12. A. Grothendieck, "Sur quelques points d'algèbre homologique," Tôhoku Math. J., 9, No. 2, 119-183; No. 3, 185-221 (1957).
13. R. Hartshorne, "Residues and duality," Lecture Notes Math., 20 (1966).
14. V. Molotkov, "Sheaves of automorphisms and invariants of Banach supermanifolds," in: Mathematics and Mathematics Education. Proceedings of Fifteenth Conference of the Bulgarian Mathematical Society, Sl"nchev Bryag, April 6-9, 1986, Sophia (1986), pp. 281-283.
15. I. B. Penkov, "$\mathcal{D}$-modules on supermanifolds," Invent. Math., 71, No. 3, 501-512 (1983).
16. I. B. Penkov and I. A. Skornyakov, "Cohomologie des $\mathcal{D}$-modules tordus typiques sur les supervariétés de drapeaux," C. R. Acad. Sci., Ser. I, 299, No. 20, 1005 (1985).
17. D. Quillen, "Superconnections and the Chern character," Topology, 24, No. 1, 89-95 (1985).
18. M. Rothstein, "Deformations of complex supermanifolds," Proc. Amer. Math. Soc., 95, No. 2, 255-260 (1985).
19. T. Schmitt, "Coherent sheaves on analytic supermanifolds," Seminar Analysis, 1983/84, Acad. Wiss. DDR, Inst. Math. (1984), pp. 94-112.

SUPERCELL PARTITIONS OF FLAG SUPERSPACES

A. A. Voronov and Yu. I. Manin

UDC 512.743

A description is given of the partition of flat superspaces, which correspond to classical simple Lie supergroups, into Schubert supercells. The relative positions of Schubert supervarieties are studied and their singularities are resolved.

The goal of this paper is to achieve further understanding of the structure of flag superspaces, an important class of supermanifolds which arises quite naturally. We begin (Secs. 1-4) by constructing the partition of superspaces of complete flags into Schubert supercells, requiring the latter to satisfy a certain universality condition which is trivially valid in the classical case. The Weyl supergroups that arise in this context, among whose elements are reflections with respect to odd roots, index the supercells, and the dimension of each supercell equals the superlength of a suitable element of the Weyl supergroup. The superlength is defined combinatorially; that the definition is legitimate is a nontrivial combinatorial fact, for which we furnish a geometric proof.

Before generalizing the results to incomplete flags, which are superanalogs of the spaces G/P, where P is a parabolic subgroup of a simple algebraic group G (in Sec. 6), we describe the structure of parabolic subgroups of supergroups of types SL, OSp and Q in terms of root systems — see Sec. 5. This description shows that the spaces of incomplete G-flags constitute all G-equivariant factors of superspaces of complete flags.

In Sec. 7 we define an order in the Weyl supergroup, and prove that this order corresponds to the inclusion relation among Schubert supervarieties, i.e., closures of Schubert supercells.

Schubert supervarieties furnish natural examples of supervarieties with singularities. In Sec. 8 we shall present a construction that resolves these singularities, generalizing the classical Bott–Samelson construction. In the purely even case the Bott–Samelson construction enables one to prove that the singularities of Schubert varieties are rational; in supergeometry, however, the very existence of an analog of the construction is apparently a nontrivial property of singularities of Schubert supervarieties. The question of the rationality of singularities of Schubert supervarieties has not yet been fully investigated. At present it is not even clear just what a rational singular point of a supervariety is. (Recall that in the classical case every rational singular point is normal, implying that the local ring of the point contains no nilpotents.)

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennye Problemy Matematiki, Noveishie Dostizheniya, Vol. 32, pp. 27-70, 1988.

0090-4104/90/5101-2083$12.50

This paper may also be considered a first contribution to the study of algebraic topology of flag superspaces. A great many questions still remain open. For example, for a meaningful generalization to the supercase of such results of Bernshtein, Gel'fand and Gel'fand [1] and Demazure [14] as the combinatorial intersection index of Schubert varieties and comparison of classes of Schubert varieties with characteristic classes of flag sheaves, one needs, first and foremost, a regular cohomology theory, in which the classes of Schubert supercells would lie. The role of such a theory might possibly be fulfilled by bordism theory for supermanifolds (see Voronov and Zorich [5]) or by the cohomology theory of Barabov, Shvarts, Voronov and Zorich (see [4]).

In the exposition of our results we have favored purely geometric arguments, reducing the use of group-theoretic constructions to a minimum. We have thereby avoided the difficulties involved in factorization modulo the action of a supergroup. Moreover, the group-theoretic point of view does not always produce correct notions in the supercase. Thus, representation theory (see Kac [17]), Borel–Weil–Bott theory (see Penkov and Skornyakov [19], Penkov [11]) and the results of this paper indicate that the analogs of Borel subgroups are generally not maximal solvable subgroups but stabilizers B of complete flags. In contradistinction to the classical case, not all subgroups B are conjugates of one another, and consequently the superspace of complete flags splits into connected components (another manifestation of the difference is that one cannot define a system of representatives of the Weyl supergroup in G). The components appearing in this paper are somewhat more numerous than the conjugate classes of subgroups B — we find it more convenient to consider certain components corresponding to the same subgroup B as distinct. It should also be noted that the intuitive view of root systems in the supercase does not always accord with reality, since the root system of a simple Lie superalgebra need not be an abstract root system.

The role played by Schubert cells in classical geometry and representation theory determines the application of the results of this paper. In this connection we note that Schubert supercells have proved useful in understanding the geometry of supergravity (see [9]) and in the construction of reflections with respect to odd roots (see [11, 19]).

The main results of this paper were announced in [2, 3].

We are indebted to I. B. Penkov and I. A. Skornyakov, with whom we were in constant contact during the preparation of the paper.

1. Classical Supergroups and Flag Superspaces

1. Classical Supergroups. Let $T\simeq\mathbf{C}^{m|n}$ be the space of the standard representation of a classical algebraic supergroup G of type SL, OSp, ΠSp or Q. A type OSp group leaves invariant a nonsingular even symmetric form $b:T\to T^*$, a type ΠSp group — a nonsingular odd antisymmetric form $b:T\to T^*$, a type Q group — an odd involution $p:T\to T$, $p^2 = \mathrm{id}$. Henceforth we shall assume that the corresponding morphism b or p is fixed. In cases ΠSp and Q, the stipulated properties of this morphism imply that m = n. All these supergroups G, with the sole exception of OSp(2r, 2s) are connected. The group OSp(2r, 2s) splits into two connected components.

2. Connected Components of Flag Superspaces. Let ${}^{SL}I$ be the set of all sequences of the form $(\delta_1,\ldots,\delta_r)$ $\delta_i=p|q$, $\sum_{i=1}^{r}\delta_i=m|n$, $r\leqslant m+n$. Fix $I\in{}^{SL}I$.

Definition. Let S be a superscheme (over $\mathbf{C}$ — henceforth we shall take this for granted). A flag $0=\mathscr{S}_0\subset\mathscr{S}_1\subset\ldots\subset\mathscr{S}_{r-1}\subset\mathscr{S}_r=T_S=T\otimes_{\mathbf{C}}\mathcal{O}_S$ of locally free locally direct subsheaves in T_S is of type I if $\mathrm{rk}\,\mathscr{S}_i-\mathrm{rk}\,\mathscr{S}_{i-1}=\delta_i$ for all i: $1\leqslant i\leqslant r$.

Thus the rank of the i-th constituent of the flag is $d_i=\sum_{k=1}^{i}\delta_k$.

Definition – Lemma [7]. The functor ${}^{G}F_I$ on the category of superschemes over $\mathbf{C}$ that associates to each superscheme S the set of flags of type I in T_S satisfying the conditions (for G ≠ SL)

$$(b\otimes id_S)(\mathscr{S}_0\subset\ldots\subset\mathscr{S}_r)=\mathscr{S}_r^{\perp}\subset\ldots\subset\mathscr{S}_0^{\perp}\ \text{for OSp and}\ \Pi\text{Sp},$$
$$(p\otimes id_S)(\mathscr{S}_0\subset\ldots\subset\mathscr{S}_r)=\mathscr{S}_0\subset\ldots\subset\mathscr{S}_r\ \text{for Q},$$

is represented by the superspace ${}^{G}F_I$ of flags of type I. □

3. LEMMA. If I is the type of a flag in ${}^{G}F_I$, then

$$\delta_i(I)=\begin{cases}\delta_{r+1-i}(I) & \text{for } \mathrm{OSp},\\ \delta^c_{r+1-i}(I) & \text{for } \Pi\mathrm{Sp},\\ \delta^c_i(I) & \text{for } Q,\end{cases}$$

where $(a|b)^c=(b|a)$.

The proof follows directly from the definitions. □

4. Complete Flags. A G-flag is said to be complete (or full) if it has the maximum possible length r. The set of types of complete G-flags is denoted by ${}^{G}I_n$. The structure of the set ${}^{G}I_n$ is given by the following simple lemma.

LEMMA.

$${}^{SL}I_n=\left\{(\delta_1,\ldots,\delta_{m+n})\,|\,\delta_i=1|0 \text{ or } 0|1,\ \sum_{i=1}^{m+n}\delta_i=m|n\right\},$$
$${}^{OSp}I_n=\{(\delta_1,\ldots,\delta_{m+n})\in{}^{SL}I_n\,|\,\delta_i=\delta_{m+n+1-i}\},$$
$${}^{\Pi Sp}I_n=\{(\delta_1,\ldots,\delta_{m+n})\in{}^{SL}I_n\,|\,\delta_i=\delta^c_{m+n+1-i}\},$$
$${}^{Q}I_n=\{(\underbrace{1|1,\ldots,1|1}_{m})\}.\ \square$$

Definition. The superspace of complete G-flags is defined as the disconnected union

$${}^{G}F=\coprod_{I\in{}^{G}I_n}{}^{G}F_I.$$

Remark. All the ${}^{G}F_I$ are connected, except in the case G = OSp(2r, 2s), where ${}^{G}F_I$ splits into two components for each I. In the case G = Q(m), ${}^{G}F$ is connected.

2. Relative Position of a Pair of Complete Flags

1. Definition. Let $\mathcal{F}_{I,\cdot}$, $\mathcal{F}_{J,\cdot}$ be two G-flags of types $I, J\in{}^{G}I_n$. We shall say that they are regularly positioned relative to each other if for all i, j $\mathcal{F}_{I,i}\cap\mathcal{F}_{J,j}$ are locally direct locally free subsheaves in T_S of constant rank. The relative-position type of flags $\mathcal{F}_{I,\cdot}$ and $\mathcal{F}_{J,\cdot}$ is the matrix with elements

$$d_{ij}=\mathrm{rk}(\mathcal{F}_{I,i}\cap\mathcal{F}_{J,j}),\quad 0\leqslant i,j\leqslant r,$$

where r = m + n for G = SL, OSp, ΠSp, r = m for G = Q.

2. LEMMA. a) The matrix (d_{ij}) has the following properties:

$$d_{0j}=0|0,\quad d_{i0}=0|0,\quad d_{rj}=d_j(J),\quad d_{ir}=d_i(I),$$
$$0|0\leqslant d_{ij}-d_{i-1,j}\leqslant\delta_i(I),\quad 0|0\leqslant d_{ij}-d_{i,j-1}\leqslant\delta_j(J),$$
$$d_{i,j_0}\neq d_{i-1,j_0}\Rightarrow d_{ij}\neq d_{i-1,j}\quad\text{for}\quad j\geqslant j_0,$$
$$d_{i_0,j}\neq d_{i_0,j-1}\Rightarrow d_{ij}\neq d_{i,j-1}\quad\text{for}\quad i\geqslant i_0,$$

where $a|b\leqslant a'|b'$ if $a\leqslant a'$ and $b\leqslant b'$.

b) In cases G = OSp, ΠSp the matrix (d_{ij}) has the following additional symmetry properties:

$$d_{ij}=d_{m+n-i,m+n-j}-m|n+d_{m+n-i}(I)+d_{m+n-j}(J)\quad\text{for } G=\mathrm{OSp},$$
$$d_{ij}=d^c_{2m-i,2m-j}-m|m+d^c_{2m-i}(I)+d^c_{2m-j}(J)\quad\text{for } G=\Pi\mathrm{Sp}.$$

The proof is by a direct check. □

3. Weyl Supergroups. Define an action of S_{m+n} on ${}^{SL}I_n$ by the formula $\delta_i(wI)=\delta_{w^{-1}(i)}(I)$. The action of S_m on the singleton ${}^{Q}I_n$ is by definition trivial – the only possibility.

Definition. The Weyl supergroup ${}^{G}W$ of G is defined as

a) S_{m+n} for G = SL(m, n),

b) $\{w\in{}^{SL(m,n)}W\,|\,w({}^{G}I_n)={}^{G}I_n\}$ for G = OSp(m, n),

c) $\{w\in{}^{SL(m,m)}W\,|\,w({}^{G}I_n)={}^{G}I_n\}$ for G = ΠSp(m),

d) S_m for G = Q(m).

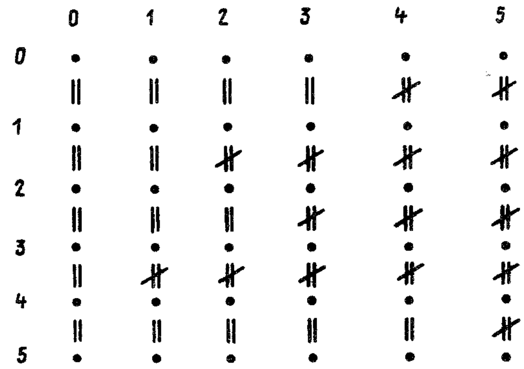

Fig. 1

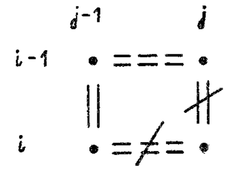

Fig. 2

$0|1$ or $1|0$, up to the $(m + n)$-th element of the first column, which is $d_1(J) = \delta_1(J) = 0|1$ or $1|0$. Hence there is exactly one jump in the first column – of $\delta_1(J)$. To its right, by property 2, there are also inequality signs.

Suppose now that there are $k - 1$ inequality signs in the $(k - 1)$-th column and to the right of each of them along the rows there are only inequality signs. Then, since the zeroth element of the k-th column is $0|0$, the last is $d_k(J)$ and the vertical jumps are at most $\delta_i(I) = 1|0$ or $1|0$, it follows that in this case there are exactly k inequality signs. In other words, exactly one new inequality sign will appear in the k-th column, and it will be the leftmost in its row, proving our assertion.

d) <u>Completion of Proof.</u> By step (c), given any matrix (d_{ij}) with properties 2, we can construct a corresponding triple (I, J, w). We shall prove that this is a map of the sets 2) → 3) in the statement of the lemma, i.e., that $w(J) = I$. In the notation of part (b), we have to show that $\delta_i(J)=\delta_{w^{-1}(i)}(I)=\delta_{k_i}(I)$. This follows from the following property of the matrix (d_{ij}): if $d_{ij}\neq d_{i-1,j}$ and $d_{i,j-1}=d_{i-1,j-1}$, then $d_{ij}\neq d_{i,j-1}$ and $d_{i-1,j-1}=d_{i-1,j}$ (see Fig. 2; this is a simple corollary of properties 2). Indeed, by the construction of k_i $d_{k_i,i}\neq d_{k_i-1,i}$ and $d_{k_i,i-1}=d_{k_i-1,i-1}$, and so $d_{k_i,i}\neq d_{k_i,i-1}$ and $d_{k_i-1,i-1}=d_{k_i-1,i}$, whence $\delta_{k_i}(I)=d_{k_i,i}-d_{k_i-1,i}=d_{k_i,i}-d_{k_i,i-1}=\delta_i(J)$. We thus have constructed a map 2) → 3), which is easily seen to be the inverse of the map 3) → 2) constructed in part (a).

The inverse 1) → 3) of the map 3) → 1) constructed in (a) is constructed as follows. The matrix of the relative position type of two complete flags has properties 2. The map 2) → 3) constructed above carries this matrix into a triple (I, J, w) such that $w(I) = J$.

We have thus proved the Combinatorial Lemma in the case G = SL.

<u>II. Cases G = OSp(m, n), ΠSp(m).</u> As we have already proved the lemma for the group SL(m, n), we can associate to every matrix (d_{ij}) with properties 2(a) corresponding to G a triple (I, J, w), $I, J\in {}^{SL}I_n$, $w\in {}^{SL}W$, $J=w(I)$. Since $I=(d_{i,m+n})$, $J=(d_{m+n,j})$, it is clear that I, $J\in {}^{G}I_n$. Properties 2(b) of the matrix (d_{ij}) guarantee that the permutation w carries a flag type in ${}^{SL}I_n$ which is symmetric about the midpoint into a similar type, i.e., $w\in {}^{G}W$. Thus we have a map 2) → 3). The inverse of the latter is precisely the map 3) → 2) constructed for G = SL(m, n). The fact that a triple (I, J, w) corresponding to G yields a matrix (d_{ij}) with properties 2(a), (b) is clear from the construction. The reasoning for the maps 1) → 3) and 3) → 1) is the same as for G = SL.

<u>III. Case G = Q(m).</u> The proof here is the same as for G = SL(m, n), except for some slight modifications concerning the dimensions of the flag constituents. □

<u>3. Schubert Supercells: Definition</u>

1. Throughout this section we fix G and write $F = {}^{G}F$, $W = {}^{G}W$. For each element $w\in W$, let $\mathbf{d}_{ij,w}$ denote the following function on $\coprod_{J=w(I)} F_I\times F_J$ with values in $\mathbf{Z}|\mathbf{Z}$, constant on each $F_I \times F_J$:

$$\mathbf{d}_{ij,w}|_{F_I\times F_J}=d_{ij,w,IJ}$$

– the matrix elements corresponding to the triple (I, J, w) by the Combinatorial Lemma. Let $\mathcal{F}_\cdot$ be the tautological flag on F. Consider the sheaves $\mathcal{F}_i\cap\mathcal{F}_j=p_1^*\mathcal{F}_i\cap p_2^*\mathcal{F}_j\subset T_{F\times F}=T\otimes_{\mathbf{C}}\mathcal{O}_{F\times F}$, where p_1, p_2 are the projections of F × F onto the first and second factors, respectively. Define

$$|Y_w| \subset \{\mathbf{C}\text{- points of } \coprod_{J=w(I)} F_I \times F_J\}$$

as the set of all $\mathbf{C}$-points x over which $\dim_x(\mathscr{S}_i \cap \mathscr{S}_j) = \mathbf{d}_{ij,w}$. By the Combinatorial Lemma, $\coprod_{w\in W} |Y_w|$ exhausts the set of all $\mathbf{C}$-points of F × F. We introduce a subsuperscheme structure on $|Y_W|$ following the general construction of the flattening partition [18].

