0\text{ is a commutative}\\ \text{locally ringed space}\\ \text{with the filtration of}\\ \text{the structure sheaf and}\\ \psi:\operatorname{gr}X_0\simeq X^0_0\text{ is an iso-}\\ \text{morphism}\end{array}\right\}\Big/\left\{\begin{array}{l}\text{isomor-}\\ \text{phisms}\\ \text{of pairs}\end{array}\right\}$$

with distinguished point $(X^0_0, \operatorname{id})$.

8. THEOREM. There exist a canonical pointed filtration on $\sigma_{X^0_0}$

$$\sigma_{X^0_0}=\sigma^0_2\supset\sigma^0_4\supset\ldots\supset\left(\begin{smallmatrix}\text{distingu-}\\ \text{ished point}\end{smallmatrix}\right)=(X^0_0,\operatorname{id})$$

and a sequence of maps

$$c^0_{2j}:\sigma^0_{2j}\to H^1(\mathcal{T}_{X_{\text{red}}}\otimes S^{2j}(E)),$$

such that for any $j \in \mathbf{N}$ $(X_0, \varphi_0) \in \sigma^0_{2j+2}$ if and only if $(X_0, \varphi_0) \in \sigma^0_{2j}$ and $c^0_{2j}(X_0, \varphi_0) = 0$. □

Thus, if $\mathrm{gr}\, X_0 \simeq X^0_0$, then the existence of a projection $X \to X_{\mathrm{red}}$ is equivalent to vanishing of all classes $c^0_{2j}(X_0, \varphi_0)$ for some (equivalently: for any) isomorphism $\varphi_0: \mathrm{gr}\, X_0 \tilde{\to} X^0_0$.

9. Remark. There exist a canonical restriction morphism

$$\mathrm{Res}: \mathscr{H} \to \mathscr{H}^0,$$

and a diagram

$$\begin{array}{ccc} \sigma_{2j} & \xrightarrow{H^1(\mathrm{Res})} & \sigma^0_{2j} \\ {\scriptstyle c_{2j}}\downarrow & & \downarrow{\scriptstyle c^0_{2j}} \\ H^1(\mathscr{T}^{2j}_n) & \xrightarrow{H^1(\mathrm{res}_j)} & H^1(\mathscr{T}_{X_{\mathrm{red}}} \otimes S^{2j}(E)), \end{array}$$

in which res_j is the canonical restriction morphism $\mathscr{T}^{2j}_n \to \mathscr{T}_{X_{\mathrm{red}}} \otimes S^{2j}(E)$, which is commutative for any $j \in \mathbf{N}$.

10. Component Analysis of Locally Free $\mathcal{O}$-Modules. Let X be an arbitrary supermanifold, $\mathscr{E}'$ a locally free $\mathcal{O}_{X_{\mathrm{red}}}$-module. It is natural to ask: Under what conditions can $\mathscr{E}'$ be extended to a locally free $\mathcal{O}_X$-module $\mathscr{E}$ such that $\mathscr{E}_{\mathrm{red}} := \mathscr{E}/\mathscr{N}_X\mathscr{E} = \mathscr{E}'$? And if such an extension exists, how can one describe all nonisomorphic extensions? The answer to both questions is provided by a superanalog of a well-known theorem of Griffiths [11] about infinitesimal extensions of vector bundles on manifolds.

Assume that X is fixed, and let $X^{(i)} := (\mathfrak{X}, \mathcal{O}^{(i)} := \mathcal{O}_X/\mathscr{N}^{i+1}_X)$ denote the i-th "infinitesimal neighborhood" of the submanifold X_{red} in X. $X^{(i)}$ is not a supermanifold (for $i \neq 0, N, N+1$, where $M|N = \dim X$), but if X is algebraic (complex analytic), then $X^{(i)}$ is a superscheme (analytic superspace). Let $\mathscr{E}'$ be a locally free $\mathcal{O}_{\mathrm{red}}$-module and $\mathscr{E}^{(i)}$ a locally free $\mathcal{O}^{(i)}$-module such that $\mathscr{E}^{(i)}/\mathscr{N}_X\mathscr{E}^{(i)} = \mathscr{E}'$.

11. THEOREM. a) There exists a unique cohomology class

$$c(\mathscr{E}^{(i)}) \in H^2((\mathcal{E}nd\, \mathscr{E}' \underset{\mathcal{O}_{\mathrm{red}}}{\otimes} \mathscr{N}^{i+1}_X/\mathscr{N}^{i+2}_X)_0),$$

such that the locally free $\mathcal{O}^{(i+1)}$-module $\mathscr{E}^{(i+1)}$ with $\mathscr{E}^{(i+1)}/(\mathscr{N}^{i+1}_X/\mathscr{N}^{i+2}_X)\mathscr{E}^{(i+1)} = \mathscr{E}^{(i)}$ exists if and only if $c(\mathscr{E}^{(i)}) = 0$.

b) If $c(\mathscr{E}^{(i)}) = 0$, then the group $H^1((\mathcal{E}nd\, \mathscr{E}' \underset{\mathcal{O}_{\mathrm{red}}}{\otimes} \mathscr{N}^{i+1}_X/\mathscr{N}^{i+2}_X)_0)$ acts transitively on the set of isomorphim classes of locally free $\mathcal{O}^{(i+1)}$-modules $\mathscr{E}^{(i+1)}$ with $\mathscr{E}^{(i+1)}/(\mathscr{N}^{i+1}_X/\mathscr{N}^{i+2}_X)\langle \mathscr{E}^{(i+1)} = \mathscr{E}^{(i)}$. If there exists an extension $\tilde{\mathscr{E}}^{(i+1)}$ of $\mathscr{E}^{(i)}$ such that the natural restriction map $H^0(\mathcal{E}nd(\tilde{\mathscr{E}}^{(i+1)})) \to H^0(\mathcal{E}nd(\mathscr{E}^{(i)}))$ is surjective, then the action of $H^1((\mathcal{E}nd\, \mathscr{E}' \otimes \mathscr{N}^{i+1}_X/\mathscr{N}^{i+2}_X)_0)$ is effective.

The proof is practically the same as in the case of ordinary infinitesimal extensions of vector bundles – see [5, Chap. 2, Sec. 6], or the original paper of Griffiths [11]. □

12. COROLLARY. a) Given a locally free $\mathcal{O}_{\mathrm{red}}$-module $\mathscr{E}'$, there exists a locally free $\mathcal{O}_X$-module $\mathscr{E}$ with $\mathscr{E}_{\mathrm{red}} = \mathscr{E}'$ if and only if there exists a sequence $\mathscr{E}^{(0)} = \mathscr{E}', \mathscr{E}^{(1)}, \ldots, \mathscr{E}^{(N-1)}$ of extensions of $\mathscr{E}'$ such that $c(\mathscr{E}^{(i)}) = 0$ for $0 \leqslant i \leqslant N-1$. In particular, $\mathscr{E}$ exists if $H^2(\mathcal{E}nd\, \mathscr{E} \otimes \mathscr{N}^k_X/\mathscr{N}^{k+1}_X) = 0$ for all $k \in \mathbf{Z}_+$.

b) If $H^1(\mathcal{E}nd\, \mathscr{E}' \otimes \mathscr{N}^k_X/\mathscr{N}^{k+1}_X) = 0$ for all $k \in \mathbf{Z}_+$ and there exists a locally free $\mathcal{O}_X$-module $\mathscr{E}$ with $\mathscr{E}/\mathscr{N}_X\mathscr{E} = \mathscr{E}'$, then $\mathscr{E}$ is unique up to isomorphism. □

13. COROLLARY. If X is C^∞-smooth, complex analytic with Stein X_{red} or algebraic with affine X_{red}, then any locally free $\mathcal{O}_{\mathrm{red}}$-module $\mathscr{E}'$ is uniquely (up to isomorphism) extendable to a locally free $\mathcal{O}_X$-module $\mathscr{E}$. Moreover, there exists an isomorphism $\mathscr{E} \simeq \mathscr{E}' \otimes_{\mathcal{O}_{\mathrm{red}}} \mathcal{O}_X$, where the $\mathcal{O}_{\mathrm{red}}$-module structure on $\mathcal{O}_X$ is defined by any projection $X \to X_{\mathrm{red}}$. □