2. THEOREM. a) On each $|Y_W|$ there exists a canonical structure of a locally closed subsuperscheme $Y_w \hookrightarrow F \times F$ such that the morphism $\coprod_{w\in W} Y_w \hookrightarrow F \times F$ is a flattening partition for the system of sheaves $\{\mathscr{S}_i \cap \mathscr{S}_j\}$. This means that an arbitrary morphism of superschemes $g: S \to F \times F$ (S is Noetherian) has the property: {all $g^*(\mathscr{S}_i \cap \mathscr{S}_j)$ are locally direct locally free subsheaves in T_S of ranks $\mathbf{d}_{ij,w}$} if and only if g factors through the embedding $\coprod_{w\in W} Y_w \hookrightarrow F \times F$.

b) All Y_W are bundles over F: $Y_w \xrightarrow{p_2} F$, each fiber $p_2^{-1}(x)$ being isomorphic to the supercell $\mathbf{C}^{e/s}$ (p_2 is the natural projection onto the second factor in the product F × F).

c) Y_W is a functor from the category of Noetherian superschemes over $\mathbf{C}$ to the category of sets, which associates to any superscheme S the set of S-points of the superscheme $\coprod_{J=w(I)} F_I \times F_J$ over which $\mathscr{S}_i \cap \mathscr{S}_j$ are locally direct locally free subsheaves in T_S of ranks $\mathbf{d}_{ij,w}$.

Remark. Part (c) is obviously a reformulation of part (a).

The proof relies on the construction of the supercell partition of a relative projective superspace, to which we now proceed.

3. Supercell Partition of a Projective Superspace. Let X be a fixed Noetherian superscheme, $\mathscr{T}$ a locally free sheaf of rank $m|n$ on it, and $\mathscr{S}.$ a fixed complete flag in $\mathscr{T}$. In the relative projective superspace $\mathbf{P}_X(1|0;\mathscr{T})$, consider the following chain of embedded subsuperschemes — projectivizations of the bundles $\mathscr{S}_i$:

$$\varnothing \subset \mathbf{P}_X(1|0;\mathscr{S}_1) \subset \mathbf{P}_X(1|0;\mathscr{S}_2) \subset \ldots \subset \mathbf{P}_X(1|0;\mathscr{T}).$$

On each of the nonempty open subsets $\mathbf{P}_X(1|0;\mathscr{S}_k)_{\mathrm{red}} \setminus \mathbf{P}_X(1|0;\mathscr{S}_{k-1})_{\mathrm{red}}$ of $\mathbf{P}_X(1|0;\mathscr{S}_k)_{\mathrm{red}}$, define the natural structure of an open subsuperscheme $Z_k \subset \mathbf{P}_X(1|0;\mathscr{S}_k)$.

Proposition. a) The morphism $\coprod_k Z_k \hookrightarrow \mathbf{P}_X(1|0;\mathscr{T})$ is a relative flattening partition for the system of sheaves $\{\tilde{\mathscr{S}} \cap \mathscr{S}_i\}$, where $\tilde{\mathscr{S}}$ is the $\mathcal{O}(-1)$-tautological sheaf on $\mathbf{P}_X(1|0;\mathscr{T})$.

b) Z_k is a relative affine space of dimension $\mathrm{rk}\,\mathscr{S}_k - 1|0$ over X. In other words, the fiber of the natural projection $Z_k \to X$ is isomorphic to $\mathbf{C}^{\mathrm{rk}\mathscr{S}_k - 1|0}$.

Proof. Part (b) is obvious from the construction: Z_k is a big cell in $\mathbf{P}_X(1|0;\mathscr{S}_k)$.

To prove (a), we have to show that for any morphism of schemes over X, $g: S \to \mathbf{P}_X(1|0;\mathscr{T})$ (S Noetherian), the sheaves $g^*(\tilde{\mathscr{S}} \cap \mathscr{S}_i)$ are locally direct locally free subsheaves in $\mathscr{T}_S$ of rank $0|0$ for $i < k$ and of rank $1|0$ for $i \geq k$ if and only if g factors through the embedding $Z_k \hookrightarrow \mathbf{P}_X(1|0;\mathscr{T})$. Necessity follows from the fact, known from classical geometry, that the reduction g_{red} of such a morphism g factors through $\mathbf{P}_X(1|0;\mathscr{S}_k)_{\mathrm{red}} \setminus \mathbf{P}_X(1|0;\mathscr{S}_{k-1})_{\mathrm{red}} = (Z_k)_{\mathrm{red}}$, and from the obvious fact that g factors through $\mathbf{P}_X(1|0;\mathscr{S}_k)$.

Sufficiency becomes obvious if one notes that $\tilde{\mathscr{S}} \cap \mathscr{S}_i$ are locally direct locally free subsheaves in $\mathscr{T}_{Z_k}$ of the indicated ranks. This completes the proof. □

4. COROLLARY. Under the assumptions of Subsec. 3, there exists a relative flattening partition $\coprod_k Y_k$ of the supermanifold $\mathbf{P}_X(1|0;\mathscr{T}) \times_X F_X$ for the system of sheaves $\{\tilde{\mathscr{S}} \cap \tilde{\mathscr{S}}_i\}$, where $\tilde{\mathscr{S}}$ is the tautological sheaf on $\mathbf{P}_X(1|0;\mathscr{T})$, $\tilde{\mathscr{S}}.$ the tautological flag on F_X — the superspace of complete flags in $\mathscr{T}$ over X. Under these conditions, the fiber $p_2^{-1}(x)$ of the projection $p_2: \coprod Y_k \to F_X$ over the X-point $x \in F_X(X)$ represented by the flag $\mathscr{S}.$ in $\mathscr{T}$ is canonically isomorphic to $\coprod_k Z_k$. □

5. Proof of Theorem 2 I. Case G = SL. The proof proceeds by induction, considering the superspace of complete flags as a relative superspace of complete flags of smaller length over a projective superspace. To make the inductive step possible, we shall prove a relative version of Theorem 2, that is, we shall work in the category of superschemes over a certain Noetherian superscheme X on which a locally free sheaf $\mathscr{T}$ of rank $m|n$ is defined. [The formulation of parts (a) and (c) of the theorem is the same, except that all morphisms are un-

derstood as morphisms of superschemes over X and all products as fibered products over X. The formulation of part (b) needs no modification.]

If m + n = 1 the space F × F is X, and the flattening partition obviously consists of a single component $Y_w = F \times F$, $w = e$, $d_{11,w} = m|n = 1|0$ or $0|1$.

Suppose the theorem has been proved for m + n = ℓ − 1. We shall prove it for m + n = ℓ. Consider the natural projection $F \to \mathbf{P}$, $\mathbf{P} = \mathbf{P}_X(1|0; \mathcal{T})$ if m > 0 and $\mathbf{P} = \mathbf{P}_X(0|1; \mathcal{T})$ if m = 0. Relative to this projection, F is the superspace $F'_{\mathbf{P}}$ of complete flags in $\mathcal{T}/\tilde{\mathcal{S}}$ over $\mathbf{P}$, where $\tilde{\mathcal{S}}$ is the tautological sheaf on $\mathbf{P}: F = F'_{\mathbf{P}}$. The inductive hypothesis gives us a relative flattening partition (roughly speaking, a flattening partition in a fiber) of the superspace $F'_{\mathbf{P}}$ on $\mathbf{P}$ for every flag $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_{m+n}$ on X, since the flag $\mathcal{S}_\cdot$ induces a flag $\mathcal{S}_1/\tilde{\mathcal{S}} \subseteq \ldots \subseteq \mathcal{S}_{m+n}/\tilde{\mathcal{S}}$ of length m + n − 1 on $\mathbf{P}$. By Proposition 3, the same flag $\mathcal{S}_\cdot$ yields a flattening partition of $\mathbf{P}$ over X. Thus we obtain a supercell partition of F over X, corresponding to $\mathcal{S}_\cdot$ – an X-point of the supermanifold F over X.

In order not to consider a functor of points, we describe the construction of a flattening partition of the product $F \underset{X}{\times} F = F'_{\mathbf{P}} \underset{X}{\times} F'_{\mathbf{P}}$. The correspondence $q: \mathcal{S}_\cdot \mapsto \mathcal{S}_\cdot/\tilde{\mathcal{S}}$, where $\mathcal{S}_\cdot$ is the tautological flag on F, $\tilde{\mathcal{S}}$ the taugological sheaf on $\mathbf{P}$, determines a morphism $F'_{\mathbf{P}} \underset{X}{\times} F'_{\mathbf{P}} \xrightarrow{\mathrm{id} \times q} F'_{\mathbf{P}} \underset{\mathbf{P}}{\times} F'_{\mathbf{P}}$ whose base, by the inductive hypothesis, admits a flattening partition. The natural projection $p: F'_{\mathbf{P}} \to \mathbf{P}$ determines a morphism $F'_{\mathbf{P}} \underset{X}{\times} F'_{\mathbf{P}} \xrightarrow{p \times \mathrm{id}} \mathbf{P} \underset{X}{\times} F'_{\mathbf{P}} = \mathbf{P} \underset{X}{\times} F$, whose base, by Corollary 4, admits a flattening partition. Taking the inverse images of the components of both flattening partitions under the respective morphisms, we obtain two partitions of the superspace $F \underset{X}{\times} F$. The components of the required partition are now defined as the (scheme) intersections of the components of both partitions. It remains to index the components of the partition by the elements of the Weyl supergroup ${}^{SL}W$ and to prove parts (a) and (b) of the theorem.

Fix $w \in {}^{SL}W$ and let (d_{ij}) be the matrix corresponding to w by the Combinatorial Lemma. Let (d'_{ij}) be the matrix obtained from (d_{ij}) by deleting the first row and subtracting $\mathrm{rk}\,\tilde{\mathcal{S}}$ from d_{ij} if i > 2, and deleting the k-th column and subtracting $\mathrm{rk}\,\tilde{\mathcal{S}}$ from d_{ij} if j > k, k = w(1). As this matrix has properties 2.2(a), there exists a corresponding element w' of the Weyl supergroup ${}^{SL}W'$ of SL(m − 1, n) or SL(m, n − 1). By the inductive hypothesis, w' determines a component $Y'_{w'}$ of a flattening partition of $F'_{\mathbf{P}} \underset{\mathbf{P}}{\times} F'_{\mathbf{P}}$. Consider also the component of the flattening partition of the superscheme $\mathbf{P} \underset{X}{\times} F$ corresponding to the first row of (d_{ij}), i.e., Y_k/ Then $\mathrm{rk}\,(\tilde{\mathcal{S}} \cap \mathcal{S}_j)|_{Y_k} = 0|0$ if j < k and $\mathrm{rk}\,(\tilde{\mathcal{S}} \cap \mathcal{S}_j)|_{Y_k} \neq 0|0$ if j ≥ k (cf. the proof of Proposition 3 and Corollary 4). Define the component Y_w of the flattening partition of $F \underset{X}{\times} F$ as $Y_w = (\mathrm{id} \times q)^{-1} Y_{w'} \cap (p \times \mathrm{id})^{-1} Y_k$. It is clear that, given any pair (w', k), we can construct w in such a way that this correspondence is the inverse of the correspondence w → (w', k) described previously. To prove that Y_w satisfies condition (a) of Theorem 2, it suffices to observe, first, that $g^*(\mathcal{S}_i \cap \mathcal{S}_j)$ are locally direct locally free subsheaves in $\mathcal{T}_S$ of ranks d_{ij} (for the notation see the assumptions of the theorem) if and only if $((\mathrm{id} \times q) \circ g)^* (\mathcal{S}_i \cap \mathcal{S}_j / \tilde{\mathcal{S}})$ are locally direct locally free subsheaves in $\mathcal{T}_S / ((\mathrm{id} \times q) \circ g)^* \tilde{\mathcal{S}}$ of ranks d'_{ij} and $(p \times \mathrm{id}) \circ g)^* (\tilde{\mathcal{S}} \cap \mathcal{S}_j)$ are locally direct locally free subsheaves in $\mathcal{T}_S$ of ranks 0|0 if j < k and 1|0 or 0|1 if $j \geqslant k$. Second, g factors through the embedding $Y_w \to F \underset{X}{\times} F$ if and only if $(\mathrm{id} \times q) \circ g$ and $(p \times \mathrm{id}) \circ g$ factor, respectively, through the embeddings $Y_{w'} \to F'_{\mathbf{P}} \underset{\mathbf{P}}{\times} F'_{\mathbf{P}}$ and $Y_k \to \mathbf{P} \underset{X}{\times} F$ – this follows from the construction of Y_w.

We now prove (b). The general fiber of the projection $p_2: F \underset{X}{\times} F \to F$ is the space of complete flags, which can be represented as the relative space of complete flags of smaller length over a projective superspace. Viewed thus, every Schubert supercell in $p_2^{-1}(x)$, $x \in F(\mathbf{C})$ is a Schubert supercell of the relative space of complete flags over a Schubert supercell of the projective superspace. Hence one can use Proposition 3 and proceed by induction.

<u>II. Cases G = OSp, ΠSp, Q.</u> The same proof goes through, provided that the statements of Proposition 3 and Corollary 4 are suitably modified: for G = OSp, ΠSp one replaces the projective superspace $\mathbf{P}_X(1|0; \mathcal{T})$ by the superspace $\mathbf{P}_X(1|0; \mathcal{T}, b)$ of subsheaves of rank 1|0 in $\mathcal{T}$ isotropic relative to the form $b: \mathcal{T} \to \mathcal{T}^*$; for G = Q one must consider the super-Grassmannian $G_X(1|1; \mathcal{T}, p)$ of p-symmetric subsheaves of rank 1|1 in $\mathcal{T}$, where $p: \mathcal{T} \to \mathcal{T}$ is the appropriate odd involution. The exact formulation is left to the reader. □

In classical geometry, the partition of a flag space F into Schubert cells is simply the partition of F = G/B into B-orbits, where G is a suitable simple algebraic group and B a Borel subgroup. In our approach this clearly corresponds to the partition of F × F into G-orbits. [Indeed, having fixed a point $x \in F(\mathbf{C})$ in the second factor, we fix a subgroup B of G — the stabilizer of the point. Consequently, the fiber of a G-orbit Y over x is a B-orbit in F.]

The next lemma shows that Schubert supercells are also G-orbits in the superscheme sense.

6. Transitivity Lemma. Let S be a superscheme, $TS = T \otimes \mathcal{O}S$. Let $\mathcal{F}'_\cdot$, $\mathcal{F}''_\cdot$ be two complete G-flags in TS (G = SL, OSp, ΠSp or Q) with the following properties (cf. Definition 2.1): 1) $\mathcal{F}'_\cdot$ and $\mathcal{F}''_\cdot$ are regularly positioned relative to each other, 2) the type of their relative position is (d_{ij}). Let $\tilde{\mathcal{F}}'_\cdot$, $\tilde{\mathcal{F}}''_\cdot$ be another pair of complete G-flags with the same properties [and the same (d_{ij})], with the type of $\mathcal{F}'_\cdot$ equal to that of $\tilde{\mathcal{F}}'_\cdot$ and the type of $\mathcal{F}''_\cdot$ equal to that of $\tilde{\mathcal{F}}''_\cdot$ (cf. Definition 1.2). Then every point s ∈ S has an affine neighborhood U = Spec A such that there exists an element $g \in G(A)$ carrying the pair of flags $\mathcal{F}'_\cdot|_U$, $\mathcal{F}''_\cdot|_U$ in T_U into the pair $\tilde{\mathcal{F}}'_\cdot|_U$, $\tilde{\mathcal{F}}''_\cdot|_U$.

Proof. I. Case G = SL. Let U = Spec A be a neighborhood of $s \in S$ such that the sheaves $\mathcal{F}'_i|_U$, $\mathcal{F}''_j|_U$, $\tilde{\mathcal{F}}'_i|_U$, $\tilde{\mathcal{F}}''_j|_U$, $(\mathcal{F}'_i \cap \mathcal{F}''_j)|_U$, $(\tilde{\mathcal{F}}'_i \cap \tilde{\mathcal{F}}''_j)|_U$ are free and are direct subsheaves in T_U. To simplify the notation, we shall henceforth omit the symbol $|_U$ and identify all sheaves over U with their spaces of global sections over U.

We shall construct the required element $g \in G(A)$ explicitly, using induction on the indexes i of the constituents of the flag $\mathcal{F}'_\cdot$. On $\mathcal{F}'_1$ we define the map g so that $\mathcal{F}'_1 \mapsto \tilde{\mathcal{F}}'_1$ (this is possible because the flags $\mathcal{F}'_\cdot$ and $\tilde{\mathcal{F}}'_\cdot$ have the same type). When this is done the restriction of the (as yet unconstructed) map g to $\mathcal{F}'_1$ carries $\mathcal{F}'_1 \cap \mathcal{F}''_\cdot$ into $\tilde{\mathcal{F}}'_1 \cap \tilde{\mathcal{F}}''_\cdot$ (the intersection $\mathcal{F}'_1 \cap \mathcal{F}''_\cdot$ is treated as a degenerate flag $\mathcal{F}'_1 \cap \mathcal{F}''_1 \subseteq \mathcal{F}'_1 \cap \mathcal{F}''_2 \subseteq \ldots \subseteq \mathcal{F}'_1 \cap \mathcal{F}''_{m+n-1}$), since the elements d_{ij} are the same for the pairs of flags $\mathcal{F}'_\cdot$, $\mathcal{F}''_\cdot$ and $\tilde{\mathcal{F}}'_\cdot$, $\tilde{\mathcal{F}}''_\cdot$.

Now suppose that g has been constructed on the constituent $\mathcal{F}'_k$ and carries $\mathcal{F}'_1 \subset \ldots \subset \mathcal{F}'_k$ into $\tilde{\mathcal{F}}'_1 \subset \ldots \subset \tilde{\mathcal{F}}'_k$ and $\mathcal{F}'_k \cap \mathcal{F}''_\cdot$ into $\tilde{\mathcal{F}}'_k \cap \tilde{\mathcal{F}}''_\cdot$. We shall construct an extension of g to $\mathcal{F}'_{k+1}$, satisfying the same conditions for k = k + 1.

Let j_0 be the least j for which $d_{k+1,j} \neq d_{kj}$. In the A-module $\mathcal{F}'_{k+1} \cap \mathcal{F}''_{j_0}$ take an element e_{k+1} in the complement of $\mathcal{F}'_k \cap \mathcal{F}''_{j_0}$, and in the A-module $\tilde{\mathcal{F}}'_{k+1} \cap \tilde{\mathcal{F}}''_{j_0}$ an element $\tilde{e}_{k+1}$ in the complement of $\tilde{\mathcal{F}}'_k \cap \tilde{\mathcal{F}}''_{j_0}$ (dimensional arguments show that this is legitimate). Define g on e_{k+1}: $ge_{k+1} = \tilde{e}_{k+1}$. Extended by linearity to $\mathcal{F}'_{k+1}$, g carries $\mathcal{F}'_1 \subset \ldots \subset \mathcal{F}'_{k+1}$ into $\tilde{\mathcal{F}}'_1 \subset \ldots \subset \tilde{\mathcal{F}}'_{k+1}$.