14. Any locally free $\mathcal{O}_{\mathrm{red}}$-module $\mathscr{E}'$ can also be extended to a locally free $\mathcal{O}_X$-module $\mathscr{E}$ under weaker assumptions concerning X: it is sufficient to demand the existence of a projection $p: X \to X_{\mathrm{red}}$ – then $\mathscr{E} = p^*(\mathscr{E}')$. In this context it is important to note that even if the supermanifold X is split it does not follow that any locally free $\mathcal{O}_X$-module $\mathscr{E}$ is isomorphic to $p^*(\mathscr{E}_{\mathrm{red}})$, where $p: X \to X_{\mathrm{red}}$ is the natural projection. In fact, one need only take a split X such that the sheaf $\mathcal{O}_{\mathrm{red}}$ has a nontrivial extension by means of the $\mathcal{O}_{\mathrm{red}}$-module $S^2(N_X^*) = \mathrm{gr}_2\mathcal{O}_X$. Adding $\mathrm{gr}_i\mathcal{O}_X$, $i \neq 0, 2$ to this extension as direct summands, we obtain an $\mathcal{O}_X$-module $\mathscr{E}$ such that $\mathscr{E}_{\mathrm{red}} \simeq \mathcal{O}_{\mathrm{red}}$. But $\mathscr{E} \not\simeq \mathcal{O} \simeq p^*(\mathscr{E}_{\mathrm{red}})$, since $\dim \Gamma(X, \mathscr{E}) < \dim \Gamma(X, \mathcal{O})$.

Since there does not exist a "fibered" functor gr, the considerations of this section have no immediate generalization to the relative case.

15. Remark. It is clear that the space of supermanifolds X with fixed underlying manifold is a subspace of the space of versal deformations of the supermanifold gr X. Thus, the noncommutative cohomology of Subsec. 1 implies a certain approach to deformation theory. On the other hand, deformation theory is in a certain sense a generalization of the structure theory of supermanifolds. These remarks also apply to the structure theory of coherent sheaves on supermanifolds. As to the general theory of deformations on supermanifolds, we refer the reader to Vaintrob's paper [2] in this volume.

16. Remark. The structure theory of supermanifolds and bundles on them has been rediscovered and rewritten many times. We mention two publications which describe the obstruction to the extension of a splitting (or bundle) from one infinitesimal neighborhood of the underlying manifold to the next: Palamodov [7] and Manin [5, Chap. 4, Sec. 2], and two that adopt an approach close to ours: Molotkov [14] and Rothstein [18].

4. Invertible Sheaves

1. Sheaves of Rank 1|0 and 0|1. Possible analogs of the Picard group for an algebraic supermanifold X are the group $[\mathrm{Pic}_0 X]$ of locally free sheaves of rank 1|0, the group [pic X] of locally free sheaves of rank 1|0 or 0|1 (both relative to tensor multiplication) and, finally, the set $[\mathrm{Pic}^{\Pi} X]$ of locally free sheaves of rank 1|1 with Π-symmetry. Obviously, the natural embedding $[\mathrm{Pic}_0 X] \hookrightarrow [\mathrm{Pic}\, X]$ can be completed to a short exact sequence

$$1 \to [\mathrm{Pic}_0 X] \to [\mathrm{Pic}\, X] \to \mathbf{Z}_2 \to 0,$$

which splits canonically via the homomorphism $\mathbf{Z}_2 \to [\mathrm{Pic}\, X]$, $0 \mapsto \mathcal{O}_X$, $1 \mapsto \Pi\mathcal{O}_X$. In this sense, consideration of the group [Pic X] is uninteresting, with the exception of those cases in which it appears naturally.

The group $[\mathrm{Pic}_0 X]$ is entirely analogous to the classical Picard group, but its structure is less trivial in the cases to which we are accustomed (projective superspaces, flag superspaces, SUSY-curves etc. – see [8, 9, 16]); the set $[\mathrm{Pic}^{\Pi} X]$ is a new invariant of the supermanifold (see [4, p. 311]).

We begin with an investigation of even (odd) invertible sheaves on a supermanifold X; by definition, they are locally free $\mathcal{O}_X$-modules of rank 1|0 (0|1). Clearly, the set of all invertible sheaves on a fixed X has the natural structure of a strictly commutative Picard category or a categorical Abelian group (on the formalism of Picard categories see, e.g., [10]) relative to tensor multiplication. Note that the parity of an invertible sheaf defines a canonical functor from this category into the category $\mathbf{Z}_2$ (regarded as a groupoid of two objects). Let $\mathrm{Pic}_0 X$ denote the kernel of the functor (i.e., the Picard category of all even invertible sheaves on X), then $[\mathrm{Pic}_0 X]$ is the group of isomorphism classes of the objects in $\mathrm{Pic}_0 X$. As in the usual case, one has a group isomorphism

$$[\mathrm{Pic}_0 X] \simeq H^1(\mathcal{O}_0^*),$$

where $\mathcal{O}_0^*$ is the subsheaf of invertible sections of $\mathcal{O}_X$, considered as a sheaf of multiplicative groups.

We now define the relative Picard functor. Considering the category $\mathbf{Ssch}_{\mathcal{M}}$ of superschemes (the category $\mathbf{San}_{\mathcal{M}}$ of analytic superspaces) over $\mathcal{M}$ in the standard way, where $\mathcal{M}$ is a superscheme (analytic superspace), we define for fixed $Y \to \mathcal{M} \in \mathrm{Ob}\,(\mathbf{Ssch}_{\mathcal{M}})$ [fixed $\mathrm{Ob}\,(\mathbf{San}_{\mathcal{M}})$] a functor

$$\mathbf{Pic}_0\,(Y/\mathcal{M}) : \mathbf{Ssch}_{\mathcal{M}}\,(\mathbf{San}_{\mathcal{M}}) \to \mathbf{Ab}$$

by the equality

$$\mathbf{Pic}_0\,(Y/\mathcal{M})\,(Z/\mathcal{M}) := [\mathrm{Pic}_0\,(Y \times_{\mathcal{M}} Z)] / p_2^*[\mathrm{Pic}_0\, Z],$$

where $\mathbf{Ab}$ is the category of Abelian groups, $Z \to \mathcal{M}$ an object of the appropriate category and $p_2 : Y \times_{\mathcal{M}} Z \to Z$ the natural projection. In particular, we define the Picard group of Y over $\mathcal{M}$ by

$$[\mathrm{Pic}_0\,(Y/\mathcal{M})] := \mathbf{Pic}_0\,(Y/\mathcal{M})\,(\mathcal{M}/\mathcal{M}).$$

Note that the important question as to whether the functor $\mathbf{Pic}_0$ is representable is nontrivial even in the absolute case (i.e., when $\mathcal{M}$ is a point) and requires further investigation.

2. Sheaves of Rank 1|1 with Π-Symmetry. Let X be a superspace. Define $\mathrm{Pic}^{\Pi} X$ as the category whose objects are pairs (S, p), where S is a locally free $\mathcal{O}_X$-module of rank 1|1, $p: S \tilde{\to} S$ is a Π-symmetry on S (i.e., an odd morphism such that $p^2 = -\mathrm{id}_S$), and the morphisms are the elements of the group $\underline{\mathrm{Hom}}(S_1, S_2)$ that define isomorphisms of $\mathcal{O}_X$-modules and commute with the Π-symmetry. There is a natural completely univalent functor

$$\alpha: \mathrm{Pic}_0 X \to \mathrm{Pic}^{\Pi} X,$$
$$\alpha(\mathscr{L}) := \mathscr{L} \oplus \Pi \mathscr{L},$$

but, unlike $\mathrm{Pic}_0 X$, $\mathrm{Pic}^{\Pi} X$ does not have the natural structure of a Picard category. In this case $[\mathrm{Pic}^{\Pi} X]$ is a pointed set of isomorphism classes of objects of the category $\mathrm{Pic}^{\Pi} X$ (with basis point $\mathcal{O}_X \oplus \Pi \mathcal{O}_X$). A standard construction enables us to identify pointed sets

$$\alpha_1: [\mathrm{Pic}^{\Pi} X] \tilde{\to} H^1({}^{\Pi}\underline{Aut}(\mathcal{O}_x \oplus \mathcal{O}_x)),$$

where ${}^{\Pi}\underline{Aut}(\mathcal{O}_X \oplus \mathcal{O}_X)$ is the sheaf of groups of local automorphisms of the object $\mathcal{O}_X \oplus \Pi \mathcal{O}_X$ of the category $\mathrm{Pic}^{\Pi} X$. As a sheaf of groups, ${}^{\Pi}\underline{Aut}(\mathcal{O}_X \oplus \Pi \mathcal{O}_X)$ is canonically isomorphic to the multiplicative group $\mathcal{O}_X^*$ of invertible sections of $\mathcal{O}_X$. Thus, there is an isomorphism of pointed sets

$$\alpha_2: [\mathrm{Pic}^{\Pi} X] \tilde{\to} H_1(\mathcal{O}_X^*).$$

Moreover, it is easy to verify the commutativity of the diagram

$$\begin{array}{ccc} [\mathrm{Pic}^{\Pi} X] & \overset{\alpha_2}{\tilde{\to}} & H^1(\mathcal{O}_X^*) \\ {\scriptstyle [\alpha]}\uparrow & & \uparrow {\scriptstyle H^1(\beta)} \\ [\mathrm{Pic}_0 X] & \tilde{\to} & H^1(\mathcal{O}_0^*), \end{array}$$

where [α] is the embedding induced by α and β is embedding $\mathcal{O}_0^* \hookrightarrow \mathcal{O}_X^*$.

For a more detailed study of the mutual relationships between $\mathrm{Pic}_0 X$, $\mathrm{Pic}^{\Pi} X$ and the set of sheaves of rank 1|1, the reader is referred to the supplement to this section.

Supplement to Sec. 4: General Properties of 1|1-Sheaves

0. Basic Objects. Let X be a complex supermanifold. In addition to the group $[\mathrm{Pic}_0 X]$ and pointed set $[\mathrm{Pic}^{\Pi} X]$ introduced in Sec. 4, our investigation of 1|1-sheaves (i.e., locally free $\mathcal{O}_X$-modules of rank 1|1) will need the pointed sets

$$F^{1|1}X := \{\text{isomorphism classes of } 1|1\text{-sheaves}\},$$

$$SF^{1|1}X := \left\{\begin{array}{l}\text{isomorphism classes of } 1|1\text{-sheaves, endowed with the} \\ \text{trivialization } t: \mathrm{Ber}\, S \tilde{\to} \mathcal{O}_X,\ t \text{ is even} \\ \text{if } \mathrm{rk}\,\mathrm{Ber}\, S = 1|0, \text{ and odd if } \mathrm{rk}\,\mathrm{Ber}\, S = 0|1\end{array}\right\}.$$

In this supplement we let $\mathrm{Pic}_0 X$ denote the group $[\mathrm{Pic}_0 X]$ and $\mathrm{Pic}^{\Pi} X$ the pointed set $[\mathrm{Pic}^{\Pi} X]$.

1. The Two Endomorphisms of $F^{1|1}X$. For each $S \in F^{1|1}X$, consider the two relative flag superspaces of relative dimension 0|1: $X_+ := \mathrm{Fl}_X(1|0; S) \overset{\pi_+}{\to} X$ and $X_- := \mathrm{Fl}_X(0|1; S) \overset{\pi_-}{\to} X$. Put $u^+(S) := (\pi_+)_* \mathcal{O}_{X_+}$ and $u^-(S) := (\pi_-)_* \mathcal{O}_{X_-}$. Clearly, $u^{\pm}$ are endomorphisms of the pointed set $F^{1|1}X$, $u^{\pm}(\Pi S) = u^{\mp}(S)$.

LEMMA. There are two canonical exact sequences

$$0 \to \mathcal{O}_X \to u^{\pm}(S) \to (\mathrm{Ber}\, S)^{\pm 1} \to 0.$$

The proof proceeds by direct evaluation of Čech cocycles for $u^{\pm}(S)$ in terms of the cocycle defining S. □

2. Cohomological Interpretation. Let G be a complex group superscheme. Define the sheaf G_X of groups of X as the sheaf associated with the presheaf

$$U \mapsto G(\Gamma(U, \mathcal{O}_X)),$$

$U \subset X$ is an open subset.

We also define the following group schemes: for an arbitrary complex algebra A

$$\mathbf{G}_m^{1|0}(A) := A_0^* \text{ relative to multiplication},$$
$$\mathbf{G}_m(A) := A^* \text{ relative to multiplication},$$
$$\mathbf{G}_a^{0|1}(A) := A_1 \text{ relative to addition}.$$

Proposition. One has canonical isomorphisms of the groups

$$\mathrm{Pic}_0 X = H^1(X, \mathbf{G}_{mX}^{1|0})$$

and pointed sets

$$\mathrm{Pic}^{\Pi} X = H^1(X, \mathbf{G}_{mX}),$$
$$F^{1|1} X = H^1(X, \mathrm{GL}(1|1)_X),$$
$$SF^{1|1} X = H^1(X, \mathrm{SL}(1|1)_X).$$

The proof is standard. □

For the rest of this supplement we shall write simply $H_1(G)$ instead of $H^1(X, G_X)$.

The following class may be regarded in a sense as a purely odd analog of the Chern class of a 1|1-sheaf with Π-symmetry.

Definition. With the morphism of group superschemes $c: \mathbf{G}_m \to \mathbf{G}_a^{0|1}$ given by the formula

$$c(a_0 + a_1) := a_1/a_0,$$

we associate a corresponding morphism of pointed sets

$$\mathrm{Pic}^{\Pi} X \tilde{\to} H^1(\mathbf{G}_m) \xrightarrow{c} H^1(\mathbf{G}_a^{0|1}) = H^1(\mathcal{O}_1).$$

Note that the morphisms $u^{\pm}: F^{1|1}X \to F^{1|1}X$ defined in Subsec. 1 are induced by idempotent endomorphisms of GL(1|1):

$$u^{+}: \begin{pmatrix} a & b \\ c & d \end{pmatrix} = H \mapsto \begin{pmatrix} \mathrm{Ber}\, H & b/d \\ 0 & 1 \end{pmatrix},$$
$$u^{-}: \begin{pmatrix} a & b \\ c & d \end{pmatrix} = H \mapsto \begin{pmatrix} 1 & 0 \\ c/a & (\mathrm{Ber}\, H)^{-1} \end{pmatrix}.$$

Hence it follows that $(u^{\pm})^2 = u^{\pm}$.

Note that if $H = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(1|1)(A)$, then $b/a = b/d$, $c/a = c/d$, and the endomorphisms $u^{\pm}|_{\mathrm{SL}(1|1)}$ may be regarded as homomorphisms

$$u^{\pm}: \mathrm{SL}(1|1) \to \mathbf{G}_a^{0|1},$$
$$u^{+}: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b/d,$$
$$u^{-}: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto c/d.$$

Define a homomorphism

$$u: \mathrm{SL}(1|1) \to \mathbf{G}_a^{0|1}$$

by the formula

$$u := u^{+} - u^{-}.$$

Then it is easy to see that the following sequence is exact:

$$1 \to \mathbf{G}_m \xrightarrow{\alpha} \mathrm{SL}(1|1) \xrightarrow{u} \mathbf{G}_a^{0|1} \to 0, \tag{1}$$

where $\alpha(a_0 + a_1) := \begin{pmatrix} a_0 & a_1 \\ a_1 & a_0 \end{pmatrix}$.

3. Composition of Elements of Pic^{Π} X. Le (S, p), (S', p') be two 1|1-sheaves with Π-symmetry. We extend p and p' to pairs of sup rcommuting symmetries $\tilde{p} := p \otimes \mathrm{id}$ and $\tilde{p}' := \mathrm{id} \otimes p'$ on $S \otimes S'$.