We claim that $\mathcal{F}'_{k+1} \cap \mathcal{F}''_\cdot$ is carried by g into $\mathcal{F}'_{k+1}$ in $\tilde{\mathcal{F}}'_{k+1} \cap \tilde{\mathcal{F}}''_\cdot$. If $j < j_0$ it follows from dimensional arguments that $\mathcal{F}'_{k+1} \cap \mathcal{F}''_j = \mathcal{F}'_k \cap \mathcal{F}''_j$ and $\tilde{\mathcal{F}}'_{k+1} \cap \tilde{\mathcal{F}}''_j = \tilde{\mathcal{F}}'_k \cap \tilde{\mathcal{F}}''_j$, and so, by the inductive hypothesis, $g(\mathcal{F}'_{k+1} \cap \mathcal{F}''_j) = \tilde{\mathcal{F}}'_{k+1} \cap \tilde{\mathcal{F}}''_j$. $g(\mathcal{F}'_{k+1} \cap \mathcal{F}''_{j_0}) = \tilde{\mathcal{F}}'_{k+1} \cap \tilde{\mathcal{F}}''_{j_0}$ by construction. Finally, if $j > j_0$,

$$\mathcal{F}'_{k+1} \cap \mathcal{F}''_j = Ae_{k+1} \oplus (\mathcal{F}'_k \cap \mathcal{F}''_j), \tag{1}$$

and moreover $e_{k+1} \in \mathcal{F}'_{k+1} \cap \mathcal{F}''_{j_0} \subset \mathcal{F}'_{k+1} \cap \mathcal{F}''_j$, but $e_{k+1} \notin \mathcal{F}'_k \supset \mathcal{F}'_k \cap \mathcal{F}''_j$. By (1), $g(\mathcal{F}'_{k+1} \cap \mathcal{F}''_j) = A\tilde{e}_{k+1} \oplus \tilde{\mathcal{F}}'_k \cap \tilde{\mathcal{F}}''_j$, which by construction is equal to $\tilde{\mathcal{F}}'_{k+1} \cap \tilde{\mathcal{F}}''_j$.

The last step of the inductive construction yields a map g with the desired properties, except for Ber g = 1. This may be remedied by correcting g, e.g., at the last step: $\hat{g}(e_k) = (\mathrm{Ber}\, g)^{\pm 1} \cdot \tilde{e}_k$ (the exponent will be −1 if e_k is even, +1 otherwise).

II. Case G = OSp, ΠSp. Up to [(m + n + 1)/2], where m|n = dim T, the inductive construction is the same as for G = SL. One then chooses e_k and $\tilde{e}_k$ so that $b(e_k, e_{m+n+1-k}) = b(\tilde{e}_k, \tilde{e}_{m+n+1-k}) = 1$, $b(e_k, e_j) = b(\tilde{e}_k, \tilde{e}_j) = 0$ for all other j (b is a suitable bilinear form). It is now clear that the element g taking e_k into $\tilde{e}_k$ for all k lies in G(A), since it preserves the Gram matrix of the bilinear form. By construction, g has the desired properties.

III. Case G = Q. The construction is the same as for G = SL, except that at each inductive step the dimension of the space $\mathcal{F}'_k$ on which g is being defined is increased by 1|1. More precisely, considering the complement to $\mathcal{F}'_k \cap \mathcal{F}''_{j_0}$ in $\mathcal{F}'_{k+1} \cap \mathcal{F}''_{j_0}$, one simultaneously chooses two elements: an even one e_{k+1} and an odd one $p(e_{k+1})$ (the same for $\tilde{\mathcal{F}}_\cdot$). These are carried into elements $\tilde{e}_{k+1}$, $p(\tilde{e}_{k+1})$, respectively, and thus $g \in Q(A)$, i.e., $g \circ p = p \circ g$. □

4. Schubert Supercells: Dimension

1. Definition. A basis reflection σ is one of the following elements of the group ${}^GW$:

a) $G = SL$: σ is a permutation of neighbors in $\{1, 2, \ldots, m + n\}$, $\sigma_i = (i, i + 1)$;

b) $G = OSp$, ΠSp: σ is a simultaneous permutation of neighbors in the left half of the sequence $\{1, 2, \ldots, m + n\}$ and of mirror neighbors in the right half, $\sigma_i = (i, i + 1)(m + n + 1 - i, m + n - i)$, $i+1 \leqslant \left[\frac{m+n}{2}\right]$, or the transposition of $l=\left[\frac{m+n}{2}\right]$ with its mirror reflection, $\tau_l = (l, m+n+1-l)$;

c) $G=Q: \sigma_i = (i, i+1) \in S_m$.

2. We can now define the superlength of an element of the Weyl group.

Definition. a) Let $I, J \in {}^GI_n$, and let $\sigma \in {}^GW$ be a basis reflection such that $J = \sigma(I)$. If $\sigma = \sigma_i$, then

$$l_{IJ}(\sigma) = \begin{cases} 1\,|\,0, & \text{if } I=J,\ G=\mathrm{SL},\ \mathrm{OSp},\ \Pi\mathrm{Sp}, \\ 0\,|\,1, & \text{if } I\neq J,\ G=\mathrm{SL},\ \mathrm{OSp},\ \Pi\mathrm{Sp}, \\ 1\,|\,1, & \text{if } G=\mathrm{Q}, \end{cases}$$

and if $\sigma = \tau_l$, then

$$l_{IJ}(\sigma) = \begin{cases} 1\,|\,0, & \text{if } \delta_l(J)=1\,|\,0,\ G=\mathrm{OSp}\,(2r+1, 2s), \\ 1\,|\,1, & \text{if } \delta_l(J)=0\,|\,1,\ G=\mathrm{OSp}\,(2r+1, 2s), \\ 0\,|\,0, & \text{if } \delta_l(J)=1\,|\,0,\ G=\mathrm{OSp}\,(2r, 2s), \\ 1\,|\,0, & \text{if } \delta_l(J)=0\,|\,1,\ G=\mathrm{OSp}\,(2r, 2s), \\ 0\,|\,1, & \text{if } \delta_l(J)=1\,|\,0,\ G=\Pi\mathrm{Sp}\,(m), \\ 0\,|\,0, & \text{if } \delta_l(J)=0\,|\,1,\ G=\Pi\mathrm{Sp}\,(m). \end{cases}$$

b) Under the same conditions, let $w=\sigma^k\ldots\sigma^1 \in {}^GW$ be a reduced factorization into basis reflections (i.e., the number k of basis reflections into which w is factored is minimal), $J = w(I)$. Put $I_i = \sigma^i \ldots \sigma^1(I)$. Then the superlength $l_{IJ}(w)$ is defined as the pair of numbers

$$l_{IJ}(w) := \sum_{i=0}^{k-1} l_{I_i, I_{i+1}}(\sigma^{i+1}),$$

and the length as the number k.

3. THEOREM. If $J = w(I)$, then

$$\dim Y_w \cap (F_I \times F_J) = l_{IJ}(w) + \dim F_I.$$

Remark. Using induction on $m + n$ and formulas for the dimension of G-Grassmannians (cf. [8, Theorem 5.6.3]), one can verify that

$$\dim^G F_I = \begin{cases} \left(\frac{m(m-1)}{2} + \frac{n(n-1)}{2} \,\Big|\, mn\right), & G=\mathrm{SL}\,(m, n), \\ (r^2+s^2 \,|\, s(2r+1)), & G=\mathrm{OSp}\,(2r+1, 2s), \\ (r(r-1)+s^2 \,|\, 2rs), & G=\mathrm{OSp}\,(2r, 2s), \\ \left(\frac{m(m+1)}{2} \,\Big|\, \frac{m(m+1)}{2}\right), & G=\mathrm{Q}\,(m), \end{cases}$$

this dimension is independent of I, and

$$\dim^G F_I = \left(rs + \frac{r(r-1)}{2} + \frac{s(s-1)}{2} \,\Big|\, \frac{(r+1)r}{2} + \frac{s(s-1)}{2} + rs\right),$$
$$G=\Pi\mathrm{Sp}\,(m),$$

where $r|s=d_m(I)$ is the dimension of the maximal isotropic constituent in a flag of type I, $r + s = m$.

Before proving the theorem we state two corollaries.

4. COROLLARY. The superlength $l_{IJ}(w)$ is independent of the choice of the reduced factorization $w = \sigma^k \ldots \sigma^1$. □

5. COROLLARY.

$$l_{IJ}(w) + \dim F_I = l_{JI}(w^{-1}) + \dim F_J.$$

Proof. $Y_w \cap (F_I \times F_J) \simeq Y_{w^{-1}} \cap (F_J \times F_I)$. This follows from the definition of the space Y_w — the isomorphism is established by permuting the factors of the product $F_I \times F_J$. □

6. The proof of the theorem proceeds by induction on the length k of the element $w = \sigma^k \ldots \sigma^1$ of the Weyl group. Denote the tautological flags on the first and second factors of $F_I \times F_J$ by $\mathcal{S}'_{I,\cdot}$ and $\mathcal{S}''_{J,\cdot}$, respectively.

Let $k = 0$, i.e., $w=e$, $J=I$, $l_{II}(e)=0|0$, and it will suffice to prove that $Y_e \cap (F_I \times F_I) \simeq F_I$. Now Y_e is the diagonal Δ in $F \times F$. Indeed, by Sec. 2.5(a), $d_{ij,e,II}=d_{\min(i,j)}(I)$. On the other hand, $(\mathcal{S}'_{I,i} \cap \mathcal{S}''_{I,j})|_\Delta = \mathcal{S}'_{I,\min(i,j)}|_\Delta$, and so these sheaves are locally free locally direct subsheaves in T_Δ of ranks $d_{\min(i,j)}(I)$. Therefore, if $g: S \to F \times F$ is a morphism of superschemes which factors through $\Delta \hookrightarrow F \times F$, then all $g^*(\mathcal{S}'_{I,i} \cap \mathcal{S}''_{I,j})$ are locally direct locally free subsheaves in T_S of ranks $d_{ij,e,II}$. Conversely, if all $g^*(\mathcal{S}'_{I,i} \cap \mathcal{S}''_{I,j})$ are locally free of ranks $d_{ij,e,II}$, then $\operatorname{rk} g^*(\mathcal{S}'_{I,i} \cap \mathcal{S}''_{I,j}) = d_i(I)$ for all i and I, that is, $g^*(\mathcal{S}'_{I,i}) = g^*(\mathcal{S}''_{I,i})$. It is clear that in this case g factors through $\Delta \hookrightarrow F \times F$.

Now for the induction step. Suppose the dimension formula true for any $w = \sigma^k \ldots \sigma^1$ (reduced factorization into basis elements). We have to prove the formula for elements of the Weyl supergroup of length $k+1: w = \sigma^{k+1}\sigma^k \ldots \sigma^1$. Put $w_0 = \sigma^k \ldots \sigma^1$, $J_0 = w_0(I)$. We shall need the following lemma.

7. LEMMA. 1) Let $\sigma^{k+1} = \sigma_q$ be a basis reflection. Then $w_0^{-1}(q) < w_0^{-1}(q+1)$.

2) Let $\sigma^{k+1} = \tau_l$, $l = \left[\frac{m+n}{2}\right]$. Then $w_0^{-1}(l) < w_0^{-1}(m+n+1-l)$.

To prove the lemma, we observe that each of the Weyl supergroups under consideration is isomorphic to a classical Weyl group, and moreover the basis reflections correspond to reflections with respect to elements of some system of simple roots. Hence Lemma 7 is simply a restatement, in terms of permutations, of the following classical lemma:

7'. LEMMA (see [1, 20]). Let w be an element of a classical Weyl group of type A, B or C, γ a positive root. If $l(w) = l(\sigma_\gamma w) - 1$, where ℓ denotes length, then $w^{-1}(\gamma)$ is a positive root.

We introduce new notation: if $\sigma^{k+1} = \sigma_q$, then $a = w_0^{-1}(q)$, $b = w_0^{-1}(q+1)$ and if $\sigma^{k+1} = \tau_l$, then $a = w_0^{-1}(l)$, $b = w_0^{-1}(m+n+1-l)$. By Lemma 7, $a < b$.

I. Case $\sigma^{k+1} = \sigma_q$. We compare the matrices $(d_{ij,w,IJ})$ and (d_{ij,w_0,IJ_0}).

8. LEMMA. $d_{iq,w,IJ} = d_{iq,w_0,IJ_0} + \delta_{q+1}(J_0) - \delta_q(J_0)$ if $i \geqslant b$,

$$d_{iq,w,IJ} = d_{iq,w_0,IJ_0} - \delta_q(J_0) \quad \text{if} \quad a \leqslant i < b,$$

with symmetric relations for G = OSp or ΠSp,

$$d_{ij,w,IJ} = d_{ij,w_0,IJ_0} \quad \text{for other} \quad i, j.$$

The proof follows at once from the definition of $d_{ij,w,IJ}$ [see Sec. 2.5(a)].

Thus, we have formulas

$$\operatorname{rk}(\mathcal{S}'_{I,i} \cap \mathcal{S}''_{J,q})|_{Y_w} = \operatorname{rk}(\mathcal{S}'_{I,i} \cap \mathcal{S}_{J_0,q})|_{Y_{w_0}} - \delta_q(J_0) + \delta_{q+1}(J_0) \quad \text{if} \quad i > b, \tag{1}$$

$\operatorname{rk}(\mathcal{S}'_{I,i} \cap \mathcal{S}''_{J,q})|_{Y_w} = \operatorname{rk}(\mathcal{S}'_{I,i} \cap \mathcal{S}''_{J_0,q})|_{Y_{w_0}} - \delta_q(J_0)$ if $a \leqslant i < b$, with symmetric relations for G = OSp, ΠSp,

$$\operatorname{rk}(\mathcal{S}'_{I,i} \cap \mathcal{S}''_{J,j})|_{Y_w} = \operatorname{rk}(\mathcal{S}'_{I,i} \cap \mathcal{S}''_{J_0,j})|_{Y_{w_0}}$$

for other i, j.

Consider the natural projections $F_{J_0} \to \tilde{F}$ and $F_J \to \tilde{F}$ onto the superspace $\tilde{F}$ of incomplete G-flags obtained by "forgetting" the q-th constituents $\mathcal{S}''_{J_0,q}$ and $\mathcal{S}''_{J,q}$ and the dual constituents in the cases G = OSp, ΠSp. Formulas (1) show that the projections of the supermanifolds Y_{w_0} and Y_w under $p_0: F_I \times F_{J_0} \to F_I \times \tilde{F}$ and $p: F_I \times F_J \to F_I \times \tilde{F}$ coincide: $p_0(Y_{w_0}) = p(Y_w) = Y$. (Recall that the partition on Y_w was defined as a flattening partition for the system of sheaves $\{\mathcal{S}'_{I,i} \cap \mathcal{S}''_{J,j}\}$; it is obvious that the projections of Y_w onto $F_I \times \tilde{F}$ form a flattening partition for $\{\mathcal{S}'_{I,i} \cap \mathcal{S}''_{\tilde{J},j}\}$, where $\mathcal{S}''_{\tilde{J},\cdot}$ is the tautological flag on $\tilde{F}$, $p_0^* \mathcal{S}''_{\tilde{J},j} = \mathcal{S}''_{J_0,j}$, $p^* \mathcal{S}''_{\tilde{J},j} = \mathcal{S}''_{J,j}$, $j \neq q$.) In fact, Y_{w_0} is isomorphically projected onto Y, while Y_w is a big cell relative to a projective superspace of dimension $l_{J_0 J}(\sigma^{k+1})$ over Y (if G = Q — the super-Grassmannian of p-symmetric 1|1-dimensional planes in a 2|2-dimensional superspace with odd involution p). This again follows from the properties of the matrices (d_{ij}) and formulas (1), rewritten as follows:

$$\operatorname{rk}\left(\mathcal{S}'_{I,i}\cap\mathcal{S}''_{J_0,q}/\mathcal{S}''_{J_0,q-1}\right)|_{Y_{w_0}}=\begin{cases}0|0 & \text{if } i<a,\\ \delta_q(J_0) & \text{if } i\geqslant a,\end{cases} \tag{2}$$

$$\operatorname{rk}\left(\mathcal{S}'_{I,i}\cap\mathcal{S}''_{J_0,q+1}/\mathcal{S}''_{J_0,q-1}\right)|_{Y_{w_0}}=\begin{cases}0|0 & \text{if } i<a,\\ \delta_q(J_0) & \text{if } a\leqslant i<b,\\ \delta_q(J_0)+\delta_{q+1}(J_0) & \text{if } i\geqslant b,\end{cases} \tag{3}$$

$$\operatorname{rk}\left(\mathcal{S}'_{I,i}\cap\mathcal{S}''_{J,q}/\mathcal{S}''_{J,q-1}\right)|_{Y_w}=\begin{cases}0|0 & \text{if } i<b,\\ \delta_{q+1}(J_0) & \text{if } i\geqslant b,\end{cases} \tag{4}$$

$$\operatorname{rk}\left(\mathcal{S}'_{I,i}\cap\mathcal{S}''_{J,q+1}/\mathcal{S}''_{J,q-1}\right)|_{Y_w}=\operatorname{rk}\left(\mathcal{S}'_{I,i}\cap\mathcal{S}''_{J_0,q+1}/\mathcal{S}''_{J_0,q-1}\right)|_{Y_{w_0}} \quad \text{for all } i. \tag{5}$$

Indeed, formula (2) implies

$$\mathcal{S}''_{J_0,q}/\mathcal{S}''_{J_0,q-1}|_{Y_{w_0}}\subset\mathcal{S}'_{I,a}\cap\mathcal{S}''_{J_0,q+1}/\mathcal{S}''_{J_0,q-1}|_{Y_{w_0}},$$

whence, by (3), the last inclusion becomes an equality. This mean that the tautological sheaf $\mathcal{S}''_{J_0,q}/\mathcal{S}''_{J_0,q-1}$ of the superspace Y_{w_0} over Y is exactly the sheaf $\mathcal{S}'_{I,a}\cap\mathcal{S}''_{J_0,q+1}/\mathcal{S}''_{J_0,q-1}$ lifted from Y, and therefore $\dim Y_w-\dim Y=0|0$. Similarly, formulas (4), (5) and (3) yield the conclusion that the tautological sheaf $\mathcal{S}''_{J,q}/\mathcal{S}''_{J,q-1}$ of rank $\delta_{q+1}(J_0)$ of the superspace Y_w over Y must be embedded in the sheaf $\mathcal{S}'_{I,b}\cap\mathcal{S}''_{J,q+1}/\mathcal{S}''_{J,q-1}$ of rank $\delta_q(J_0)+\delta_{q+1}(J_0)$ lifted from Y, and that it cannot intersect the sheaf $\mathcal{S}'_{I,b-1}\cap\mathcal{S}''_{J,q+1}/\mathcal{S}''_{J,q-1}$ of rank $\delta_q(J_0)$, also lifted from Y. Thus, $\dim Y_w-\dim Y_{w_0}=\dim Y_w-\dim Y=l_{J_0J}(\sigma^{k+1})$, and finally $\dim Y_w=l_{IJ}(w)+\dim F_I$ by the inductive hypothesis.

<u>II. Case $\sigma^{k+1}=\tau_\ell$.</u> Consider the natural projections $F_{J_0}\to\tilde{F}$ and $F_J\to\tilde{F}$ onto the superspace $\tilde{F}$ of incomplete G-flags obtained by "forgetting" the ℓ-th constituents $\mathcal{S}''_{J_0,l}$ and $\mathcal{S}''_{J,l}$, respectively, and the dual constituents in the case G = OSp(2r + 1, 2s) (this is the only supergroup G for which the constituent dual to $\mathcal{S}''_{J_0,l}$ is not $\mathcal{S}''_{J_0,l}$ itself). Reasoning as in the previous case, one can show that the projections of the super-manifolds Y_{w_0} and Y_w under $p_0: F_I\times F_{J_0}\to F_I\times\tilde{F}$ and $p: F_I\times F_J\to F_I\times\tilde{F}$ coincide $(p_0(Y_{w_0})=p(Y_w)=Y)$ and that $\dim Y_{w_0}=\dim Y$, while Y_w is a big cell of the relative (over Y) projective superspace $\mathbf{P}(\delta_l(J);\ \delta_l(J)+\delta_{l+1}(J)+\delta_{l+2}(J),\ b)$ for $G=\mathrm{OSp}(2r+1,2s)$ and $\mathbf{P}(\delta_l(J);\ \delta_l(J)+\sigma_{l+1}(J),\ b)$ for other G. Here $\mathbf{P}(x;\ y,\ b)$ denotes the superspace of x-dimensional isotropic (relative to b) lines in a y-dimensional superspace. The relative dimensions of these superspaces are given in Table 1 below (cf. [8, Theorem 5.6.3]). This dimension coincides with $l_{J_0J}(\sigma^{k+1})$ and by the inductive hypothesis $\dim Y_w=l_{IJ}(w)+\dim F_I$. □

5. Structure of Parabolic Subgroups

The list of known homogeneous spaces of a complex simple algebraic supergroup G, beginning with the spaces of complete G-flags described above, can be extended by including also spaces of incomplete G-flags, including super-Grassmannians. In the purely even situation these exhaust all possible homogeneous spaces, if the latter are defined as complete quotient spaces of G modulo closed subgroups. In the supercase, there are more homogeneous spaces than flag spaces; their structure is described by the following simple proposition.

<u>Proposition.</u> Let G be a complex algebraic supergroup, whose underlying group G_{red} is reductive. Let G act transitively (in the superscheme sense – see Sec. 3.6 or [6, Sec. 4.1.17]) on some superscheme X, and let P be the stationary subgroup of a closed point X (in this case we write X = G/P). The variety X_{red} is complete if and only if P_{red} is a parabolic subgroup of G_{red}. The structure of parabolic subgroups of reductive groups was described in [15] (cf. [13]). □

In classical geometry there is an equivalent definition of homogeneous spaces, as quotient spaces modulo parabolic subgroups, i.e., closed subgroups containing a Borel subgroup. Our topic in this section is superanalogs of parabolic subgroups. It will follow from our results that they are all stationary subgroups of incomplete G-flags. As an application we shall derive the following proposition (see [11]): The reflection of an invertible sheaf on a space of G-flags with respect to an odd root, used in proving the superversion of the Borel–Weil–Bott theorem, carries the sheaf into an invertible sheaf, again on a space of G-flags.

TABLE 1

Group	Projective superspace	Dimension of projective superspace
$\mathrm{OSp}(2r+1,2s)$	$P(1\mid 0;\ 3\mid 0,\ b)$ $P(0\mid 1;\ 1\mid 2,\ b)$	$1\mid 0$ $1\mid 1$
$\mathrm{OSp}(2r,\ 2s)$	$P(1\mid 0;\ 2\mid 0,\ b)$ $P(0\mid 1;\ 0\mid 2,\ b)$	$0\mid 0$ $1\mid 0$
$\Pi\mathrm{Sp}(m)$	$P(1\mid 0;\ 1\mid 1,\ b)$ $P(0\mid 1;\ 1\mid 1,\ b)$	$0\mid 1$ $0\mid 0$

1. Borel and Parabolic Subgroups. A Borel subgroup of a simple algebraic supergroup G of type SL, OSp, ΠSp or Q is the stabilizer of a complete G-flag in the space of the standard representation T. In all cases except Q, every Borel subgroup is obviously represented by a subgroup of upper triangular matrices $\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$ in $G\hookrightarrow \mathrm{GL}_{\mathbf{C}}(T)$. If G = Q the Borel subgroups are represented by subgroups in $G\hookrightarrow \mathrm{GL}_{\mathbf{C}}(T)$ of the following form:

$$\left(\begin{array}{cc|cc} * \ddots & * & * \ddots & * \\ 0 & \ddots * & 0 & \ddots * \\ \hline * \ddots & * & * \ddots & * \\ 0 & \ddots * & 0 & \ddots * \end{array}\right)$$

(the block subdivision corresponds to parity). A parabolic subgorup of G is a closed subgroup containing a Borel subgroup. Below we shall describe these subgroups (for G ≠ ΠSp and under some less essential restrictions) in terms of root systems; Borel subgroups will be treated in Subsecs. 5, 8 and parabolic subgroups in Subsecs. 6, 8. However, before we can deal more thoroughly with roots, we must lay the ground accordingly.

2. Root Systems. From now until Subsec. 7, $\mathfrak{g}$ will denote a classical Lie superalgebra over $\mathbf{C}$ of type A(m, n), m ≠ n, $m, n\geqslant 0$, $B(m,n)$, $m\geqslant 0$, $n>0$, $C(n)$, $n\geqslant 2$, $D(m,n)$, $m\geqslant 2$, $n>0$, D(2, 1; α), F(4) or G(3). [Type A(m, n) corresponds to the Lie superalgebra $\mathfrak{sl}(m+1, n+1)$, type B(m, n) to $\mathfrak{osp}(2m+1, 2n)$, type C(n) to $\mathfrak{osp}(2, 2n-2)$, type D(m, n) to $\mathfrak{osp}(2m, 2n)$, types D(2, 1; α), F(4) and G(3) are exceptional; type Q(n), which corresponds to the Lie superalgebra $\mathfrak{q}(n+1)$, will be considered in Subsec. 8.] If we fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, then $\mathfrak{g}$ factors into root subspaces (see [16, 2.5.3]): $\mathfrak{g}=\bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$, where Δ is the root system of $\mathfrak{g}$. This factorization has the following properties.

Proposition (Kac, [16, Proposition 2.5.5]).

(a) $\mathfrak{g}_0=\mathfrak{h}$;

(b) $\dim \mathfrak{g}_\alpha=1\ \forall\alpha\neq 0$;

(c) up to a multiplicative factor, there exists on $\mathfrak{g}$ exactly one nonsingular invariant symmetric bilinear form (,);

(d) (1) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]=\mathfrak{g}_{\alpha+\beta}\Leftrightarrow\alpha, \beta, \alpha+\beta\in\Delta, \alpha+\beta\neq 0$;

(2) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta)=0\ \forall\alpha\neq -\beta$;

(3) the form (,) defines a nonsingular pairing of $\mathfrak{g}_\alpha$ with $\mathfrak{g}_{-\alpha}$;

(4) the form (,) is nonsingular on $\mathfrak{h}$;

(5) $[e_\alpha, e_{-\alpha}]=(e_\alpha, e_{-\alpha})h_\alpha$, where $h_\alpha \neq 0$ is defined by $(h_\alpha, h)=\alpha(h)$, $h\in\mathfrak{h}$, $e_{\pm\alpha}\in\mathfrak{g}_{\pm\alpha}$;

(6) $\alpha\in\Delta\Rightarrow -\alpha\in\Delta$. □

3. Systems of Simple Roots and Positive Roots. Definition (Kac, [16, 2.5.4]). A subset $\Pi=\{\alpha_1,\dots,\alpha_r\}\subset\Delta$ is called the system of simple roots if there exist vectors $e_i\in\mathfrak{g}_{\alpha_i}$, $f_i\in\mathfrak{g}_{-\alpha_i}$, $h_i\in\mathfrak{h}$ such that $[e_i, f_j] = \delta_{ij}\delta_i$, the vectors e_i and f_i, i = 1,...,r, generate $\mathfrak{g}$, and Π is the minimal system with these properties.

Examples show that for all classical Lie superalgebras except A(m, n) the root system $\Delta \setminus \{0\}$ is not an abstract system of complex roots. Nevertheless, Proposition 2 and the following proposition furnish sufficient properties of root systems to carry the classical proof of the fundamental result (see [13, Chap. VIII, Sec. 3.4]) over to the supercase.

Proposition. The system of simple roots Π has the following properties:

1) Π is a basis of the vector space $\mathfrak{h}^*$;

2) all roots $\beta \in \Delta$ may be expressed as linear combinations $\beta = \sum_{\alpha \in \Pi} m_\alpha \cdot \alpha$ with integer coefficients m_α of the same sign (i.e., either all $m_\alpha \geqslant 0$ or all $m_\alpha \leqslant 0$).

Proof. We begin with the second part.

2) Since e_i and f_i, $i = 1, \ldots, r$ generate $\mathfrak{g}$, any element $g \in \mathfrak{g}_\beta$, $g \neq 0$ can be written as $g = \Sigma c_j \sigma_j (e_1, \ldots, e_r, f_1, \ldots, f_r)$, where σ_j is a commutator of elements of Π. Since $[e_i, f_j] = \delta_{ij} h_i$ and $[h_j, e_i] = \alpha_i(h_j) e_i, [h_j, f_i] = -\alpha_i(h_j) f_i, [h_i, h_j] = 0$ it follows that $g = \Sigma d_j \delta_j (e_1, \ldots, e_r) + \Sigma d'_k \delta'_k (f_1, \ldots, f_r) + \Sigma a_i h_i$, where δ_j and δ'_k are commutators of the elements $e_1, \ldots, e_r$ and $f_1, \ldots, f_r$, respectively. If $g \in \mathfrak{g}_\beta$, $\beta \neq 0$, $\dim \mathfrak{g}_\beta = 1$ by Proposition 1(b), and therefore $g = a\delta_j(e_1, \ldots, e_r)$ or $a\delta'_k(f_1, \ldots, f_r)$ for some δ_j and δ'_k. Hence $\beta = \sum_{i=1}^r m_i \alpha_i$, where m_i is the degree of the monomial δ_j in e_i (or of δ'_k in f_i), i.e., $m_i \in \mathbf{Z}$ and all $m_i \geqslant 0$ (or $\leqslant 0$).

1) That the elements of Π are linearly independent becomes obvious if one looks at the systems of simple roots written out [16, 2.5.4].

The set Π spans $\mathfrak{h}^*$ as a vector space over $\mathbf{C}$. Indeed, for any $\alpha \in \mathfrak{h}^*$ one can define h_α by requiring that $(h_\alpha, h) = \alpha(h)$, $h \in \mathfrak{h}$, and the fact that $\mathfrak{g}$ is spanned by the elements $e_1, \ldots, e_r$ and $f_1, \ldots, f_r$ implies that $h_\alpha = \sum_{i=1}^r c_i h_i$. Since $h_i = [e_i, f_i] = (e_i, f_i) h_{\alpha_i}$, where $(h_{\alpha_i}, h) = \alpha_i(h)$ for all $h \in \mathfrak{h}$ [Proposition 2(d), (5)], it follows that $h_\alpha = \sum_{i=1}^r c_i \cdot (e_i, f_i) h_{\alpha_i}$, whence $\alpha = \sum_{i=1}^r c_i \cdot (e_i, f_i) \alpha_i$, as required. □

We define the set of positive roots Δ^+ as the set of nonzero linear combinations of simple roots with nonnegative integer coefficients: $\Delta^+ = (\Delta \cap \mathbf{Z}_+ \Pi) \setminus \{0\}$. A symmetric definition gives $\Delta^-: \Delta^- = (\Delta \cap (-\mathbf{Z}_+)\Pi) \setminus \{0\}$. By Proposition 3, $\Delta = \Delta^+ \cup \Delta^- \cup \{0\}$, $\Delta^+ \cap \Delta^- = \varnothing$.

4. Indecomposable and Simple Roots. Definition. A root $\alpha \in \Delta^+$ is said to be decomposable if $\exists \beta, \gamma \in \Delta^+ : \alpha = \beta + \gamma$; otherwise we shall say that α is an indecomposable root.

LEMMA. The set of indecomposable roots is precisely the set of simple roots Π.

Proof. 1) Each element $\alpha_j \in \Pi$ is indecomposable, for if $\alpha_j = \beta + \gamma$, $\beta = \Sigma b_i \alpha_i$, $\gamma = \Sigma c_i \alpha_i$, $b_i \geqslant 0$, $c_i \geqslant 0$, $\alpha_i \in \Pi$, then $\alpha_j = \Sigma (b_i + c_i) \alpha_i$, and since the simple roots are linearly independent we get $b_i + c_i = 0$ $\forall i \neq j$, $b_j + c_j = 1$. Since the coefficients b_k, c_k are nonnegative integers, it follows that $\beta = 0$ or $\gamma = 0$, contrary to the assumption that $\beta, \gamma \in \Delta^+$.

2) Let $\alpha \in \Delta^+$, $\alpha \notin \Pi$. We claim that α is decomposable. Indeed, consider any nonzero vector $g \in \mathfrak{g}_\alpha$. Then $g = x \cdot \delta(e_1, \ldots, e_r)$ by the definition of Π [here $x \in \mathbf{C}$, $\delta(e_1, \ldots, e_r)$ is some commutator of the elements $e_1, \ldots, e_r$ figuring in the definition of the system of simple roots). Since $\alpha \notin \Pi$, $\delta(e_1, \ldots, e_r) = [\delta_1(e_1, \ldots, e_r), \delta_2(e_1, \ldots, e_r)]$ for certain commutators δ_1 and δ_2. Since $[\mathfrak{g}_\gamma, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\beta+\gamma}$ we have $\alpha = \beta + \gamma$, where β and γ are such that $\delta_1(e_1, \ldots, e_r) \in \mathfrak{g}_\beta$, $\delta_2(e_1, \ldots, e_r) \in \mathfrak{g}_\gamma$, and hence $\beta, \gamma \in \Delta^+$. □

5. Borel Subalgebras and Borel Subgroups. The Borel subalgebra of a classical Lie superalgebra $\mathfrak{g}$ [relative to a fixed Cartan subalgebra – see above, Subsec. 2] is the subalgebra $\mathfrak{b} = \bigoplus_{\alpha \in \Delta^+ \cup \{0\}} \mathfrak{g}_\alpha$. A parabolic subalgebra is any subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$.

Let G be a complex algebraic supergroup of type SL(m, n), m ≠ n, OSp(m, n) its component of the identity [G^0 = G always, except in the case G = OSp(2r, 2s)], $\mathfrak{g}$ the corresponding Lie superalgebra.

Proposition (Skornyakov, cf. Kac [17]). Under the natural one-to-one correspondence between subalgebras of $\mathfrak{g}$ and closed subgroups of G^0, Borel subalgebras correspond in one-to-one fashion to Borel subgroups in the sense of Subsec. 1, and parabolic subalgebras to parabolic subgroups.

Proof. 1) Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}$ corresponding to the set of positive roots Δ^+. There is a fuller analog of Lie's theorem for the Lie superalgebra $\mathfrak{b}$ than for arbitrary solvable Lie superablgebras: Every finite-dimensional irreducible representation of $\mathfrak{b}$ is one-dimensional. Indeed, any highest weight vector of an arbitrary finite-dimensional representation of $\mathfrak{b}$ is a characteristic vector, and thus any finite-dimensional irreducible representation of $\mathfrak{b}$ is one-dimensional. Applying this fact successively to the restriction to $\mathfrak{b}$ of the standard representation T of $\mathfrak{g}$ and its quotient representations, we infer that $\mathfrak{b}$ leaves invariant some complete SL-flag f in T. If G = SL this means that the closed subgroup B of G with Lie superalgebra $\mathfrak{b}$ is contained in the stabilizer $St_G(f)$ of the flag, i.e., in a Borel subgroup. In fact, $B = St_G(f)$, for otherwise the root system of the Lie superalgebra of $St_G(f)$ would contain roots α and $-\alpha$ for some $\alpha \in \Delta \setminus \{0\}$. This is impossible, as is readily seen, e.g., by considering the matrix representation.

If G = OSp, the analog of Lie's theorem for $\mathfrak{b}$ must be applied in a somewhat different way. To this end, we note that every weight vector in the standard representation of $\mathfrak{g}$ is isotropic (this is also true in every subfactor of the restriction of this representation to $\mathfrak{b}$). Indeed, none of the weights of the standard representation vanish, and if $v \in T$ is a vector of weight α, then $0 = b(hv, v) + b(v, hv) = 2\alpha(h) b(v, v)$ for all $h \in \mathfrak{h}$, whence it follows that b(v, v) = 0 (b is the symmetric $\mathfrak{g}$-invariant bilinear structure form of T), that is, v is isotropic. We now use induction to construct a complete G-flag in T that is invariant under $\mathfrak{b}$, beginning with a $\mathfrak{b}$-invariant one-dimensional weight subspace V (which, as just shown, is isotropic), and considering the subfactor $V^\perp/V$ of the representation T of $\mathfrak{b}$, where $V^\perp$ is the orthogonal complement of V relative to b. We then apply the same arguments to $V^\perp/V$, and so on. The final result is a complete isotropic flag in $f = \{0 \subset U \subset \ldots \subset V^\perp \subset T\}$. As in the case G = SL, the closed subgroup B of G^0 corresponding to $\mathfrak{b}$ is precisely $St_{G^0}(f)$, that is to say, a Borel subgroup.

2) Let B be a Borel subgroup of G^0. As already remarked, if α is a root of its Lie superalgebra $\mathfrak{b}$, then $-\alpha$ is not a root of $\mathfrak{b}$. Therefore, if Δ^+ denotes the system of nonzero roots of $\mathfrak{b}$ and Δ^- the complement of $\Delta^+ \cup \{0\}$ in the root system Δ of $\mathfrak{g}$, then $\Delta^+ = -\Delta^-$, $\Delta = \Delta^+ \cup \Delta^- \cup \{0\}$. Let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be the set of indecomposable elements of Δ^+. We claim that Π is the system of simple roots in the sense of Definition 3. Choose nonzero vectors $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ and put $h_i := [e_i, f_i]$ – by Proposition 2(d), (5) this is an element of $\mathfrak{h}$. In addition, if $i \neq j$, $[e_i, f_j] = 0$, for otherwise $\alpha_i - \alpha_j = \beta \in \Delta \setminus \{0\}$, contrary to the assumption that α_i and α_j are indecomposable. Obviously, the vectors e_i span the subalgebra $\bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$; symmetrically, the vectors f_i span the subalgebra $\bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$. The vectors h_i span the space $\mathfrak{h}$, since Π generates $\mathfrak{h}^*$, and by Proposition 2(d), (5) the vectors h_i are dual relative to the form (,) to the roots α_i (up to a multiplicative factor). Thus, the elements e_i, f_i span $\mathfrak{g}$. Finally, Π is the minimal set with these properties, since a minimal subset $\Pi' \subseteq \Pi$ would be a system of simple roots and the corresponding Borel subalgebra $\mathfrak{b}'$ would be contained in $\mathfrak{b}$. But in the first part of the proof we proved that in this situation $\mathfrak{b}' \subset \mathfrak{b}$ implies $\mathfrak{b}' = \mathfrak{b}$, whence it follows that $\Pi' = \Pi$. Thus Π is the system of simple roots and $\mathfrak{b}$ is indeed a Borel subalgebra.

3) The assertion concerning parabolic subgroups and subalgebras follows trivially from the proven result for Borel subgroups and subalgebras. □

6. THEOREM. Let $\mathfrak{g}$ be a classical Lie superalgebra, $\mathfrak{b}$ a Borel subalgebra, Π the corresponding system of simple roots. Then for any parabolic subalgebra $\mathfrak{p} \supset \mathfrak{b}$ there exists a subset $I \subset \Pi$ such that $\mathfrak{p} = \mathfrak{b} \oplus \left(\bigoplus_{\alpha \in \Delta_I^-} \mathfrak{g}_\alpha \right)$, where Δ_I^- is the subset of the set Δ^- of negative roots generated by I as a semigroup.