THEOREM. With the above notation

a) $S \otimes S' = (S \otimes S')_{+} \oplus (S \otimes S')_{-}$, where $(S \otimes S')_{\pm}$ are 1 ·-sheaves which are characteristic subspaces of the endomorphism $\tilde{p} \circ \tilde{p}'$ with char teristic values ±i,

b) $\mathrm{Ber}(S \otimes S')_{\pm}$ are trivial and

$$u^{\varkappa}((S \otimes S')_{\varkappa'}) = c(S') - i\varkappa\varkappa' c(S) \in H^1(\mathcal{O}_1),\ \varkappa,\ \varkappa' \in \{+, -\}.$$

Proof. (a) follows from the fact that $\tilde{p}\circ\tilde{p}'$ is an even endomorphism with square $-\mathrm{id}_{S\otimes S'}$.

(b) is proved by direct evaluation of Čech cocyles. □

4. Possibility of Introducing Π-Symmetry on 1|1-Sheaves. Using Theorem 3, we can associate to a pair of 1|1-sheaves with Π-symmetry a pair of 1|1-sheaves on which Π-symmetry can be introduced under certain cohomological restrictions – see Corollary 4 below.

Since a Π-symmetry defines a trivialization of the Berezinian [see Proposition 2 and formula (1)], a necessary condition for the existence of a Π-symmetry on a 1|1-sheaf S is the triviality of its Berezinian: $\mathrm{Ber}\, S \tilde{\to} \Pi\mathcal{O}$.

The freedom in the choice of a trivialization is measured by the group $H^0(\mathcal{O}_0^*)$, since the orbits of the latter's action on $H^1(SL(1|1))$ are the fibers of the map $H^1(\mathrm{SL}(1|1))\to H^1(\mathrm{GL}(1|1))$ induced by the short exact sequence $1\to \mathrm{SL}(1|1)\to \mathrm{GL}(1|1)\xrightarrow{\mathrm{Ber}} \mathbf{G}_m^{1|0}\to 1$.

The obstruction to introduction of the Π-symmetry induced by a given trivialization of the Berezinian on a 1|1-sheaf S, as implied by the exact sequence

$$H^1(\mathbf{G}_m)\to H^1(\mathrm{SL}(1|1))\xrightarrow{u} H^1(\mathbf{G}_a^{0|1})$$

[see (1)], is the class $u(S)\in H^1(\mathbf{G}_a^{0|1})$.

On the other hand, it is easy to see that

$$u^{\pm}(\alpha\mu)=\alpha^{\pm 1}u^{\pm}(\mu),$$

where $\alpha\in H^0(\mathcal{O}_0^*)$, $\mu\in H^1(\mathrm{SL}(1|1))$ and the multiplication on the right is the natural multiplication $H^0(\mathcal{O}_0^*)\otimes H^1(\mathcal{O}_1)\to H^1(\mathcal{O}_1)$. We have thus proved

Proposition. Let S be a 1|1-sheaf such that $\mathrm{Ber}\, S \simeq \Pi\mathcal{O}$. Then one can introduce a Π-symmetry on S if and only if there exists $\alpha\in H^0(\mathcal{O}_0^*)$ such that $\alpha^2 u^+(S)=u^-(S)$. □

COROLLARY. Under the assumptions of Theorem 3, the 1|1-sheaves $(S\otimes S')_{\pm}$ can be endowed with a Π-symmetry if and only if there exists $\alpha\in H^0(\mathcal{O}_0^*)$ such that $(\alpha^2+1)c(S)=(\alpha^2-1)c(S')$. □

5. Obstructions to Decomposition of a 1|1-Bundle with Π-Symmetry as the Direct Sum of Two Bundles Isomorphic Relative to Π. The pointed set of such obstructions is the image of the morphism c in the exact noncommutative cohomology sequence

$$\mathrm{Pic}_0 X\to \mathrm{Pic}^{\Pi} X\xrightarrow{c} H^1(\mathcal{O}_1), \tag{2}$$

induced by the exact sequence

$$1\to \mathbf{G}_m^{1|0}\to \mathbf{G}_m\xrightarrow{c} \mathbf{G}_a^{0|1}\to 0$$

(see Subsec. 2). The group $\mathbf{G}_m^{1|0}$ is central in $\mathbf{G}_m$, and therefore the exact sequence (2) may be extended one more term to the right ([12, Corollary to Proposition 3.4.2]):

$$\mathrm{Pic}_0 X\to \mathrm{Pic}^{\Pi} X\xrightarrow{c} H^1(\mathcal{O}_1)\xrightarrow{\partial} H^2(\mathcal{O}_0^*).$$

The set Im c in which we are interested is equal to Ker ∂. We shall evaluate Ker ∂ explicitly.

Proposition. The following diagram is commutative:

$$\begin{array}{ccc} H^1(\mathcal{O}_1) & \xrightarrow{\alpha\mapsto\alpha\smile\alpha} & H^2(\mathcal{O}_0) \\ & \searrow^{\partial} & \downarrow{\scriptstyle exp} \\ & & H^2(\mathcal{O}_0^*) . \end{array}$$

Less formally: $\mathrm{Ker}\,\partial=\{\alpha\in H^1(\mathcal{O}_1)\,|\,\exp(\alpha^2)=1\}$.

Proof. Let $\{U_i\}$ be a sufficiently fine cover of X and let $\varphi_{ij}\in\mathcal{O}_1(U_i\cap U_j)$ define a Čech 1-cocycle. The coboundary morphism can be evaluated explicitly:

$$\partial(\{\varphi_{ij}\})=\{1-\varphi_{jk}\varphi_{ik}+\varphi_{jk}\varphi_{ij}-\varphi_{ik}\varphi_{ij}\}=\{\exp(-\varphi_{jk}\varphi_{ik}+\varphi_{jk}\varphi_{ij}-\varphi_{ik}\varphi_{ij})\}.$$

Clearly, the collection $-\varphi_{jk}\varphi_{ik}+\varphi_{jk}\varphi_{ij}-\varphi_{ik}\varphi_{ij}\in\mathcal{O}_0(U_i\cup U_j\cap U_k)$ defines a Čech 2-cocycle which is the product $\{\varphi_{ij}\}\smile\{\varphi_{ij}\}$. □

Note that in the case of algebraic X

$$\operatorname{Ker}\left(H^2(\mathcal{O}_0)\xrightarrow{\exp}H^2(\mathcal{O}_0^*)\right)=\operatorname{Im}(H^2(X_{\text{red}},\mathbf{Z})\to H^2(\mathcal{O}_0))=$$

{the group of obstructions to realization of singular 2-cycles on X_{red} by Cartier divisors}.

Thus, the obstructions to decomposition of 1|1-sheaves with Π-symmetry as direct sums of two Π-isomorphic locally free sheaves are the square roots of the obstructions exhibited above.