Proof. Since $\mathfrak{h}$ is an Abelian subalgebra of $\mathfrak{p}$, $\mathfrak{p} = \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha$ for some subset $A \subset \Delta$. We must show that $A = \Delta^+ \cup \{0\} \cup \Delta_I^-$ for some $I \subset \Pi$.

Define I to be the set of all simple roots appearing in the decomposition of roots in $A \cap \Delta^-$ as sums of simple roots. We claim that $I \subset A$. Let $\alpha \in A \cap \Delta^-$ and let $-\alpha$ be decomposable, i.e., for some $\beta, \gamma \in \Delta^+$ $-\alpha = \beta + \gamma$. Then $-\beta, -\gamma \in A \cap \Delta^-$, since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{-\gamma}$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\gamma] = \mathfrak{g}_{-\beta}$ by Proposition 2(d), (1), and as $\beta, \gamma \in \Delta^+ \subset A$ it follows that $-\beta, -\gamma \in A$. It now follows by induction that $I \subset A$. Hence $\Delta_I^- \subset A$. By the definition of I $A \cap \Delta^- \subset \Delta_I^-$, and therefore $A = \Delta^+ \cup \{0\} \cup \Delta_I^-$. □

7. COROLLARY (Skornyakov). All connected parabolic subgroups of G are stabilizers of G-flags in T (G of type SL or OSp).

Proof. Let B be a connected Borel subgroup which is the stabilizer $\mathrm{St}_{G^0}(f)$ of some complete G-flag $f=\{0\subset\mathscr{F}_1\subset\ldots\subset\mathscr{F}_{m+n}=T\}$, where $T^{m|n}$ is the space of the standard representation of G. We distinguish two cases.

I. Case $G=\mathrm{SL}(m,n)$, $\mathrm{OSp}(2r+1,2s)$ or $\mathrm{OSp}(2r,2s)$; if $G=\mathrm{OSp}(2r,2s)$ $\dim\mathscr{F}_{r+s}-\dim\mathscr{F}_{r+s-1}=0|1$. Let f' be a (not necessarily complete) G-flag that can be extended to f (i.e., all constituents of f' are constituents of f). It is clear that $\mathrm{St}_{G^0}(f')\supset\mathrm{St}_{G^0}(f)=B$, i.e., $\mathrm{St}_{G^0}(f')$ is a (connected) parabolic subgroup. Thus the set of stabilizers of G-flags extending to f is a subset of the finite set of (connected) parabolic subgroups containing B. It will suffice to prove that these sets contain the same number of elements. By Proposition 5 and Theorem 6, the number of parabolic subgroups containing B is equal to the number of subsets of the corresponding system of simple roots Π. The number of elements of Π for G = SL(m, n) is m + n − 1, and for G = OSp(m, n) − [(m + n)/2] (see [16, 2.5.4]). Clearly, for every G this is precisely the number of (isotropic in the case G = OSp) constituents of the complete G-flag f, not counting 0 and T. The number of G-flags f' extending to f is obviously equal to the number of subsets of the set of (isotropic) constituents of f, not counting 0 and T. In addition, to different f_1' and f_2' correspond different subgroups $\mathrm{St}_{G^0}(f_1')$ and $\mathrm{St}_{G^0}(f_2')$. Otherwise these subgroups would both coincide with the subgroup $\mathrm{St}_{G^0}(f')$ for the flag f' composed of all constituents of f_1' and f_2'. This would imply that the space of G-flags of the same type as f' is precisely the space of G-flags of the same type as f_1' (recall that the type of a flag is the ordered sequence of dimensions of its successive factors); but this is impossible, because constituents of a flag f' not occurring in f_1' can always be infinitesimally displaced in such a way that the resulting flag is still a G-flag (at this point it is essential that we are in Case I). We have thus proved that the number of stabilizers of G-flags that can be extended to a fixed complete G-flag f equals the number of connected parabolic subgroups containing B = $\mathrm{St}_{G^0}(f)$. This proves the corollary for Case I.

II. Case $G=\mathrm{OSp}(2r,2s)$, $\dim\mathscr{F}_{r+s}-\dim\mathscr{F}_{r+s-1}=1|0$. Here the argument is somewhat more complicated, since different G-flags f_1' and f_2' extending to the same complete G-flag f may have the same stabilizers $\mathrm{St}_{G^0}(f_1')$ and $\mathrm{St}_{G^0}(f_2')$. Example: f_2' = f, f_1' = {f without the Lagrangian (maximal isotropic) constituent $\mathscr{F}_{r+s}$}. If $\mathrm{St}_{G^0}(f_1')\supsetneqq\mathrm{St}_{G^0}(f_2')$, then the Lagrangian constituent of f_2' can be infinitesimally displaced, which is impossible – in fact, f_1' can be extended to a complete flag by adding a Lagrangian subspace in exactly two ways. Hence it follows, besides, that the same Borel subgroup B is the stabilizer in G^0 of two distinct complete G-flags: $f=\{0\subset\mathscr{F}_1\subset\ldots\subset\mathscr{F}_{r+s-1}\subset\mathscr{F}_{r+s}\subset\mathscr{F}_{r+s+1}\subset\ldots\subset T\}$ and $f'=\{0\subset\mathscr{F}_1\subset\ldots\subset\mathscr{F}_{r+s-1}\subset\mathscr{F}'_{r+s}\subset\mathscr{F}_{r+s+1}\subset\ldots\subset T\}$, the latter differing from f only in its Lagrangian constituent. We shall make use of this fact in proving the remainder of the corollary.

As in Case I, we note that the set of stabilizers of G-flags that can be extended to f or to f' is a subset of the set of connected parabolic subgroups containing $B=\mathrm{St}_{G^0}(f)=\mathrm{St}_{G^0}(f')$. These sets coincide, since they are finite and contain the same number of elements. Indeed, the second set contains $2^{|\Pi|}=2^{r+s}$ elements, where Π is the system of simple roots of the supergroup G = OSp(2r, 2s), while the number of elements in the first is obtained by adding the following: 1) the number of G-flags containing the constituent $\mathscr{F}_{r+s}$ but not $\mathscr{F}_{r+s-1}$; 2) the number of G-flags containing $\mathscr{F}'_{r+s}$ but not $\mathscr{F}_{r+s-1}$; 3) the number of G-flags containing $\mathscr{F}_{r+s-1}$ and $\mathscr{F}_{r+s}$; 4) the number of G-flags containing neither $\mathscr{F}_{r+s-1}$, $\mathscr{F}_{r+s}$ nor $\mathscr{F}_{r+s}$ (in all cases the flags in question are assumed to be extendable to f or f'). Obviously, the stabilizers of all these flags are distinct and each of the four components of the sum equals 2^{r+s-2}.□

8. In this subsection we briefly summarize some results concerning parabolic subgroups of the supergroup G = Q(n). The corresponding Lie superalgebra is $\mathfrak{g}=\mathfrak{q}(n)=\{X\in\mathfrak{gl}(n,n)\,|\,[X,p]=0\}$, where p is a given Π-symmetry in the standard representation $T^{n|n}$, $p^2=1$. The proofs, which will be omitted, are analogous to those for G = SL and S = OSp.

8.1. Let $\mathfrak{g}=\mathfrak{q}(n)$, $n\geqslant3$, and let $\mathfrak{h}_{\bar0}$ be a Cartan subalgebra of the even part $\mathfrak{g}_{\bar0}\simeq\mathfrak{gl}(n)$. For $\mathfrak{h}_{\bar0}$ we have a decomposition $\mathfrak{g}=\bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$ into root subspaces (see Penkov [10]), where Δ is the root system of the reductive Lie algebra $\mathfrak{g}_{\bar0}$, so that Δ is a root system of type A_{n-1}. We define the Cartan subalgebra of $\mathfrak{g}$ to be the subalgebra $\mathfrak{h}=\mathfrak{h}_{\bar0}\oplus\mathfrak{h}_{\bar1}:=\mathfrak{g}_0$. The root decomposition has the following properties.

Proposition (Penkov [10]).

(a) $\dim\mathfrak{g}_\alpha=1|1\quad\forall\alpha\neq0$;

(b) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]=\mathfrak{g}_{\alpha+\beta} \Leftrightarrow \alpha, \beta, \alpha+\beta \in \Delta, \alpha+\beta \neq 0.\square$

8.2. Since $\Delta \subset \mathfrak{h}_{\bar 0}^*$ is the root system of the Lie algebra $\mathfrak{gl}(n)$, Δ is an abstract system of complex roots in the subspace of linear forms on $\mathfrak{h}_{\bar 0}$ that vanish on the center of $\mathfrak{gl}(n)$ (see [12, 13]). We may therefore call a subset $\Pi=\{\alpha_1,\ldots,\alpha_r\}\subset\Delta$ the system of simple roots of the superalgebra $\mathfrak{g}$ if it is an abstract system of simple roots (see [12]). We also define the set of positive roots $\Delta^+ := (\Delta \cap \mathbf{Z}_+\Pi)\setminus\{0\}$ and the set of negative roots $\Delta^- := (\Delta\cap(-\mathbf{Z}_+)\Pi)\setminus\{0\}$. Then $\Delta=\Delta^+\cup\Delta^-\cup\{0\}$, $\Delta^+\cap\Delta^-=\varnothing$.

8.3. The Borel subalgebra of $\mathfrak{g}$ is defined as $\mathfrak{b}=\bigoplus_{\alpha\in\Delta^+\cup\{0\}}\mathfrak{g}_\alpha$. A parabolic subalgebra is any subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$.

Proposition. Under the natural one-to-one correspondence between subalgebras in $\mathfrak{g}$ and closed subgroups of G, the Borel subalgebras correspond in one-to-one fashion to the Borel subgroups in the sense of Subsec. 1, and the parabolic subalgebras to the parabolic subgroups. $\square$

8.4. THEOREM. Let $\mathfrak{g}=\mathfrak{q}(n)$, $n\geqslant 3$, and let $\mathfrak{b}$ be a Borel subalgebra and Π the corresponding system of simple roots. Then for any parabolic subalgebra $\mathfrak{p}\supset\mathfrak{b}$ there exists a subset $I\subset\Pi$ such that $\mathfrak{p}=\mathfrak{b}\oplus(\bigoplus_{\alpha\in\Delta_I^-}\mathfrak{g}_\alpha)$, where Δ_I^- is the subset of the set Δ^- of negative roots generated by I as a semigroup.

Proof. The set of roots of the superalgebra $\mathfrak{p}$ relative to $\mathfrak{h}_{\bar 0}$ is indeed a parabolic set of roots, and the theorem easily follows from the description of parabolic sets of roots — see [12, Proposition VI.1.7.20]. $\square$

8.5. COROLLARY. All parabolic subgroups of G = Q(n) are stabilizers of G-flags in T. $\square$

Remark. The truth of the theorem and the corollary is readily verified for the superalgebras $\mathfrak{q}(1)$ and $\mathfrak{q}(2)$. If $\mathfrak{sq}(n)\subset\mathfrak{q}(n)$ is the subsuperalgebra consisting of all endomorphisms of $T^{n|n}$ that commute with p and have zero odd trace, the assertions are true (and the proofs are the same word for word) for $n\geqslant 3$, but for n = 2 there is a counterexample: the parabolic subalgebra of $\mathfrak{sq}(2)$ of all elements

$$\left(\begin{array}{cc|cc} * & * & * & * \\ 0 & * & * & * \\ \hline * & * & * & * \\ * & * & 0 & * \end{array}\right).$$

9. The structure of the parabolic subgroups in the case G = ΠSp(n) is not known. It should be noted that the root system of the superalgebra $\mathfrak{g}=\pi\mathfrak{sp}(n)$ is even less similar to an abstract root system than that of the superalgebra $\mathfrak{osp}(m,n)$. An idea of the difficulties arising here may be gained from the root description of Borel subgroups of $\mathfrak{g}$ presented by Penkov [11].

6. Superspaces of Incomplete Flags

In this section we summarize results relating to Schubert supercells in the case of superspaces of incomplete flags.

1. The basic objects of our investigation will be G-flags. We retain the notation of Sec. 1: $T^{m|n}$ is the space of the standard representation of G, ${}^GI$ is the set of types of G-flags, i.e., the set of sequences $(\delta_1,\ldots,\delta_r)$ such that for some G-flag $0=\mathcal{P}_0\subset\mathcal{P}_1\subset\ldots\subset\mathcal{P}_r=T$, $\delta_i=\dim\mathcal{P}_i-\dim\mathcal{P}_{i-1}$. If $I\in{}^GI$, then ${}^GF_I$ denotes the superspace of G-flags of type I. As in the case of complete flags, all the supermanifolds are connected, with the exception of G = OSp(2r, 2s), when ${}^GF_I$ may spit into two components.

LEMMA.

$$^{SL}I=\left\{(\delta_1,\ldots,\delta_r)\,|\,1\leqslant r\leqslant m+n,\ \delta_i\geqslant 0|0,\ \sum_{i=1}^r\delta_i=m|n\right\},$$

$$^{OSp}I=\{(\delta_1,\ldots,\delta_r)\in{}^{SL}I\,|\,\delta_i=\delta_{r+1-i}\},$$

$$^{\Pi Sp}I=\{(\delta_1,\ldots,\delta_r)\in{}^{SL}I\,|\,\delta_i=\delta^c_{r+1-i}\},$$

$$^{Q}I=\{(\delta_1,\ldots,\delta_r)\in{}^{SL}I\,|\,\delta_i=a_i\,|\,a_i\}.\ \square$$

Fix the number of constituents r and a sequence of natural numbers $\mathfrak{u}=(u_1,\ldots,u_r)$ such that $\sum_{i=1}^{r}u_i=m+n$. Let ${}^G\Theta={}^G\Theta(r,\mathfrak{u})$ denote the subset of ${}^GI$ consisting of all $I=(\delta_1,\ldots,\delta_r)$ such that $|\delta_i|=u_i$ for all i, where $|(a|b)|=a+b$. The superspace of incomplete G-flags corresponding to ${}^G\Theta$ will be denoted by ${}^GF_\Theta$:

$$ {}^GF_\Theta:=\coprod_{I\in{}^G\Theta}{}^GF_I. $$

2. Position of an Incomplete Flag Relative to a Complete Flag. Definition. Let $\mathcal{F}_{I,\cdot}$ be a complete G-flag in T_S of type $I\in{}^GI_n$, $\mathcal{F}_{J,\cdot}$ a G-flag of type $I\in{}^G\Theta$ (S is a superscheme over $\mathbf{C}$). We shall say that these flags are regularly positioned relative to each other if for all i, j $\mathcal{F}_{I,i}\cap\mathcal{F}_{J,j}$ are locally direct locally free subsheaves in T_S of constant rank. The type of the position of $\mathcal{F}_{J,\cdot}$ relative to $\mathcal{F}_{I,\cdot}$ is the matrix with components $d_{ij}=\mathrm{rk}\,(\mathcal{F}_{I,i}\cap\mathcal{F}_{J,j})$, $0\leqslant j\leqslant r$, $0\leqslant i\leqslant t$, where

$$ t=m+n\quad\text{for } G=\mathrm{SL},\ \mathrm{OSp}, $$
$$ t=2m\ \text{for}\ G=\Pi\mathrm{Sp}, $$
$$ t=m\ \text{for}\ G=Q, $$

– this notation will be retained throughout this section.

3. Properties of Relative Position Matrices. LEMMA. The matrix $(d_{ij})_{\substack{0\leqslant i\leqslant t\\ 0\leqslant j\leqslant r}}$ of the position of an incomplete G-flag relative to a complete flag has the following property: (d_{ij}) is obtained from some matrix $(d_{ij})_{0\leqslant i,\,j\leqslant t}$ which is the relative position type of complete G-flags by deleting the $t-r$ rows with the same indexes as in the natural "forgetful" map ${}^GI_n\to{}^G\Theta$.

The proof follows from the fact that any incomplete flag can be extended to a complete one. □

4. Definition. Let $\Theta={}^G\Theta(r,\mathfrak{u})$, as in Subsec. 1.

a) W_Θ is the subgroup of ${}^GW$ whose elements are the permutations that carry each of the sets $\{1,\ldots,i_1\}$, $\{i_1+1,\ldots,i_2\}$, $\ldots$, $\{i_{r-1}+1,\ldots,t\}$ into itself, where $i_k:=\sum_{j=1}^{k}u_j$.

b) w(I) will denote the image of a pair (w,I), $w\in{}^GW/W_\Theta$, $I\in{}^GI_n$ under the map $({}^GW/W_\Theta)\times{}^GI_n\to{}^G\Theta$ induced by the action ${}^GW\times{}^GI_n\to{}^GI_n$.

5. Combinatorial Lemma. There exists a bijection between the following sets:

a) the geometrically realizable types of position of incomplete flags of all types $I\in{}^G\Theta$ relative to complete flags;

b) the matrices $(d_{ij})_{\substack{0\leqslant i\leqslant t\\ 0\leqslant j\leqslant r}}$ with properties 3, hence satisfying the following symmetry conditions in cases G = OSp or ΠSp:

$$ \mathrm{OSp}:\ d_{ij}=d_{t-i,r-j}-m|n+d_{t-i}(I)+d_{r-j}(J), $$
$$ \Pi\mathrm{Sp}:\ d_{ij}=d^c_{t-i,r-j}-m|m+d^c_{t-i}(I)+d^c_{r-j}(J); $$

c) the triples $\{(I,J,w)\,|\,I\in{}^GI_n,\ J\in{}^G\Theta,\ w\in{}^GW/W_\Theta,\ J=w(I)\}$.

Proof. It is readily seen that each of these sets is obtained from the corresponding set for complete flags by a suitable "forgetful" projection. The fibers of these projections are mapped bijectively onto one another by the bijections whose existence was established in the Combinatorial Lemma for complete flags. □

6. Schubert Supercells. Put $F:={}^GF$, $F':={}^GF_\Theta$, $W:={}^GW$. Let $\mathcal{F}'_\cdot$ be the tautological flag on F', $\mathcal{F}_\cdot$ the tautological flag on F.

For every class $w\in W/W_\Theta$ let $\mathrm{d}_{ij,w}$ denote the function on $\coprod_{J=w(I)}F_I\times F'_J$ with values in $\mathbf{Z}\times\mathbf{Z}$ which is constant on each $F_I\times F'_J$:

$$ \mathrm{d}_{ij,w}|_{F_I\times F'_J}:=d_{ij,w,IJ} $$

– the matrix component corresponding to the triple (I, J, w) by virtue of the Combinatorial Lemma.

Let $|Y_w| \subset (\coprod_{J=w(I)} F_I \times F'_J)(\mathbf{C})$ be the set of all $\mathbf{C}$-points x over which $\dim_x(\mathcal{F}_i \cap \mathcal{F}'_j) = \mathbf{d}_{ij,w}$. Obviously, $\coprod_{w \in W/W_\Theta} |Y_w|$ covers all points of $(F \times F')_{\mathrm{red}}$.

THEOREM. a) On each $|Y_W|$ there exists a canonical structure of a locally closed sub-superscheme $Y_w \subset F \times F'$ such that the morphism $\coprod_{w \in W/W_\Theta} Y_w \hookrightarrow F \times F'$ is a flattening partition for the sheaf system $\{\mathcal{F}_i \cap \mathcal{F}'_j\}$. This means that an arbitrary morphism of superschemes $g: S \to F \times F'$ (S Noetherian) has the property:

$$\left\{\begin{array}{l}\text{all } g^*(\mathcal{F}_i \cap \mathcal{F}'_j) \text{ — are locally direct locally free sub-} \\ \text{sheaves in } T_S \text{ of ranks } \mathbf{d}_{ij,w}\end{array}\right\}$$

if and only if g factors through the embedding $\coprod_{w \in W/W_\Theta} Y_w \hookrightarrow F \times F'$.

b) All Y_W are bundles over F: $Y_w \xrightarrow{p_1} F$, and the typical fiber $p_1^{-1}(x)$ of the bundle is isomorphic to the open supercell $\mathbf{C}^{r|s}$.

c) Y_W is a functor from the category of Noetherian superschemes over $\mathbf{C}$ into the category of sets: given a superscheme S, it determines the set of S-points of the superscheme $\coprod_{J=w(I)} F_I \times F'_J$ over which $\mathcal{F}_i \cap \mathcal{F}'_j$ are locally direct locally free subsheaves in T_S of ranks $\mathbf{d}_{ij,w}$.

d) $\dim Y_w = \dim F + \min_{w' \in w \subset W} l(w')$, $w \in W/W_\Theta$.

The proof is based on the observation that the fibers of the natural projection $F_I \times F_{J_1} \to F_I \times F'_J$ are superspaces of complete G-flags (where $J_1 \in {}^G I_n$ is an extension of J to the type of a complete G-flag). □

7. As in the case of complete flags, Schubert supercells will be G-orbits in the product of the space of complete flags and a space of incomplete flags.

Transitivity Lemma. Let S be a superscheme, $T_S = T \otimes \mathcal{O}_S$, $\mathcal{F}'_\cdot$, $\mathcal{F}''_\cdot$ two regularly relatively positioned G-flags in T_S of types $I \in {}^G I_n$ and $J \in {}^G \Theta$, respectively, and let the type of the position of $\mathcal{F}''_\cdot$ relative to $\mathcal{F}'_\cdot$ be (d_{ij}). Let $\tilde{\mathcal{F}}'_\cdot$, $\tilde{\mathcal{F}}''_\cdot$ be another pair of G-flags with the same properties [the same (d_{ij}), the same types: types: $\mathcal{F}'_\cdot$ = type $\tilde{\mathcal{F}}'_\cdot$, type $\mathcal{F}''_\cdot$ = type $\tilde{\mathcal{F}}''_\cdot$]. Then every point $s \in S$ has an affine neighborhood $U \simeq \operatorname{Spec} A$ such that there exists an element g of the group G(A) of A-points of G carrying the pair of flags

$$\mathcal{F}'_\cdot|_U,\ \mathcal{F}''_\cdot|_U \text{ in } T_U$$

into the pair of flags

$$\tilde{\mathcal{F}}'_\cdot|_U,\ \tilde{\mathcal{F}}''_\cdot|_U.$$

The proof coincides almost word for word with that of the Transitivity Lemma for complete flags. □

7. Order in the Weyl Supergroup; Relative Position of Schubert Supercells

The Schubert supercells of superspaces of complete flags are indexed by the elements of the Weyl supergroup, and also by the relative position matrices (see Secs. 2, 3). Hence the relation $Y_{w',IJ} \subset \bar{Y}_{w,IJ}$ (where $\bar{Y}$ is the superscheme closure, $Y_{w,IJ} := Y_w \cap {}^G F_I \times {}^G F_J$) defines an order in the Weyl supergroup and on the set of relative position matrices. In this section we shall give an intrinsic description of these orders. Using the fact that all $Y_{w,IJ}$ are G-orbits, one readily shows that $(Y_{w,IJ})_{\mathrm{red}} = |Y_{w,IJ}|$ are the Schubert cells of the space $(F_I)_{\mathrm{red}} \times (F_J)_{\mathrm{red}}$. It is therefore clear that the Schubert supercells are "distributed" over $F \times F$ in the same way as the Schubert cells over $F_{\mathrm{red}} \times F_{\mathrm{red}}$.

It remains unclear whether a single supercell which lies in the closure of another at the underlying level can be of higher odd dimension than the other supercell. If this were possible, the partition into Schubert supercells would not be a supercell complex in a reasonable supersense. The fact that this is nevertheless not the case follows from one of our results (see Theorem 3):

$$(Y_{w',IJ})_{\mathrm{red}} \subset (\bar{Y}_{w,IJ})_{\mathrm{red}} \Leftrightarrow Y_{w',IJ} \subset \bar{Y}_{w,IJ}.$$

1. Let I, $J \in {}^G I_n$ $W_{IJ} = \{w \in {}^G W \mid w(I) = J\}$, and let $l(w) = l_0(w) | l_1(w) \in \mathbf{Z} \times \mathbf{Z}$ be the superlength of an element w of the Weyl supergroup ${}^G W$. It is evident from the definition of the Weyl

supergroup that there is an isomorphism of groups $W_{JJ}\simeq{}^{G_{\mathrm{red}}}W$, hence also a bijection of sets $W_{IJ}\simeq{}^{G_{\mathrm{red}}}W$. In fact, we shall define an order relation not on ${}^{G}W$ but rather on the subset $W_{IJ}\subset{}^{G}W$, which indexes the Schubert supercells in the space ${}^{G}F_I\times{}^{G}F_J$.

Let us call $w\in{}^{G}W$ a <u>reflection</u> if w is conjugate in ${}^{G}W$ to a basis reflection (see Definition 4.1).

<u>Definition.</u> (For G ≠ OSp(2r, 2s) – for this supergroup see the remark following the definition.)

(i) Let $w_1,\ w_2\in W_{IJ},\ \sigma\in W_{JJ}$ a reflection. Then $w_1\xrightarrow{\sigma}w_2$ means that $\sigma w_1=w_2$ and $l_0(w_2)=l_0(w_1)+1$.

(ii) We put w < w' if there exists a chain $w=w_1\to w_2\to\ldots\to w_k=w'$.

<u>Remark.</u> In the case G = OSp(2r, 2s) the group G_{red} = O(2r) × Sp(2s) is the union of two connected components. By definition, the Weyl group is ${}^{G_{\mathrm{red}}}W$ = N(H)/C(H), H a maximal torus in G_{red}, N and C its normalizer and centralizer, respectively, in G_{red}. Let W^0 denote the Weyl group of the component of the identity $(G_{\mathrm{red}})^0$ of G_{red}. W^0 is a subgroup of index 2 in ${}^{G_{\mathrm{red}}}W\simeq W_{JJ}$. For our purposes, it will be convenient to define an order not on the set W_{IJ} but on each of the two left cosets W_{IJ} and $W_{IJ}^2\subset W_{IJ}$ modulo the subgroup $W^0\subset W_{JJ}$. Thus, in the case G = OSp(2r, 2s) the above definition must be modified as follows: instead of W_{IJ} take W_{IJ}^i, i = 1 or 2, and instead of W_{JJ} the group W^0.

2. Before formulating the theorem, we present a few definitions which are simple generalizations of the classical ones to the supercase.

<u>Definition.</u> a) Let $\varphi:Y\to Z$ be a morphism of superschemes. The superscheme image of Y under φ is defined as the closed subsuperscheme $\varphi(Y)$ of Z uniquely determined by the properties: φ factors through the natural embedding $\varphi(Y)\hookrightarrow Z$, and if $X\hookrightarrow Z$ is a closed embedding through which φ factors, then the embedding $\varphi(Y)\hookrightarrow Z$ also factors through $X\hookrightarrow Z$:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, "\varphi"] & X \arrow[d, hook] \\ \varphi(Y) \arrow[r, hook] \arrow[ur, dashed] & Z \end{tikzcd}.$$

b) Let Y be a locally closed subsuperscheme in Z. Its closure $\bar{Y}$ is defined as the superscheme image of Y under the natural embedding $Y\hookrightarrow Z$.

c) Let X and Y be locally closed subsuperschemes in Z. We shall say that $X\subset\bar{Y}$ if the embedding $X\hookrightarrow Z$ factors through $\bar{Y}\hookrightarrow Z$.

<u>3. THEOREM.</u> If $I,\ J\in{}^{G}I_n,\ w,w'\in W_{IJ}$, the following conditions are equivalent:

(i) $w'\leqslant w$ in the sense of Definition 1;

(ii) $d_{ij,w',IJ}\geqslant d_{ij,w,IJ}\ \forall i,\ j$, where $(d_{ij,w,IJ})$ is the relative position matrix corresponding to the triple (I, J, w) according to the Combinatorial Lemma 2.5;

(iii) $Y_{w',IJ}\subset\overline{Y_{w,IJ}}$,

(iv) $(Y_{w',IJ})_{\mathrm{red}}\subset\overline{(Y_{w,IJ})_{\mathrm{red}}}$.

<u>Remarks.</u> a) The equivalence (i) ⇔ (iv) means, in particular, that the order in the group W_{JJ} coincides with the standard order (see [1, 20]) in the Weyl group W^0 of the component of the identity in G_{red}, if W^0 is identified with W_{JJ} as in Subsec. 1.

b) Recall that, according to the remark at the end of Subsec. 1, if G = OSp(2r, 2s) we replace the set W_{IJ} throughout by one of the sets W_{IJ}^i, i = 1, 2, and W_{JJ} by the group W^0. In order to avoid complicated notation, we adopt the following convention. When dealing with (super)manifolds which are connected if G ≠ OSp(2r, 2s) and disconnected if G = OSp(2r, 2s), we shall always refer in the latter case not to the manifolds themselves but to either one of their components. In that case G_{red} should be replaced by G_{red}^0.

<u>4. Scheme of the Proof:</u>

$$\begin{tikzcd} (i) \arrow[r, Leftrightarrow] & (iv) \arrow[d, Rightarrow] & (iii) \arrow[l, Rightarrow] \\ & (ii) \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] & \end{tikzcd}.$$

5. (i) ⇔ (iv).

LEMMA. a) There exists a unique element $w_0 \in W_{IJ}$ such that $l_0(w_0)=0$.

b) $(Y_{ww_0})_{\mathrm{red}} = {}^{G_{\mathrm{red}}}Y_w$ (under the canonical isomorphism ${}^{G}F_{\mathrm{red}} \simeq {}^{G_{\mathrm{red}}}F$) for all $w \in W_{JJ} = W^0$.

c) The order in the set W_{IJ} coincides with the standard order in the group W (see [1, 20]) under the identification $W^0 \xrightarrow{\sim} W_{IJ}: w \mapsto ww_0$ for all $w \in W^0 = W_{JJ}$.

Proof of the Lemma. Observe that if $G = \mathrm{Q}(n)$, $I = J = (1|1, \ldots, 1|1)$, $w_0 = e$, $W_{JJ} = {}^{G}W$ and the lemma is obvious. We may therefore assume that $G \neq Q(n)$.

a) The Schubert supercells $Y_{w,IJ}$ form a partition of the supermanifold ${}^{G}F_I \times {}^{G}F_J$ and so the set of underlying manifolds $(Y_{w,IJ})_{\mathrm{red}}$ is precisely the set of Schubert cells Y_w of the space ${}^{G_{\mathrm{red}}}F \times {}^{G_{\mathrm{red}}}F$. Among the cells Y_w there is exactly one of dimension $0 + \dim {}^{G_{\mathrm{red}}}F$. But this is just the dimension of the underlying manifold of the corresponding supercell $Y_{w_0,IJ}$, which by Theorem 4.3 is equal to $l_0(w_0) + \dim({}^{G}F_I)_{\mathrm{red}}$, and so $l_0(w_0) = 0$.

c) is a corollary of (b), since under the assumptions of (b) $l_0(ww_0) = \dim(Y_{ww_0,IJ})_{\mathrm{red}} - \dim({}^{G}F_I)_{\mathrm{red}} = \dim {}^{G_{\mathrm{red}}}Y_w - \dim {}^{G_{\mathrm{red}}}F = N(w)$ is the classical length (see [1, 20]) of the element $w \in W^0 = {}^{G_{\mathrm{red}}}W$ relative to the set of basis reflections corresponding to choice of the Borel subgroup $B_J = \mathrm{St}_{G_{\mathrm{red}}}(f_J)$, where f_J is a standard G-flag of type J [see part a) in the proof of Lemma 2.5]. Then the orders in W^0 and W_{IJ} coincide by definition.

b) The idea of the proof is based on the observation that the correspondence $W_{IJ} \to {}^{G_{\mathrm{red}}}W$ under which $w \mapsto w'$, $w \in W_{IJ}$ and w' is uniquely determined by the equality ${}^{G_{\mathrm{red}}}Y_{w'} = ({}^{G}Y_{w,IJ})_{\mathrm{red}}$ may be expressed as $w \mapsto w' = w\tilde{w}_0$. Then, by the uniqueness part of (a), $\tilde{w}_0 = w_0^{-1}$.

Thus, let $w \in W_{IJ}$, $w' \in {}^{G_{\mathrm{red}}}W$ and ${}^{G_{\mathrm{red}}}Y_{w'} = ({}^{G}Y_{w,IJ})_{\mathrm{red}}$. It is clear that w' is determined by the relative position type of the pair of G_{red}-flags $\varphi(f_I)$, f_J in the space $T^{m|n}$ of the standard representation of G, where $(f_I, f_J) \in (Y_{w',IJ})_{\mathrm{red}}$, and $\varphi(f_I)$ is a certain rearrangement of the flag f_I into a flag of type J. This rearrangement is accomplished as follows. Each constituent $\mathcal{F}_i$ of f_I is the direct sum of its even and odd parts: $\mathcal{F}_i = (\mathcal{F}_i)_0 \oplus (\mathcal{F}_i)_1$. We now use induction on i to construct a flag $\varphi(f_I) = \mathcal{F}'_\cdot$ of type J from these components: suppose that $\mathcal{F}'_i$ has already been constructed and $\mathcal{F}'_i = (\mathcal{F}_{i_0})_0 \oplus (\mathcal{F}_{i_1})_1$. Then if $\delta_{i+1}(J) = 1|0$, we put $\mathcal{F}'_{i+1} := (\mathcal{F}_k)_0 \oplus (\mathcal{F}_{i_1})_1$, where k is the least integer such that $k > i_0$, $\delta_k(I) = 1|0$. But if $\delta_{i+1}(J) = 0|1$, then $\mathcal{F}'_{i+1} := (\mathcal{F}_{i_0})_0 \oplus (\mathcal{F}_k)_1$, where k is the least integer such that $k > i_1$, $\delta_k(I) = 0|1$. (If G = OSp or ΠSp we also demand that $i+1, k \leqslant \left[\frac{m+n+1}{2}\right]$, then considering the unique completion of the flag to an isotropic flag.) In order to find the permutation w' corresponding to the relative position matrix of the G_{red}-flags $\varphi(f_I)$ and f_J, we note that for any i the vertical jump in the i-th column of the matrix occurs between rows $w(i_0) - 1$ and $w(i_0)$ if $\delta_i(J) = 1|0$, and between rows $w(i_1) - 1$ and $w(i_2)$ otherwise. By the definition of w' this implies that

$$w'(i) = \begin{cases} w(i_0), & \text{if} \quad \delta_i(J) = 1|0, \\ w(i_1), & \text{if} \quad \delta_i(J) = 0|1. \end{cases}$$

Consequently, the permutation $\tilde{w}_0$ under which i is the image of i_0 if $\delta_i(J) = 1|0$ and of i_1 if $\delta_i(J) = 0|1$, $i = 1, \ldots, m+n$, satisfies the required equality $w' = w\tilde{w}_0$. By construction, $\tilde{w}_0$ depends on I and J but not on w and w'. This completes the proof of the lemma.

The equivalence (i) $\Leftrightarrow$ (iv) now follows easily from a theorem of Steinberg [20], which states (in our notation) that ${}^{G_{\mathrm{red}}}Y_{w'} \subset {}^{G_{\mathrm{red}}}Y_w \Leftrightarrow w' \leqslant w$, w, $w' \in {}^{G_{\mathrm{red}}}W$.

6. $(iv) \Rightarrow (ii)$. The stalk of the coherent sheaf $\mathcal{F}_{I,i} \cap \mathcal{F}_{J,j}$ over a point $y \in (Y_{w,IJ})_{\mathrm{red}} \subset ({}^{G}F_I \times {}^{G}F_J)_{\mathrm{red}}$ is of dimension $d_{ij,w,IJ}$ ($\mathcal{F}_{I,\cdot}$ and $\mathcal{F}_{J,\cdot}$ are the tautological flags on the first and second factors, respectively, of the product ${}^{G}F_I \times {}^{G}F_J$). This dimension function is upper semicontinuous, implying the desired conclusion.

7. (ii) $\Rightarrow$ (i). Consider the set of basis reflections in ${}^{G_{\mathrm{red}}}W$ corresponding to the choice in G_{red} of the Borel group $B_J = \mathrm{St}_{G_{\mathrm{red}}}(f_J)$, where f_J is a standard G-flag of type J [see part (a) in the proof of Lemma 2.5]. Let Σ denote the image of this set under the canonical isomorphism ${}^{G_{\mathrm{red}}}W \xrightarrow{\sim} W_{JJ}$.

LEMMA. For any $\sigma \in \Sigma$, if $d_{ij,w,IJ} \geqslant d_{ij,w',IJ}$ for all i, j then either $d_{ij,\sigma w,IJ} \geqslant d_{ij,w',IJ}$ or $d_{ij,\sigma w,IJ} \geqslant d_{ij,\sigma w',IJ}$ for all i, j.

Proof of the Lemma. It is readily seen that $\sigma \in \Sigma$ if and only if $\sigma = (i, j_i)$ for G = SL, $\sigma = (i\,j_i)\,(m+n+1-i,\ m+n+1-j_i)$ for $G = \mathrm{OSp}$, ΠSp, or $\sigma = (l, l+2)$, $l = r+s$ for $G = \mathrm{OSp}(2r+1, 2s)$,

$\sigma = (l, l+1)$, $l = r+s$ for $G = \mathrm{OSp}(2r, 2s)$, $\sigma = (i, i+1)$ for G = Q, where i is any integer and j_i the least integer such that $j_i > i$, $\delta_{j_i}(J) = \delta_i(J)$, and if $G = \mathrm{OSp}$, $\Pi\mathrm{Sp}$ $i, j_i \leqslant \left[\frac{m+n+1}{2}\right]$. The assertion now follows from properties 2.2 of relative matrices. The details are left to the reader.

We can now prove the implication (ii) ⇒ (i). Suppose that for all i, j $d_{ij,w',IJ} \geqslant d_{ij,w,IJ}$. We shall prove that $w' \leqslant w$, reasoning by downward induction on $l_0(w')$. By Lemma 5(c) and the corresponding classical assertion (see, e.g., [20, Lemma 53]), if $w' = s \in W_{IJ}$ has the largest $l_0(s)$, then $s \geqslant w$, and so $d_{ij,s,IJ} \leqslant d_{ij,w,IJ}$ for all i, j [since we have already proved that (i) ⇒ (ii)]. Therefore, if $d_{ij,s,IJ} \geqslant d_{ij,w,IJ}$ for all i, j, then $d_{ij,s,IJ} = d_{ij,w,IJ}$, and s = w by the Combinatorial Lemma 2.5. Thus the assertion is true if w' = s.