6. Obstructions to Decomposition of an Arbitrary 1|1-Bundle as a Direct Sum of Two Bundles Isomorphic Relative to Π. As in Subsec. 5, the pointed set of such obstructions is $\operatorname{Ker}\partial = \operatorname{Im}\lambda$ in the cohomology sequence

$$\operatorname{Pic}_0 X\to F^{1|1}X\xrightarrow{\lambda}H^1(\mathbf{G})\xrightarrow{\partial}H^2(\mathbf{G}_m^{1|0}), \tag{3}$$

induced by the exact sequence

$$1\to\mathbf{G}_m^{1|0}\to\mathrm{GL}(1|1)\xrightarrow{\lambda}\mathbf{G}\to 1,$$

where $\mathbf{G}=(\mathbf{G}_a^{0|1}\oplus\mathbf{G}_a^{0|1})\rtimes\mathbf{G}_m^{1|0}$ is the semidirect product relative to the action of $\mathbf{G}_m^{1|0}$ on $\mathbf{G}_a^{0|1}\oplus\mathbf{G}_a^{0|1}$ defined by

$$a(b,b'):=(ab,a^{-1}b'),\quad a\in\mathbf{G}_m^{1|0},\quad b,\ b'\in\mathbf{G}_a^{0|1},$$

$$\lambda\begin{pmatrix}a & b\\ c & d\end{pmatrix}:=\left(-\frac{b}{a},\ -\frac{c}{d};\ \operatorname{Ber}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right).$$

To describe $\operatorname{Ker}\partial$, we shall need yet another exact noncommutative cohomology sequence:

$$H^0(\mathbf{G}_m^{1|0})\to H^1(\mathbf{G}_a^{0|1}\oplus\mathbf{G}_a^{0|1})\xrightarrow{\mu}H^1(\mathbf{G})\to\operatorname{Pic}_0 X, \tag{4}$$

induced by the natural exact triple

$$0\to\mathbf{G}_a^{0|1}\oplus\mathbf{G}_a^{0|1}\to\mathbf{G}\to\mathbf{G}_m^{1|0}\to 1.$$

Let ψ denote the composite

$$H^1(\mathbf{G}_a^{0|1})\oplus H^1(\mathbf{G}_a^{0|1})\xrightarrow{\mu}H^1(\mathbf{G})\xrightarrow{\partial}H^2(\mathbf{G}_m^{1|0}),$$

obtained by splicing together (3) and (4). Thus, $\operatorname{Ker}\partial\supset\mu(\operatorname{Ker}\psi)\simeq\operatorname{Ker}\psi/H^0(\mathcal{O}_0^*)$.

Proposition. The following diagram is commutative:

$$\begin{array}{ccc} H^1(\mathcal{O}_1)\oplus H^1(\mathcal{O}_1) & \xrightarrow{(\alpha,\beta)\mapsto\alpha\smile\beta} & H^2(\mathcal{O}_0)\\ & \searrow^{\psi} & \downarrow \exp\\ & & H^2(\mathcal{O}_0^*). \end{array}$$

Less formally: $\operatorname{Ker}\psi=\{(\alpha,\beta)\in H^1(\mathcal{O}_1)\oplus H^1(\mathcal{O}_1)\mid\exp(\alpha\beta)=1\}$.

The proof is analogous to that of Proposition 5. □

7. Special Case: $H^0(\mathcal{O})=\mathbf{C}$, $\operatorname{Pic}_0 X=H^1(\mathcal{O}_0^*)=\{1\}$. Put $W^1:=H^1(\mathcal{O}_1)$, $W^2:=H^2(\mathcal{O}_0)$, $L:=H^2(\mathbf{Z})$. By virtue of these assumptions and Subsec. 5, $L\hookrightarrow W^2$ and L = {the group of obstructions to realization of 2-cycles on X_{red} by Cartier divisors}. By Proposition 5,

$$\operatorname{Pic}^\pi X=A\{a\in W^1\mid a_2\in L\},$$

and by Proposition 6

$$F^{1|1}X=\{(a,b)\in W^1\oplus W^1\mid ab\in L\}/\mathbf{C}^*$$

[the action of $\mathbf{C}^*$ on $W^1\oplus W^1$ is given by $\alpha(w_1,w_2)=(\alpha w_1,\alpha^{-1}w_2)$]. By Subsec. 4, the image of $\operatorname{Pic}^\Pi X$ under the embedding $\operatorname{Pic}^\Pi X\overset{\alpha}{\hookrightarrow}SF^{1|1}X\overset{\beta}{\hookrightarrow}F^{1|1}X$, where α was defined in Subsec. 2 and β is the natural embedding, is

$$\operatorname{Pic}^\Pi X=\{(a,b)\in W^1\oplus W^1\mid\exists\alpha\in\mathbf{C}^*:\alpha^2a=b,\ ab\in L\}.$$

8. Remark. The above special case includes all homogeneous superspaces G/P, where G is a complex algebraic supergroup of type $Q(n), n\geqslant 3$, and P is a parabolic subgroup – see [8].

5. Characteristic Classes and the Riemann–Roch Theorem

1. Grothendieck Groups. In this section we shall consider algebraic or analytic supermanifolds (supervarieties) X with a condition on X_{red}: the natural homomorphism $K^{\cdot}(X_{red}) \to K_{\cdot}(X_{red})$ of the Grothendieck groups of the locally free $\mathcal{O}_{red}$-modules $K^{\cdot}(X_{red})$ and coherent $\mathcal{O}_{red}$-modules $K_{\cdot}(X_{red})$ is an isomorphism (for example, X_{red} is a nonsingular algebraic variety).

Perhaps it is worth recalling the definition of the Grothendieck groups in our context.

Definition. $K_s^{\cdot}(X)$ $[K_{\cdot}^s(X)]$ is the factor group of the free Abelian group generated by all coherent locally free (simply coherent) $\mathcal{O}_X$-modules modulo the subgroup generated by all expressions of the form $\mathcal{E}-\mathcal{E}'-\mathcal{E}''$, where $0\to\mathcal{E}'\to\mathcal{E}\to\mathcal{E}''\to 0$ is a short exact sequence of locally free (coherent) $\mathcal{O}_X$-modules.

The class of a $\mathcal{O}_X$-module $\mathcal{E}$ in the appropriate Grothendieck group will be denoted by $\mathrm{cl}_s^{\cdot}(\mathcal{E})$, $\mathrm{cl}_{\cdot}^s(\mathcal{E})$, of simply $\mathrm{cl}(\mathcal{E})$ if no confusion can arise.

2. Remark. Of course, by $\mathcal{O}_X$-modules here we mean $\mathbb{Z}_2$-graded $\mathcal{O}_X$-modules, and $\mathbb{Z}_2$-grading splits each of the groups $K_s^{\cdot}(X_{red})$, $K_{\cdot}^s(X_{red})$, and $K_{\cdot}^s(X)$ into two isomorphic direct summands. We have $K_{\cdot}^s(X_{red})=K_s^{\cdot}(X_{red})=K^{\cdot}(X_{red})\oplus\Pi K^{\cdot}(X_{red})$.

3. Remark. Relative to the operation «⊗» $K_s^{\cdot}(X)$ is a commutative (not in the supersense) ring with identity $1=\mathrm{cl}_s^{\cdot}(\mathcal{O}_X)$ and element $\Pi=\mathrm{cl}_s^{\cdot}(\Pi\mathcal{O}_X)$, $\Pi^2=1$.

4. Definition. Given a proper morphism $f:X\to Y$, we define a group homomorphism $f_!^s$: $K_{\cdot}^s(X)\to K_{\cdot}^s(Y)$ by the formula

$$f_!^s(\mathrm{cl}\,\mathcal{F}):=\Sigma(-1)^i\,\mathrm{cl}\,(R^i f_*\mathcal{F}),$$

where $\mathcal{F}$ is an arbitrary coherent $\mathcal{O}_X$-module.

5. Proposition (see [19]). The homomorphism $i_!^s: K_{\cdot}^s(X_{red})\to K_{\cdot}^s(X)$, where $i: X_{red}\to X$ is the natural embedding, is an isomorphism.

Proof. It is readily verified that the inverse of $i_!^s$ is the homomorphism $j:\mathrm{cl}\,(\mathcal{E})\mapsto\mathrm{cl}\,(_{\mathrm{dr}}\mathcal{E})$. □

This proposition states that the group $K_{\cdot}^s(X)$ does not reflect the existence of a superstructure on X. The group $K_s^{\cdot}(X)$, however, is too big and difficult to evaluate, as shown by the example of a projective superspace (see below). These disadvantages are true to a lesser degree of the group KS(X), which we shall now define.