Let $l_0(w') < l_0(s)$. Again by Lemma 5(c) and the corresponding classical assertion (see, e.g., [1, Proposition 2.7]), there exists a reflection $\sigma \in \Sigma$ such that $w' < \sigma w'$. By Lemma 7, either $d_{ij,\sigma w',IJ} \geqslant d_{ij,w,IJ}$ or $d_{ij,\sigma w',IJ} \geqslant d_{ij,\sigma w,IJ}$ for all i, j. In the first case, by the inductive hypothesis, $\sigma w' \leqslant w$, so that $w' \leqslant w$. In the second case, by the inductive hypothesis $\sigma w' \leqslant \sigma w$. It follows from this inequality that either $w' \leqslant w$ or $\sigma w' \leqslant w$ (use the corresponding classical result — Lemma 2.5 in [1]). In any case we have $w' \leqslant w$, as claimed.

8. (ii) ⇒ (iii). Using the already proven equivalence (i) ⇔ (ii), one easily reduces the problem to the case $d_{ij,w,IJ} \geqslant d_{ij,\sigma w,IJ}$, $\sigma \in \Sigma \subset W_{JJ}$, Σ the set of reflections with respect to simple roots of the group G_{red}, defined in Subsec. 7. We shall deal with the case G = SL(m, n), $\sigma = (i_0, j_0)$, $i_0 < j_0$; the remaining cases are similar, only involving more complicated notation.

Consider the two projections ${}^G F_I \times {}^G F_J \xrightarrow{\psi} {}^G F_I \times F' \xrightarrow{\varphi} F_I \times F''$, defined as the identity on the first argument and by the following formulas on the second:

$$\begin{gathered}
\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_{m+n} \\
\updownarrow \\
\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_{i_0-1} \subset \mathcal{S}_{i_0} \subset \mathcal{S}_{j_0} \subset \mathcal{S}_{j_0+1} \subset \ldots \subset \mathcal{S}_{m+n} \\
\updownarrow \\
\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_{i_0-2} \subset \mathcal{S}_{i_0-1} \subset \mathcal{S}_{j_0} \subset \mathcal{S}_{j_0+1} \subset \ldots \subset \mathcal{S}_{m+n},
\end{gathered}$$

where $\mathcal{S}_{\bullet} := \mathcal{S}_{J,\bullet}$ is the tautological flag. Consider the relative position of the intersections of the Schubert supercells $Y_{w,IJ}$ and $Y_{\sigma w,IJ}$ with the fibers of these projections. We first observe that the superscheme images $\varphi\psi(Y_{w,IJ})$ and $\varphi\psi(Y_{\sigma w,IJ})$ are equal, since they are Schubert supervarieties (i.e., closures of Schubert supercells) for the space of incomplete flags F" and the matrices $(d_{ij,w,IJ})$ and $(d_{ij,\sigma w,IJ})$ defining them coincide up to the i_0-th column and after the $(j_0 - 1)$-th column.

The superscheme images $\psi(Y_{w,IJ})$ and $\psi(Y_{\sigma w,IJ})$ are again Schubert supervarieties for relative projective superspace $\mathbf{P}(\delta_{i_0}(J); \mathcal{S}_{i_0}/\mathcal{S}_{i_0-1})$ over $\varphi\psi(Y_{w,IJ}) = \varphi\psi(Y_{\sigma w,IJ})$. [Here $\delta_{i_0}(J) = \mathrm{rk}\,\mathcal{S}_{i_0} - \mathrm{rk}\,\mathcal{S}_{i_0-1}$.] The corresponding supercells are determined by the conditions of the relative position of the sheaves $\mathcal{S}_{i_0}/\mathcal{S}_{i_0-1}$ and $\mathcal{S}_{I,k}/\mathcal{S}_{i_0-1}$, $0 \leqslant k \leqslant m+n$. These conditions are uniquely determined by the matrices $(d_{ij,w,IJ})$ and $(d_{ij,\sigma w,IJ})$. The matrix $(d_{ij,w,IJ})$ gives the condition $\mathcal{S}_{i_0}/\mathcal{S}_{i_0-1} \subset (\mathcal{S}_{I,w^{-1}(i_0)} \cap \mathcal{S}_{j_0})/\mathcal{S}_{i_0-1}$ plus a certain general position condition, while $(d_{ij,\sigma w,IJ})$ gives $\mathcal{S}_{i_0}/\mathcal{S}_{i_0-1} \subset (\mathcal{S}_{I,w^{-1}(j_0)} \cap \mathcal{S}_{j_0})/\mathcal{S}_{i_0-1}$ $d_{ij,w,IJ} \geqslant d_{ij,\sigma w,IJ}$ for all i, j. This implies the inequality $w^{-1}(i_0) < w^{-1}(j_0)$ [cf. the properties of the matrix $(d_{ij,w,IJ})$ in part (d) of the proof of Lemma 2.5 and part (b) of the same lemma]. This inequality shows that $\mathcal{S}_{I,w^{-1}(i_0)} \subset \mathcal{S}_{I,w^{-1}(j_0)}$, whence $\psi(Y_{w,IJ}) \subset \psi(Y_{\sigma w,IJ}) = \overline{\psi(Y_{\sigma w,IJ})}$.

Finally, the Schubert supervarieties $\bar{Y}_w$ and $\bar{Y}_{\sigma w}$ are intersections of the same Schubert supervariety of the relative superspace of complete flags $G_{F_J} \to F'$ with the inverse images $\psi^{-1}\psi(Y_w)$ and $\psi^{-1}\psi(Y_{\sigma w})$. This follows from the fact that the horizontal equality and inequality signs between elements of neighboring columns, beginning with the i_0-th column and ending with the $(j_0 - 1)$-th, are identically placed.

Thus $\bar{Y}_w \subset \bar{Y}_{\sigma w}$ if $w \leqslant \sigma w$.

9. (iii) ⇒ (iv). Trivial. □

10. COROLLARY. Let $I, J \in {}^G I_n$, $w, w' \in W_{IJ}$. Then $l_0(w') \leqslant l_0(w) \Leftrightarrow l(w') \leqslant l(w)$.

Proof. This follows from Theorem 4.3 (dimension of supercells) and the equivalence (iii) ⇔ (iv) in Theorem 3. □

8. Resolution of Singularities of Schubert Supervarities

Singularities of Schubert varieties, i.e., closures of Schubert cells, are in a sense linearizations of fairly general singularities of algebraic varieties. To be precise, the jump in the dimension of the Zariski tangent space $U \to U'$ at a singular point may sometimes (e.g., in the case of an intersection of two nonsingular varieties) be visualized as a change in the position of a certain vector space V, whose intersection with some fixed vector space forms the Zariski tangent space U: $U = V \cap W$, into a less general position: $U' = V' \cap W$, $\dim V' = \dim V$, $\dim U' > \dim U$. This is in fact a typical case of singularity of a Schubert variety lying in the Grassmannian of the subspaces of some enveloping vector space.

An analogous situation obtains in supergeometry as well, and so one expects the investigation of singularities of Schubert supervarieties to reveal a general pattern of supersingularities which, in view of Kac's classification of simple Lie superalgebras [16] and the well-known classification of simple singularities of hyperspaces via Dynkin diagrams, promises to be of some interest.

1. Definition. Let X be a supervariety. A point $x \in X_{\text{red}}$ is said to be nonsingular if there is an open neighborhood $U \subset K$ such that $x \in U$ and U is a nonsingular supervariety. The set of singular points of X is the complement of the set of nonsingular points in X_{red}.

Recall that a Schubert supervariety is the closure $\overline{Y_{w,IJ}}$ (see Definition 7.2) of a Schubert supercell $Y_{w,IJ}$ in the superspace ${}^{G}F_{I} \times {}^{G}F_{J}$, where $I, J \in {}^{G}I_{n}$, w is an element of the Weyl supergroup W of an algebraic supergroup G, $w(I) = J$. Since $Y_{w,IJ}$ are nonsingular, Schubert supervarieties $\overline{Y_{w,IJ}}$ have singularities only at points of $(\overline{Y_{w,IJ}})_{\text{red}} \setminus (Y_{w,IJ})_{\text{red}}$, and they certainly have singularities wherever this is true of the Schubert varieties $(\overline{Y_{w,IJ}})_{\text{red}}$.

2. Bott–Samelson Superschemes. Let $w = s_k \ldots s_1$ be a fixed reduced factorization of an element $w \in W$ as a product of basis reflections, $w_i := s_i \ldots s_1$, $1 \leqslant i \leqslant k$. We define a sequence of projectivizations ${}^{G}F_{I} = Z_0 \leftarrow Z_1 \leftarrow \ldots \leftarrow Z_k$ of vector bundles by an inductive construction, as follows. Suppose that Z_j has been constructed, and let $\mathcal{F}_\cdot$ be a flag on Z_j of type $w_j(I)$ in $T_{Z_j} := T \otimes \mathcal{O}_{Z_j}$, where T is the space of the standard representation of the supergroup G. If $G = SL(m, n)$, then $s_j = (i, i+1)$ for some i, and we define Z_{j+1} to be the relative projective superspace $\mathbf{P}_{Z_j}(\delta_i(w_{j+1}(I)); \mathcal{F}_{i+1}/\mathcal{F}_{i-1})$ of relative dimension $l_{w_j(I), w_{j+1}(I)}(s_j)$. In $T_{Z_{j+1}}$ we consider a complete flag $\mathcal{F}_\cdot$, in which all $\mathcal{F}_p$ $(p \neq i)$ are lifted from Z_j, while $\mathcal{F}_i$ is uniquely defined by the tautological sheaf $\mathcal{F}_i/\mathcal{F}_{i-1} \subset \mathcal{F}_{i+1}/\mathcal{F}_{i-1}$ of the projective superspace $\mathbf{P}_{Z_j}(\delta_i(w_{j+1}(I)); \mathcal{F}_{i+1}/\mathcal{F}_{i-1})$ and the extension $0 \to \mathcal{F}_{i-1} \to \mathcal{F}_{i+1} \to \mathcal{F}_{i+1}/\mathcal{F}_{i-1} \to 0$.

If $G = OSp(m, n)$ or $\Pi Sp(m)$, then if $s_j = (i, i+1)$ $(m+n+1-i,\ m+n-i)$, $i < \left[\frac{m+n}{2}\right]$ we define Z_{j+1} to be the relative projective superspace $\mathbf{P}_{Z_j}(\delta_i(w_{j+1}(I)); \mathcal{F}_{i+1}/\mathcal{F}_{i-1})$ of relative dimension $l_{w_j(I), w_{j+1}(I)}(s_j)$. To form a flag in $T_{Z_{j+1}}$, we lift all constituents $\mathcal{F}_p$ $(p \neq i,\ m+n-i)$ from Z_j; we define $\mathcal{F}_i$, as before, by the tautological sheaf on $\mathbf{P}_{Z_j}(\delta_i(w_{j+1}(I)); \mathcal{F}_{i+1}/\mathcal{F}_{i-1})$, while $\mathcal{F}_{m+n-i}$ is defined by $\mathcal{F}_{m+n-i} := \mathcal{F}_i^{\perp}$ (relative to the bilinear structure form b in T). If $s_j = (l, m+n+1-l)$, $l = \left[\frac{m+n}{2}\right]$, we put $Z_{j+1} := \mathbf{P}_{Z_j}(\delta_l(w_{j+1}(I)); \mathcal{F}_{m+n+1-l}/\mathcal{F}_{l-1}, b)$, where the symbol "b" indicates that we are considering a superspace of isotropic lines relative to the form b. The dimension of Z_{j+1} over Z_j is $l_{w_j(I), w_{j+1}(I)}(s_j)$. The constituents $\mathcal{F}_p$ $(p \leqslant l-1$ or $p \geqslant m+n+1-l)$ are lifted onto Z_{j+1} from Z_j, while $\mathcal{F}_l$ is defined by the tautological sheaf on $\mathbf{P}_{Z_j}(\delta_l \times (w_{j+1}(I)); \mathcal{F}_{m+n+1-l}/\mathcal{F}_{l-1}, b)$. If $m + n$ is even, then $\mathcal{F}_0$ is a complete G-flag in $T_{Z_{j+1}}$; if it is odd, then $\mathcal{F}_\cdot$ is extended to a complete flag by adding $\mathcal{F}_{l+1} := \mathcal{F}_l^{\perp}$.

Finally, if $G = Q(m)$, $s_j = (i, i+1)$ for some i, and we put $Z_{j+1} = \mathrm{Gr}_{Z_j}(1|1;\ \mathcal{F}_{i+1}/\mathcal{F}_{i-1},\ p)$ – the super-Grassmannian of $1|1$-dimensional subspaces of $\mathcal{F}_{i+1}/\mathcal{F}_{i-1}$ symmetric with respect to the structure Π-symmetry p. The dimension of Z_{j+1} over Z_j is $1|1 = l_{w_j(I), w_{j+1}(I)}(s_j)$. The flag $\mathcal{F}_\cdot$ in $T_{Z_{j+1}}$ is constructed as in the case $G = SL(m, n)$.

By construction, all the Z_j are nonsingular.

Definition. $Z_w := Z_k$ is called a Bott–Samelson superscheme (it clearly depends on the choice of a reduced factorization of w).

3. Bott–Samelson Morphism. Define a morphism $\psi: Z_w \to {}^{G}F_I \times {}^{G}F_J$ as the canonical morphism under which $\psi^*(\mathcal{F}_{J,\cdot}) = \theta^*(\mathcal{F}_{I,\cdot})$, where $\theta: Z_w = Z_k \to Z_0 = {}^{G}F_I$ is the natural projection, and

$\psi^*(\mathcal{F}_{J,\cdot})=\mathcal{F}_{\cdot}$, $\mathcal{F}_{\cdot}$ is the flag constructed above of type J = w(I) on Z_w. Clearly, the diagram

$$\begin{array}{ccc} Z_w & \xrightarrow{\psi} & {}^GF_I \times {}^GF_J \\ & \searrow_{\theta} & \downarrow p_1 \\ & & {}^GF_I \end{array}$$

is commutative, i.e., ψ is a morphism of superschemes over ${}^GF_I$.

Definition. ψ is known as the Bott–Samelson morphism.

4. A morphism of algebraic supervarieties $f:X \to Y$ is said to be surjective if the corresponding morphism of algebraic varieties $f_{red}:X_{red} \to Y_{red}$ is surjective and the superscheme image f(X) is Y.

THEOREM. The Bott–Samelson morphism $\psi:Z_w \to \overline{{}^GF_I \times {}^GF_J}$ is a resolution of the singularities of the Schubert supervariety $\overline{Y_{w,IJ}}$, i.e., $\psi:Z_w \to \overline{Y_{w,IJ}}$ is defined, surjective and induces an isomorphism of an open supervariety in Z_w onto $Y_{w,IJ}$.

Proof. Step 1. We construct an open subsupervariety U_w in Z_w which is mapped isomorphically into $Y_{w,IJ}$ by ψ. We shall proceed as in Subsec. 2, i.e., constructing a sequence of open subsupervarieties $U_j \subset Z_j$, j = 0,...,k, such that $\theta_j|_{U_{j+1}}:U_{j+1} \to U_j$, where θ_j is the natural projection $Z_{j+1} \to Z_j$. Let ψ_j denote the natural morphism $Z_j \to {}^GF_I \times {}^GF_{w_j(I)}$ constructed in the same way as ψ in Subsec. 3. Put $U_0:=Z_0={}^GF_I$. Clearly, ψ_0 defines an isomorphism of U_0 onto the diagonal of ${}^GF_I \times {}^GF_I$, i.e., onto $Y_{e,II}$.

Suppose now that U_j has been constructed and let $\psi_j|_{U_j}:U_j \to Y_{w_j,I,w_j(I)}$ be an isomorphism. Define U_{j+1} to be a big supercell in the relative projective superspace (if G = Q – super-Grassmannian) $Z'_{j+1}:=\theta_j^{-1}(U_j) \to U_j$ (we are assuming that the partition into supercells is determined by a flag of type I lifted from ${}^GF_I$ relative to the morphism $Z'_{j+1} \to U_j \to U_0={}^GF_I$). Since $\psi_j|_{U_j}$ maps U_j isomorphically onto $Y_{w_j,I,w_j(I)}$ and the Schubert supercell $Y_{w_{j+1},I,w_{j+1}(I)}$ is a big supercell in a relative projective superspace over $Y_{w_j,I,w_j(I)}$ (see 4.8), and moreover ψ_{j+1} is induced by a morphism of tautological sheaves, it follows that ψ_{j+1} is an isomorphism.

Putting $U_w:=U_k$, one proves by induction that $\psi|_{U_w}=\psi_k|_{U_k}: U_w \to Y_{w,IJ}$ is an isomorphism.

Step 2. It follows from Step 1 that ψ defines a morphism $\overline{U_w} \to \overline{Y_{w,IJ}}$, and moreover $\psi(\overline{U_w})=\overline{Y_{w,IJ}}$. By construction, $Z_w = \overline{U}_w$, and all that remains is to verify that $\psi_{red}:(Z_w)_{red} \to (\overline{Y_{w,IJ}})_{red}$ is a surjective morphism. Indeed, the variety $(Z_w)_{red}$ is complete, and so the geometric (not scheme-theoretic!) image $\psi_{red}((Z_w)_{red})$ is closed in $(\overline{Y_{w,IJ}})_{red}$. On the other hand, this image contains $(Y_{w,IJ})_{red}$, implying the desired assertion. □

5. Remark. The important results obtained using Bott–Samelson schemes (see Demazure [14]) – rationality of the singularities of Schubert varieites, the Demazure character formula – depend essentially on the theorem stating that the cohomology of inverse sheaves on Schubert varieties is trivial. For Schubert supervarieties of codimension 0|0 this theorem is a particular case of a superversion of the Borel–Weil–Bott theorem (see [11, 19]). It is therefore reasonable to suppose that, in combination with the construction of Bott–Samelson superschemes, this should make it possible to prove correct superversions of Demazure's results.