6. Definition. Let $j:K_s^{\cdot}(X_{red})\to K_s^{\cdot}(X_{red})$ be the homomorphism $\mathrm{cl}\,(\mathcal{E})\mapsto\mathrm{cl}\,(\widetilde{\mathrm{gr}}\,\mathcal{E})$, where $\widetilde{\mathrm{gr}}\,\mathcal{E}:=\bigoplus_{i=0}^{\infty}\Pi^i\mathcal{N}^i\mathcal{E}/\mathcal{N}^{i+1}\mathcal{E}$, $\mathcal{N}$ is the ideal $(\mathcal{O}_x)_1\oplus(\mathcal{O}_x)_1^2$. Then, by definition, $KS(X):=\mathrm{Im}\,j\subset K_s^{\cdot}(X_{red})$. The class of the locally free $\mathcal{O}_X$-module $\mathcal{E}$ in KS(X) will be denoted by $j(\mathrm{cl}\,\mathcal{E})$ or simply $\mathrm{cl}\,(\mathcal{E})$.

7. Let $N^*:=\mathrm{cl}\,(N_X^*):=\mathrm{cl}\,(\mathcal{N}/\mathcal{N}^2)$ be the class of the conormal sheaf of the embedding $X_{red}\to X$, $\sigma_1(N^*):=\sum_{i=0}^{n}\mathrm{cl}\,(\Pi^i S^i(N_X^*))$, if $\mathrm{rk}\,N_X^*=0|n$. Then $\sigma_1(N^*)=\mathrm{cl}\,(\widetilde{\mathrm{gr}}\,\mathcal{O}_X)$ and we have

Proposition. a) $KS(X)\subset\sigma_1(N^*)\cdot K_s^{\cdot}(X_{red})$.

b) If there exists a projection $p: X\to X_{red}$, then

$$KS(X)=\sigma_1(N^*)\cdot K_s^{\cdot}(X_{red}).$$

Proof. a) For any locally free $\mathcal{O}_X$-module $\mathcal{E}$ $j(\mathrm{cl}\,\mathcal{E})=\mathrm{cl}\,(\widetilde{\mathrm{gr}}\,\mathcal{E})=\mathrm{cl}\,(\mathcal{E}_{red})\,\mathrm{cl}\,(\widetilde{\mathrm{gr}}\,\mathcal{O}_X)=\mathrm{cl}\,(\mathcal{E}_{red})\sigma_1(N^*)$.

b) For every locally free $\mathcal{O}_{red}$-module $\mathcal{E}$ $\mathrm{cl}\,(\mathcal{E})\times\sigma_1(N^*)\in KS(X)$, since $\mathrm{cl}\,(\mathcal{E})\cdot\sigma_1(N^*)=\mathrm{cl}\,(\mathcal{E})\,\mathrm{cl}\,(\widetilde{\mathrm{gr}}\,\mathcal{O}_x)=\mathrm{cl}\left(\mathcal{E}\otimes_{\mathcal{O}_{red}}\widetilde{\mathrm{gr}}\,\mathcal{O}_X\right)=j\,(\mathrm{cl}\,(p^*\mathcal{E}))$, and the $\mathcal{O}_x$-module $p^*\mathcal{E}$ is locally free. □

8. Example. Let $X=\mathbf{C}P^{m|n}$. Then $K_{\cdot}^s(X)\simeq K_{\cdot}^s(X_{red})\simeq K_s^{\cdot}(X_{red})\simeq\mathbf{Z}[t]/(1-t)^{m+1}\oplus\pi(\mathbf{Z}[t]/(1-t)^{m+1})$. The ring $K_s^{\cdot}(X)$, as far as we know, has not been evaluated, but the group [Pic X], which, as usual, is naturally embedded in the multiplicative group of invertible elements of $K_s^{\cdot}(X)$, is fairly big for m = 1: $[\mathrm{Pic}\,X]\simeq\mathbf{Z}_2\oplus\mathbf{Z}\oplus S^{n-2}(\mathbf{C}^{2|n})_0$ (see [8]). X is split (see [5, Chap. 4, Sec. 3.5]), and so $KS(X)=\sigma_1(N^*)\cdot K_s^{\cdot}(X_{red})$, moreover $\sigma_1(N^*)=(1+t^{-1})^n$, $t^{-1}=\sum_{i=0}^{m}(1-t)^i$.

9. The group KS(X) also has such familiar properties of $K^{\cdot}$ as the existence of the operations of multiplication and inverse image.

Proposition. a) The formula

$$\mathrm{cl}(\mathscr{E}_1)*\mathrm{cl}(\mathscr{E}_2):=\mathrm{cl}(\mathscr{E}_1\otimes\mathscr{E}_2)$$

for locally free $\mathcal{O}_x$-modules $\mathscr{E}_1$, $\mathscr{E}_2$ determines a well-defined ring structure on KS(X) with identity element $1=\mathrm{cl}(\mathcal{O}_x)=\sigma_1(N^*)$.

b) For any $x_1, x_2 \in KS(X)$,

$$x_1 * x_2 = x_1\cdot x_2/\sigma_1(N^*).$$

c) Define

$$f_*(x):=(f_{\mathrm{red}})_*(x)$$

where the multiplication on the right is that of the ring $K_S^{\cdot}(X_{\mathrm{red}})$. Then the formula

$$f_s^!(\mathrm{cl}(\mathscr{E})):=\mathrm{cl}(f^*(\mathscr{E}))$$

for an arbitrary morphism of supermanifolds $f:X\to Y$ and locally free $\mathcal{O}_Y$-module $\mathscr{E}$ determines a well-defined ring homomorphism $f_s^!:KS(Y)\to KS(X)$.

d) For any $y\in KS(Y)$,

$$f_s^!(y)=f'_{\mathrm{red}}(y/\sigma_1(N_Y^*))\cdot\sigma_1(N_X^*),$$

where $f^!_{\mathrm{red}}$ is the usual inverse image for the morphism $f_{\mathrm{red}}:X_{\mathrm{red}}\to Y_{\mathrm{red}}$.

Proof. Let $x_1=\mathrm{cl}(\mathscr{E}_1)$, $x_2=\mathrm{cl}(\mathscr{E}_2)$. Then $\mathrm{cl}(\mathscr{E}_1\underset{\mathcal{O}_X}{\otimes}\mathscr{E}_2)=\mathrm{cl}_s^{\cdot}(\widetilde{\mathrm{gr}}(\mathscr{E}_1\underset{\mathcal{O}_X}{\otimes}\mathscr{E}_2))=\mathrm{cl}_s^{\cdot}(\mathscr{E}_{1,\mathrm{red}}\underset{\mathcal{O}_{\mathrm{red}}}{\otimes}\mathscr{E}_{2,\mathrm{red}}\underset{\mathcal{O}_{\mathrm{red}}}{\otimes}\widetilde{\mathrm{gr}}\mathcal{O}_X)=\mathrm{cl}_s^{\cdot}(\widetilde{\mathrm{gr}}\,\mathscr{E}_1)\cdot\mathrm{cl}_s^{\cdot}(\widetilde{\mathrm{gr}}\,\mathscr{E}_2)/\mathrm{cl}_s^{\cdot}(\widetilde{\mathrm{gr}}\,\mathcal{O}_X)=x_1\cdot x_2/\sigma_1(N^*)$. This implies (b) and (a). The proof of (c) and (d) is similar. □

10. Definition. For a proper morphim $f:X\to Y$, define a group homomorphism $f_!^s:K_s^{\cdot}(X_{\mathrm{red}})\to K_s^{\cdot}(Y_{\mathrm{red}})$ by the formula

$$f_!^s(x):=(f_{\mathrm{red}})_!(x).$$

11. Remark. If one identifies $K^S_{\cdot}(X)$ with $K_S^{\cdot}(X_{\mathrm{red}})$ and $K^S_1(Y)$ with $K_S^{\cdot}(Y_{\mathrm{red}})$, Definition 4 is equivalent to Definition 10.

12. Proposition (projection formula). For a proper morphism $f:X\to Y$ and $x\in K_s^{\cdot}(X_{\mathrm{red}})$, $y\in K_s^{\cdot}(Y_{\mathrm{red}})$,

$$f_!^s(f_s^!(y)*x)=y*f_!^s(x).$$

The proof is a simple calculation. □

13. Characteristic Classes. We first describe the set of values of characteristic classes. Given a γ-filtration $F^1\supset F^2\supset\ldots$ of the ring $K^{\cdot}(X_{\mathrm{red}})$ (see [4]), we form a filtration $\{F_s^i\}=\{F^i\oplus\pi F^i\}\otimes\mathbf{Q}$ of the ring $K_s^{\cdot}(X_{\mathrm{red}})\otimes\mathbf{Q}$, which we again call a γ-filtration.