6. Example. Consider the Grassmannian X = Gr(2|0; T) of 2|0-dimensional planes in a 3|1-dimensional vector space T. Obviously, $\dim X=2|2$, $X_{red}\simeq \mathbf{P}^2$. Let $p: X \hookrightarrow \mathbf{P}(1|0;\ \Lambda^2_C(T))\simeq \mathbf{P}^{3|3}$ be the Plücker embedding. For any complex superalgebra $A=A_0\oplus A_1$, the image p(X(A)) consists of lines in $\Lambda^2_A(T)$ generated by even decomposable bivectors $Q\in\Lambda^2_A(T)$. Let $\{e_1, e_2, e_3, f\}$ be a homogeneous basis in T. We wish to find conditions in terms of the coefficients of the representation of Q in the basis

$$\{e_i\wedge e_j,\ e_i\wedge f,\ f\wedge f\}, \tag{1}$$

i, j = 1, 2, 3, i < j, of the space $\Lambda^2_A(T)$, under which the bivector $Q=Q_1+Q_2\wedge f+\lambda f\wedge f$, where $Q_1:=\lambda_{12}\, e_1\wedge e_2+\lambda_{13}\, e_1\wedge e_3+\lambda_{23}\, e_2\wedge e_3, Q_2:=\lambda_1 e_1+\lambda_2 e_2+\lambda_3 e_3,\ \lambda_{ij}\in A_0,\ \lambda_i\in A_1,\ i,j=1,2,3$, is decomposable. Decomposability of Q means that $Q=(R+af)\wedge(S+bf)=R\wedge S+(bR-aS)\wedge f-abf\wedge f$, where R and S lie in the span of e_1, e_2, e_3 and a, b $\in A_1$. Any bivector Q_1 independent of f is obviously decomposable, i.e., the equality $Q_1 = R \wedge S$ imposes no restrictions on Q_1. Assume, then, that is the case. The condition (Q_2 = bR – aS for some a and b) is equivalent to the condition $Q_1 \wedge Q_2 = 0$. Finally, the condition $\lambda = -ab$ is equivalent to $Q_1\wedge Q_2+2\lambda Q_1=0$. We finally obtain equations for the image p(X) in homogeneous coordinates $(\lambda_{12}:\lambda_{13}:\lambda_{23}:\lambda|\lambda_1:\lambda_2:\lambda_3)$ in $\mathbf{P}^{3|3}$:

$$\begin{cases}\lambda_1\lambda_{23}-\lambda_2\lambda_{13}+\lambda_3\lambda_{12}=0,\\ \lambda_1\lambda_2+\lambda\lambda_{12}=0,\\ \lambda_1\lambda_3+\lambda\lambda_{13}=0,\\ \lambda_2\lambda_3+\lambda\lambda_{23}=0.\end{cases}$$

Consider the Schubert supercell $W=\{\Lambda^{2|0}\subset T^{3|1}\,|\,\dim(\Lambda\cap V)=1|0\}\subset X$, where $V^{2|0}$ is the space spanned by vectors e_1, e_3. The corresponding Schubert supervariety is defined in $\mathbf{P}^{3|3}$ by the equations

$$\begin{cases}\lambda_1\lambda_{23}+\lambda_3\lambda_{12}=0,\\ \lambda_1\lambda_3=0,\\ \lambda_2=0,\\ \lambda=0.\end{cases}\tag{2}$$

[Indeed, it is not hard to verify that if $(\lambda_{12})_{\rm red}\in A^*_{\rm red}$ the condition $\dim(\Lambda\cap V)\geqslant 1|0$ is equivalent to $e_1\wedge e_3\wedge Q=0$, where Q is a bivector corresponding to the plane Λ. This last equation is equivalent to the system $\lambda_2=\lambda=0$. On the other hand, $W\subset\{\Lambda^{2|0}\subset T^{3|1}\,|\,\dim(\Lambda\cap V)\geqslant 1|0\}\subset\overline{W}$ and the open analytic set $\{Q\in\Lambda^2(T)\,|$ the coefficient λ_{12} in the representation of Q in the basis (1) satisfies the condition $(\lambda_{12})_{\rm red}\neq 0\}$ is dense in the affine superspace $\Lambda^2(T)$. Therefore $\overline{W}$ is defined in $X\overset{p}{\hookrightarrow}\mathbf{P}^{3|3}$ by the equations $\lambda_2=\lambda=0$.]

All points of the supervariety $\overline{W}_{\rm red}$, except the point s with homogeneous coordinates $\lambda_{13}=1$, $\lambda_{12}=\lambda_{23}=\lambda_1=\lambda_2=\lambda_3=\lambda=0$, are nonsingular. In the corresponding inhomogeneous coordinate system $(\tilde\lambda_{12}, \tilde\lambda_{23}, \tilde\lambda, \tilde\lambda_1, \tilde\lambda_2, \tilde\lambda_3)$ in the neighborhood of s, the equation of $\overline{W}$ can be written

$$\begin{cases}\tilde\lambda_1\tilde\lambda_{23}+\tilde\lambda_3\tilde\lambda_{12}=0,\\ \tilde\lambda_1\tilde\lambda_3=0,\\ \tilde\lambda_2=0,\\ \tilde\lambda=0.\end{cases}$$

Clearly, $\dim\overline{W}=2|1$.

7. Bott–Samelson Resolution of the Singular Point $s\in\overline{W}$. For the Schubert supervariety $\overline{W}$, a natural choice of the Bott–Samelson superscheme is the relative projective superspace $Z=\mathbf{P}_M(1|0;\ \mathcal{S}_3/\mathcal{S}_1)$, where $M=\mathbf{P}_{\mathbf{C}}(1|0;\ V^{2|0})\times\mathbf{P}_{\mathbf{C}}(1|0;\ T/V)$, $\mathcal{S}_1$ is the tautological sheaf on the first factor, $\mathcal{S}_3$ is a sheaf on the second factor such that $V\subset\mathcal{S}_3$ and $\mathcal{S}_3/V$ is the tautological sheaf. The Bott–Samelson morphism $\psi:\mathbf{P}_M(1|0;\ \mathcal{S}_3/\mathcal{S}_1)\to\mathrm{Gr}\,(2|0;\ T)$ is a canonical morphism such that the inverse image of the tautological sheaf on Gr (2|0; T) is a sheaf $\mathcal{S}_2:\mathcal{S}_1\subset\mathcal{S}_2\subset\mathcal{S}_3$, $\mathcal{S}_2/\mathcal{S}_1$ the tautological sheaf on $\mathbf{P}_M(1|0;\ \mathcal{S}_3/\mathcal{S}_1)$. It is clear that the supercell W is dense in the (superscheme) image $\mathrm{Im}\,\psi$ and thus $\mathrm{Im}\,\psi=\overline{W}$.

We now describe the Bott–Samelson morphism in terms of coordinates. Let $U\subset\mathbf{P}^{3|3}$ be a neighborhood of the singular point s. We shall find local equations for the Bott–Samelson superscheme $Z\subset U\times\mathbf{P}^{1|0}(1|0;V)\times\mathbf{P}^{0|1}(1|0;T/V)$ in coordinates $(l_{12}, l_{23}, l, l_1, l_2, l_3)$, $(a_1:a_2)$ and $(b|\beta)$ on the first, second and third factors, respectively (the coordinates on the first two are inhomogeneous, those on the third homogeneous). Besides the equations for the Grassmannian $X=\mathrm{Gr}\,(2|0;T)$ in U, there appear incidence relations: $\mathcal{S}_1\subset\mathcal{S}_2\subset\mathcal{S}_3$. More explicitly: if $\mathcal{S}_1=\langle a_1e_1+a_2e_3\rangle$, $\mathcal{S}_3=\langle bf^*+\beta e_2^*\rangle^{\perp}\subset T$, where $\{e_1^*, e_2^*, e_3^*, f^*\}$ is the basis dual to $\{e_1, e_2, e_3, f\}$ in T^*, and the bivector corresponding to $\mathcal{S}_2$ is $Q=l_{12}e_1\wedge e_2+e_1\wedge e_3+l_{23}e_2\wedge e_3+l_1e_1\wedge f+l_2e_2\wedge f+l_3e_3\wedge f+lf\wedge f\in\Lambda^2(T)$, then $\mathcal{S}_1\subset\mathcal{S}_2$ is equivalent to $(a_1e_1+a_2e_3)\wedge Q=0$, and $\mathcal{S}_2\subset\mathcal{S}_3$ to $i\,(bf^*+\beta e_2^*)\,Q=0$, where $i(\cdot)\cdot$ is the inner product. In the final analysis, the last two equations can be written as a system:

$$\begin{cases}a_1l_{23}+a_2l_{12}=0,\\ a_1l_3-a_2l_1=0,\\ a_1l_2=a_2l_2=0.\\ a_1l=a_2l=0,\end{cases}\qquad\begin{cases}bl_1-\beta l_{12}=0,\\ bl_3+\beta l_{23}=0,\\ 2bl+\beta l_2=0,\\ bl_2=0.\end{cases}$$

Thus the equations of Z in $U\times\mathbf{P}^{1|0}\times\mathbf{P}^{0|1}$ are

$$\begin{cases}a_1l_{23}+a_2l_{12}=0,\\ bl_1-\beta l_{12}=0,\\ bl_3+\beta l_{23}=0,\\ l=l_2=0.\end{cases}$$

It is explicitly clear from these equations that the superscheme Z is nonsingular in U × $\mathbf{P}^{1|0}\times\mathbf{P}^{0|1}$.

The Bott–Samelson morphism is expressed in coordinate form as follows: $(\tilde{\lambda}_{12}, \tilde{\lambda}_{23}, \tilde{\lambda}, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3)=\psi(l_{12}, l_{23}, l, l_1, l_2, l_3; a_1{:}a_2;\ b|\beta):=(l_{12}, l_{23}, l, l_1, l_2, l_3)$. The inverse image $E:=\psi^{-1}(s)$ is defined by the equations $l_{12}=l_{23}=l=l_1=l_2=l_3=0$, and so it is isomorphic to $\mathbf{P}^{1|0}\times\mathbf{P}^{0|1}$. Thus, we can say that in our example the Bott–Samelson morphism has glued the supervariety $\mathbf{P}^{1|0}\times\mathbf{P}^{0|1}$ to the singular point.

8. Inflation. Let M be a nonsingular complex supervariety of dimension m|n, $m\geqslant 1$, $s\in M_{\mathrm{red}}$ a point of its underlying variety, $z=(z_1,\ldots,z_m,\ \zeta_1,\ldots,\zeta_n)$ holomorphic coordinates in a neighborhood $U\subset M$ of s. An inflation of M at s is defined to be a complex supervariety $\tilde{M}$ obtained by gluing to $M\setminus\{s\}$ ($M\setminus\{s\}$ is an open subvariety in M with underlying variety $M_{\mathrm{red}}\setminus\{s\}$) the supervariety

$$\tilde{U}=\{(z, l)\in U\times\mathbf{P}^{m-1|n}\,|\,z\in l\}$$

by means of the identification $\tilde{U}\setminus\{(z, l)\,|\,z=0\}\simeq U\setminus\{s\}$, under which $(z, l)\mapsto z$. In this situation $\mathbf{P}^{m-1|n}$ is considered as the superspace of lines in $\mathbf{C}^{m|n}$ with coordinates $(z_1,\ldots,z_m,\ \zeta_1,\ldots,\zeta_n)$. The map $(z, l)\mapsto z$ extends to the natural projection $\pi:\tilde{M}\to M$, which is an isomorphism over $M\setminus\{s\}$. The inverse image $E:=\pi^{-1}(s)$ is isomorphic to $\mathbf{P}^{m-1|n}$ and is called the exceptional divisor of the inflation.

In terms of coordinates $(z_1,\ldots,z_m,\ \zeta_1,\ldots,\zeta_n)$ in U and suitable homogeneous coordinates $(l_1:\ldots:l_m|\lambda_1:\ldots:\lambda_n)$ in $\mathbf{P}^{m-1|n}$, the subvariety $\tilde{U}$ is defined by the equations

$$\begin{cases} z_i l_j=z_j l_i,\\ z_i\lambda_j=\zeta_j l_i,\\ \zeta_i\lambda_j=-\zeta_j\lambda_i \end{cases}$$

for all possible i, j.

As in classical geometry, inflation is a way of resolving singularities of supervarieties. This can be illustrated in the case of our previous example.

9. Resolution of the Singular Point $s\in\overline{W}$ Using Inflation. The supervariety $\overline{W}\subset\mathbf{P}^{3|3}$ is defined by Eqs. (2) in homogeneous coordinates $(\lambda_{12}{:}\lambda_{13}{:}\lambda_{23}{:}\lambda\,|\,\lambda_1{:}\lambda_2{:}\lambda_3)$, so we may assume that $\overline{W}$ is embedded in $\mathbf{P}^{2|2}=\{\lambda=0,\ \lambda_2=0\}$ and defined in terms of the corresponding inhomogeneous coordinates $(\lambda_{12}, \lambda_{23}, \lambda_1, \lambda_3)$ in a neighborhood $U\subset\mathbf{P}^{2|2}$ of the singular point s = (0, 0, 0, 0) by the equations

$$\begin{cases} \lambda_1\lambda_{23}+\lambda_3\lambda_{12}=0,\\ \lambda_1\lambda_3=0. \end{cases} \tag{3}$$

We now construct an inflation $\tilde{U}$ of the supervariety $U\subset\mathbf{P}^{2|2}$ at s. In $U\times\mathbf{P}^{1|2}(a_1{:}a_2\,|\,\alpha_1{:}\alpha_2)$ this inflation $\tilde{U}$ will be defined by the equations

$$\begin{cases} \lambda_{12}a_2=\lambda_{23}a_1,\ \ \lambda_{12}\alpha_1=\lambda_1 a_1,\\ \lambda_{12}\alpha_2=\lambda_3 a_1,\ \ \lambda_{23}\alpha_1=\lambda_1 a_2,\\ \lambda_{23}\alpha_2=\lambda_3 a_2,\ \ \lambda_1\alpha_2=-\lambda_3\alpha_1,\ \ \lambda_1\alpha_1=0,\ \ \lambda_3\alpha_2=0. \end{cases} \tag{4}$$

We now determine the inverse image $\pi^{-1}(\overline{W})$ under the natural projection $\pi:\tilde{U}\to U$. Consider a cover of the superspace $\mathbf{P}^{1|2}$ by charts $a_1\neq 0$ and $a_2\neq 0$. In the first chart $a_1\neq 0$ and we may put $a_1=1$. Substituting Eqs. (4) into (3), we obtain $\lambda_{12}^2(\alpha_1 a_2+\alpha_2)=0$, $\lambda_{12}^2\alpha_1\alpha_2=0$, $\lambda_{23}=\lambda_{12}a_2$, $\lambda_1=\lambda_{12}\alpha_1$, $\lambda_3=\lambda_{12}\alpha_2$ (dependent equations are omitted) – the equations of $\pi^{-1}(\overline{W})$. In the second chart, putting $a_2=1$, we obtain the following equations for $\pi^{-1}(\overline{W})$: $\lambda_{23}^2(\alpha_1+\alpha_2 a_1)=0$, $\lambda_{23}^2\alpha_1\alpha_2=0$, $\lambda_{12}=\lambda_{23}a_1$, $\lambda_1=\lambda_{23}\alpha_1$, $\lambda_3=\lambda_{23}\alpha_2$. In this situation we shall say that the inverse image $\pi^{-1}(\overline{W})$ splits into the union of the exceptional divisor $E=\{\lambda_{12}=\lambda_{23}=\lambda_1=\lambda_3=0\}\simeq\mathbf{P}^{1|2}$ (taken with multiplicity 2) and the proper inverse image of the supervariety $\overline{W}$, defined in the first and second charts on $U\times\mathbf{P}^{1|2}$, respectively, by the equations

$$\begin{cases} \alpha_1 a_2+\alpha_2=0,\\ \lambda_{23}=\lambda_{12}a_2,\\ \lambda_1=\lambda_{12}\alpha_1,\\ \lambda_3=\lambda_{12}\alpha_2 \end{cases} \quad\text{and}\quad \begin{cases} \alpha_1+\alpha_2 a_1=0,\\ \lambda_{12}=\lambda_{23}a_1,\\ \lambda_1=\lambda_{23}\alpha_1,\\ \lambda_3=\lambda_{23}\alpha_2. \end{cases}$$

These equations clearly define a nonsingular subvariety of $U\times\mathbf{P}^{1|2}$.

LITERATURE CITED

1. I. N. Bernshtein, I. M. Gel'fand, and S. I. Gel'fand, "Schubert cells and cohomology of the spaces G/P," Uspekhi Mat. Nauk, 28, No. 3, 3-26 (1973).
2. A. A. Voronov, "Relative position of Schubert supervarieties and resolution of their singularities," Funkts. Anal. Prilozhen., 21, No. 1, 72-73 (1987).
3. A. A. Voronov and Yu. I. Manin, "Schubert supercells," Funkts. Anal. Prilozhen., 18, No. 4, 75-76 (1984).
4. F. F. Voronov and A. V. Zorich, "Complex form on a supermanifold," Funkts. Anal. Prilozhen., 20, No. 2, 58-59 (1986).
5. F. F. Voronov and A. V. Zorich, "Bordism theory and homotopic properties of supermanifolds," Funkts. Anal. Prilozhen., 21, No. 3, 77-78 (1987).
6. D. A. Leites, Theory of Supermanifolds [in Russian], KF Akad. Nauk SSSR, Petrozavodsk (1983).
7. Yu. I. Manin, "Grassmannians and flags in supergeometry," in: Some Questions of Modern Analysis [in Russian], Izd. Moskov. Gos. Univ., Moscow (1984), pp. 83-101.
8. Yu. I. Manin, Gauge Fields and Complex Geometry [in Russian], Nauka, Moscow (1984).
9. Yu. I. Manin, "New dimensions in geometry," Uspekhi Mat. Nauk, 39, No. 6, 47-73 (1984).
10. I. B. Penkov, "Characters of typical irreducible finite-dimensional q(n)-modules," Funkts. Anal. Prilozhen., 20, No. 1, 37-45 (1986).
11. I. B. Penkov, "Borel–Weil–Bott theory for classical Lie supergroups," this volume.
12. N. Bourbaki, Groupes et algèbres de Lie, Chaps. IV-VI, Paris, Hermann (1968).
13. N. Bourbaki, Groupes et algèbres de Lie, Chaps. VII-VIII, Paris, Herman (1974).
14. M. Demazure, "Desingularisation des variétés de Schubert généralisées," Ann. Sci. Ecole Norm. Sup., Sér. 4, 7, No. 1, 53-88 (1974).
15. J. E. Humphreys, Linear Algebraic Groups, Springer, Berlin (1975).
16. V. G. Kac, "Lie superalgebras," Adv. Math., 26, No. 1, 8-96 (1977).
17. V. G. Kac, "Representations of classical Lie superalgebras," Lect. Notes Math., 676, 597-626 (1978).
18. D. Mumford, Lectures on curves on an algebraic surface, Princeton University Press, Princeton, N.J. (1966).
19. I. V. Penkov and I. A. Skornyakov, "Cohomologie des $\mathcal{D}$-modules tordus typiques sur les supervariétés de drapeaux," C. R. Acad. Sci., Sér. 1, 299, No. 20, 1005-1008 (1985).
20. R. Steinberg, Lectures on Chevalley Groups, Yale University Press, New Haven, Conn. (1967).

BOREL–WEIL–BOTT THEORY FOR CLASSICAL LIE SUPERALGEBRAS

I. B. Penkov

UDC 512.743

The paper is devoted to a systematic construction of the elements of Borel–Weil–Bott theory in the supercase. The main result is a presentation of the cohomology of typical irreducible G^0-sheaves on G^0/B, where G^0 is the connected component of the identity in a classical complex Lie supergroup and $B \hookrightarrow G^0$ an arbitrary Borel subsupergroup. Also presented are some simple known results concerning the cohomology of irreducible G^0-sheaves on G^0/P for a parabolic subsupergroup P.

The present paper is a survey of fairly general results known to the author relating to the cohomology of irreducible $\mathfrak{g}$-sheaves on supermanifolds G^0/B, where G^0 is the component of the identity in a classical complex Lie supergroup G, $\mathfrak{g}=\mathrm{Lie}\,G$ and $B \hookrightarrow G^0$ is a Borel subsupergroup. In the case of a complex reductive Lie group G', the irreducible $\mathfrak{g}'=\mathrm{Lie}\,G'$-sheaves on $(G'')^0/B'$ are simply invertible, and their cohomology theory is described by the now classical Borel–Weil–Bott theorem, or briefly Bott's theorem [12-14]. With details omitted, it may be stated as follows. The cohomology groups of the $\mathfrak{g}'$-sheaf $\mathcal{O}_{(G')^0/B}(\lambda)$ determined by a weight λ are all trivial, with the possible exception of one, which is an irreducible

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennye Problemy Matematiki, Noveishie Dostizheniya, Vol. 32, pp. 71-124, 1988.

0090-4104/90/5101-2108$12.50