14. Definition. Define $GK_S(X)$ as the $\mathbf{Z}$-graded ring associated with the γ-filtration of the ring $K_s^{\cdot}(X_{\mathrm{red}})\otimes\mathbf{Q}$.

15. Definition. a) Put

$$f^*(y):=f^*_{red}(y)$$

for an arbitrary morphism of supermanifolds $f:X\to Y$ and any $y\in GK_s(Y)$.

b) Put

$$f_*(x)=(f_{\mathrm{red}})_*(x)$$

for a morphism $f:X\to Y$ which is projective on the underlying manifolds and any $x\in GK_s(X)$.

It is readily verified that f^* is a ring homomorphism, f_* is a $\mathbf{Z}$-graded group homomorphism of nonzero degree, and one has the projection formula

$$f_*(f^*(y)\cdot x)=y\cdot f_*(x).$$

16. Definition. Let $\mathscr{E}$ be a locally free $\mathcal{O}_X$-module, $\mathscr{E}_{\mathrm{red}}=\mathscr{E}_0\oplus\mathscr{E}_1$ and $(\Pi):=(\Pi)^{\mathscr{E}}:=\Pi^{\mathrm{rk}\,\Pi\mathscr{E}_1}$. Then:

a) the i-th Chern class is the class $c_i(\mathscr{E}) := (\Pi)\, c_i(\mathscr{E}_0 - \Pi\mathscr{E}_1) = (\Pi)\, \gamma^i(\mathrm{cl}\,\mathscr{E}_0 - \mathrm{cl}\,\Pi\mathscr{E}_1 - \mathrm{rk}\,\mathscr{E}_0 + \mathrm{rk}\,\Pi\mathscr{E}_1) \bmod F_s^{i+1}$, where γ^i is the standard operation in $K^{\cdot}(X_{red})$ (see [4]), $c_0(\mathscr{E}) := (\Pi)$;

b) the Chern polynomial is

$$c_t(\mathscr{E}) := (\Pi)\, c_t(\mathscr{E}_0 - \Pi\mathscr{E}_1) = \sum_{i=0}^{\infty} c_i(\mathscr{E})\, t;$$

c) the exponential Chern character is the class

$$\mathrm{ch}(\mathscr{E}) := \mathrm{ch}(\mathscr{E}_0) - \Pi\,\mathrm{ch}(\Pi\mathscr{E}_1) = \sum_{i=1}^{\mathrm{rk}\mathscr{E}_0} e^{a_i(\mathscr{E}_0)} - \Pi \sum_{i=1}^{\mathrm{rk}\Pi\mathscr{E}_1} e^{-\Pi a_i(\mathscr{E}_1)},$$

where $a_i(\mathscr{E}_0)$ and $a_i(\mathscr{E}_1)$ are found from the relations $c_t(\mathscr{E}_0) = \prod_{i=1}^{\mathrm{rk}\mathscr{E}_0} (1 + a_i(\mathscr{E}_0)\, t)$ and $c_t(\mathscr{E}_1) = \prod_{i=1}^{\mathrm{rk}\Pi\mathscr{E}_1} (\Pi + a_i(\mathscr{E}_1)\, t)$, defined by virtue of the splitting principle (see [4]) for X_{red}.

The classes $c_i(\mathscr{E})$ and $\mathrm{ch}(\mathscr{E})$ are elements of $GK_s(X)$, the class $c_t(\mathscr{E})$ is an element of the multiplicative group $1 + tGK_s(X)[[t]]$.

17. The formula for $c_i(\mathscr{E})$ is motivated by the following lemma.

LEMMA. Let $\mathscr{E}$ be a locally free $\mathcal{O}_X$-module, $\mathscr{E}_{red} = \mathscr{E}_0 \oplus \mathscr{E}_1$. Then $\mathrm{cl}(\mathrm{Ber}\,\mathscr{E}) - (\Pi) \equiv (\Pi)\, \gamma^1(\mathrm{cl}\,\mathscr{E}_0 - \mathrm{cl}\,\Pi\mathscr{E}_1 - \mathrm{rk}\,\mathscr{E}_0 + \mathrm{rk}\,\Pi\mathscr{E}_1) \equiv (\Pi)\, \gamma^1(\mathrm{cl}\,\mathscr{E}_0 + \mathrm{cl}\,\Pi\,\mathscr{E}_1^* - \mathrm{rk}\,\mathscr{E}_0 - \mathrm{rk}\,\Pi\mathscr{E}_1) \bmod F_s^2$.

Proof. By the corresponding statement of even geometry, $\mathrm{cl}(\det\mathscr{E}_0) - 1 \equiv \gamma^1(\mathrm{cl}\,\mathscr{E}_0 - \mathrm{rk}\,\mathscr{E}_0) \bmod F^2$ and $\mathrm{cl}(\det(-\Pi\mathscr{E}_1)) - 1 \equiv \gamma^1(\mathrm{cl}(-\Pi\mathscr{E}_1) + \mathrm{rk}\,\Pi\mathscr{E}_1) \bmod F^2$. At the same time, $\mathrm{cl}(\mathrm{Ber}\,\mathscr{E}) - (\Pi) = \mathrm{cl}(\mathrm{Ber}\,\mathscr{E}_{red}) - (\Pi) = (\Pi)\,\mathrm{cl}(\det\mathscr{E}_0 \det(-\Pi\mathscr{E}_1)) - (\Pi) \equiv (\Pi)\,(\mathrm{cl}(\det\mathscr{E}_0) + \mathrm{cl}(\det(-\Pi\mathscr{E}_1)) - 2) \bmod F_s^2$, whence the first congruence follows. The second is proved in similar fashion, if one notes that $\mathrm{cl}(\mathrm{Ber}\,\mathscr{E}) = (\Pi)\,\mathrm{cl}(\det\mathscr{E}_0 \cdot \det\Pi\mathscr{E}_1^*)$. □

18. Remark. This lemma provides yet another way of defining characteristic classes: $c_i(\mathscr{E}) := (\Pi)\, c_i(\mathscr{E}_0 + \Pi\mathscr{E}_1^*)$, and according to the same scheme – the Chern polynomial and exponential character. Using the lemma, one readily shows that the classes thus obtained are the same as the old ones.

19. Remark. Our definition of $\mathrm{ch}\,\mathscr{E}$ coincides, up to Π, with that of Quillen [17]: if K is the curvature form of a connection on $\mathscr{E}$ (that is, of a connection on $\mathscr{E}_0 \oplus \Pi\mathscr{E}_1$), then $\mathrm{ch}\,\mathscr{E} = \mathrm{str}\, e^{iK/2\pi} \bmod (1 - \Pi)$.

20. The following proposition is an immediate consequence of the definitions and the corresponding propositions of classical geometry.

Proposition. Let $\mathscr{E}$, $\mathscr{E}_1$, $\mathscr{E}_2$ be locally free $\mathcal{O}_X$-modules, $\mathscr{L}$ a locally free $\mathcal{O}_X$-module of rank 1|0 or 0|1. Then their characteristic classes have the following properties:

a) $\mathrm{ch}\,1 = 1$, $\mathrm{ch}\,\Pi = -\Pi$;

b) $c_1(\mathscr{L}) = \begin{cases} \mathrm{cl}\,\mathscr{L} - 1, & \text{if } \mathrm{rk}\,\mathscr{L} = 1|0, \\ \Pi - \mathrm{cl}\,\mathscr{L}, & \text{if } \mathrm{rk}\,\mathscr{L} = 0|1; \end{cases}$

c) $c_1(\mathscr{L}^*) = -c_1(\mathscr{L})$, $c_1(\Pi\mathscr{E}) = -c_1(\mathscr{E})\, \Pi^{\mathrm{rk}\mathscr{E}_0 + \mathrm{rk}\Pi\mathscr{E}_1}$, $\mathrm{ch}(\Pi\mathscr{E}) = -\Pi\,\mathrm{ch}(\mathscr{E})$;

d) for any morphism of supermanifolds $f: X \to Y$,

$$c_i(f_s^!(\mathscr{E})) = f^*(c_i(\mathscr{E}));$$

e) if $0 \to \mathscr{E}_1 \to \mathscr{E} \to \mathscr{E}_2 \to 0$ is exact, then $c_t(\mathscr{E}) = c_t(\mathscr{E}_1) \cdot c_t(\mathscr{E}_2)$; in particular, $c_1(\mathscr{E}) = c_1(\mathscr{E}_1)\, (\Pi)^{\mathscr{E}_2} + c_1(\mathscr{E}_2)\, (\Pi)^{\mathscr{E}_1}$, $(\Pi)^{\mathscr{E}} = c_0(\mathscr{E}) = c_0(\mathscr{E}_1) \cdot c_0(\mathscr{E}_2) = (\Pi)^{\mathscr{E}_1} \cdot (\Pi)^{\mathscr{E}_2}$, $c_1(\mathscr{E}^*) = -c_1(\mathscr{E})$ and, in addition, $\mathrm{ch}\,\mathscr{E} = \mathrm{ch}\,\mathscr{E}_1 + \mathrm{ch}\,\mathscr{E}_2$;

f) $\mathrm{ch}(\mathscr{E}_1 \otimes \mathscr{E}_2) = \mathrm{ch}\,\mathscr{E}_1 . \mathrm{ch}\,\mathscr{E}_2$;

g) $\mathrm{ch}: KS(X) \otimes \mathbf{Q} \to GK_s(X)$ is an embedding of superrings if $F_s^d = 0$ for sufficiently large d – in particular, if X_{red} is projective and nonsingular. □

21. Remark. If we extend the character ch from KS(X) to $K_s^{\cdot}(X_{red}) \otimes \mathbf{Q}$ by the formula

$$\mathrm{ch}(x) := \mathrm{ch}(x \cdot \sigma_1^{-1}(N^*)) = \mathrm{ch}(i_s^!(x)),$$

where $x \in K_s^{\cdot}(X_{red}) \otimes \mathbf{Q}$, then under the assumptions of Proposition 20(g) $\mathrm{ch}: K_s^{\cdot}(X_{red}) \otimes \mathbf{Q} \to GK_s(X)$ is an isomorphism of superrings relative to the (commutative) multiplication in $K_s^{\cdot}(X_{red}) \otimes \mathbf{Q}$: $x_1 * x_2 := x_1 \cdot x_2 \cdot \sigma_1^{-1}(N^*)$ (compare with Proposition 9).

22. Definition. Define the Todd class

$$\mathrm{td}: KS(X) \to GK_s(X),$$

subject to the following conditions:

a) $\mathrm{td}(l) = \left(\sum_{i=1}^{\infty} (-1)^i \frac{x^{i-1}}{i!}\right)^{-1}$, $x = c_1(\mathscr{L})$, $l = \mathrm{cl}(\mathscr{L})$, $\mathrm{rk}\,\mathscr{L} = 1|0$;

b) $\mathrm{td}(l) = \mathrm{ch}\,\sigma_1(\mathscr{L}^*) = 1 + e^{\Pi x}$, $x = c_1(\mathscr{L})$, $l = \mathrm{cl}(\mathscr{L})$, $\mathrm{rk}\,\mathscr{L} = 0|1$, $\sigma_1(\mathscr{L}^*) := 1 + \mathrm{cl}(\Pi\mathscr{L}^*)$;

c) $\mathrm{td} \circ f_s^! = f^* \circ \mathrm{td}$ for any morphism f:X → Y;

d) $\mathrm{td}(x_1 + x_2) = \mathrm{td}\,x_1 \cdot \mathrm{td}\,x_2$.

(Existence and uniqueness are proved in the standard manner.)

23. Remark. The Todd class is obviously chosen in such a way as to ensure validity of the Grothendieck–Riemann–Roch theorem for the embedding $i: X_{red} \to X$:

$$\mathrm{ch}(i_!^s x) = i_*(\mathrm{ch}\,x \cdot \mathrm{td}(-N))$$

for all $x \in K_s^{\cdot}(X_{red}) \otimes \mathbf{Q}$.

24. Remark. It is immediately evident from the definition that for a locally free $\mathcal{O}^X$-module $\mathscr{E}$ $\mathrm{td}(\mathrm{cl}\,\mathscr{E}) = \mathrm{td}(i_s^!(\mathrm{cl}\,\mathscr{E})) = \mathrm{td}(\mathrm{cl}_s^{\cdot}(\mathscr{E}_{red}))$.

The class td x is extended to $K_s^{\cdot}(X_{red}) \otimes \mathbf{Q}$ by the formula

$$\mathrm{td}\,x := \mathrm{td}(x \cdot \sigma_1^{-1}(N^*)),$$

where $x \in K_s^{\cdot}(X_{red}) \otimes \mathbf{Q}$.

25. Grothendieck–Riemann–Roch Theorem. THEOREM. Let f:X → Y be a smooth morphism of nonsingular algebraic supervarieties, and assume that the morphism f_{red} and varieties X_{red} and Y_{red} are projective. Then for any $x \in K_s^{\cdot}(X_{red}) \otimes \mathbf{Q}$, including $x \in KS(X)$, the following equality holds for elements of $GK_s(X)$:

$$\mathrm{ch}(f_!^s(x)) = f_*(\mathrm{ch}(x) \cdot \mathrm{td}(\mathscr{T}_f)), \tag{1}$$

where $\mathscr{T}_f := \mathrm{cl}\,\mathscr{T}_X - \mathrm{cl}\,f^*(\mathscr{T}_Y)$ is the virtual tangent sheaf of f.

Proof. A direct check, using the definitions and the results of this section, shows that (1) reduces to the Grothendieck–Riemann–Roch theorem for the morphism f_{red}. □

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SUPERCELL PARTITIONS OF FLAG SUPERSPACES

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A description is given of the partition of flat superspaces, which correspond to classical simple Lie supergroups, into Schubert supercells. The relative positions of Schubert supervarieties are studied and their singularities are resolved.

The goal of this paper is to achieve further understanding of the structure of flag superspaces, an important class of supermanifolds which arises quite naturally. We begin (Secs. 1-4) by constructing the partition of superspaces of complete flags into Schubert supercells, requiring the latter to satisfy a certain universality condition which is trivially valid in the classical case. The Weyl supergroups that arise in this context, among whose elements are reflections with respect to odd roots, index the supercells, and the dimension of each supercell equals the superlength of a suitable element of the Weyl supergroup. The superlength is defined combinatorially; that the definition is legitimate is a nontrivial combinatorial fact, for which we furnish a geometric proof.

Before generalizing the results to incomplete flags, which are superanalogs of the spaces G/P, where P is a parabolic subgroup of a simple algebraic group G (in Sec. 6), we describe the structure of parabolic subgroups of supergroups of types SL, OSp and Q in terms of root systems — see Sec. 5. This description shows that the spaces of incomplete G-flags constitute all G-equivariant factors of superspaces of complete flags.

In Sec. 7 we define an order in the Weyl supergroup, and prove that this order corresponds to the inclusion relation among Schubert supervarieties, i.e., closures of Schubert supercells.

Schubert supervarieties furnish natural examples of supervarieties with singularities. In Sec. 8 we shall present a construction that resolves these singularities, generalizing the classical Bott–Samelson construction. In the purely even case the Bott–Samelson construction enables one to prove that the singularities of Schubert varieties are rational; in supergeometry, however, the very existence of an analog of the construction is apparently a nontrivial property of singularities of Schubert supervarieties. The question of the rationality of singularities of Schubert supervarieties has not yet been fully investigated. At present it is not even clear just what a rational singular point of a supervariety is. (Recall that in the classical case every rational singular point is normal, implying that the local ring of the point contains no nilpotents.)

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