Expressions (4) permit us to write in effective form the two-zone solution of an algebraically integrable nonlinear wave equation.

One should note that the solution of the problem of effectivization of formulas of finite-zone integration is connected with deep problems of algebraic geometry. In the approach of [2] the solution of this problem, to which Novikov gave consideration, turned out to be equivalent to the solution of the Schottky problem on description of Jacobian \( \tau \)-matrices [7-9]. Riemann, Frobenius, Netter, and others were concerned with generalization of Eq. (3) to the case of nonhyperelliptic modules; in connection with this Weil formulated a problem which was recently solved by Igusa [10, 11]. A detailed account of the results mentioned here is given in [12].

LITERATURE CITED


RELATIVE DISPOSITION OF THE SCHUBERT SUPERVARIETIES
AND RESOLUTION OF THEIR SINGULARITIES

A. A. Voronov

Let \( G \) be a complex algebraic supergroup of one of the following types: \( \text{SL}(m|n) \), \( \text{OSp}(m|n) \), \( \text{HS}(m) \) and \( \text{Q}(m) \), given together with its standard representation \( T^m \otimes T^* \). The group \( \text{OSp}(m|n) \) is the automorphism group of a nondegenerate even symmetric bilinear form \( b: T \rightarrow T^* \), \( \text{HS}(m) \) is that for an odd antisymmetric form \( p: T \rightarrow T^* \). We call a \( G \)-invariant flag

\[
0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_r = T,
\]

where \( r = m + n \) for all \( G \), except \( \text{Q}(m) \) when \( r = m \), a complete \( G \)-flag.

In [1] a decomposition of the superspace of the complete \( G \)-flags into the Schubert supercells \( X_w \), which are locally closed subsuperschemes and are numbered by the elements of the Weyl supergroup \( W \), introduced there, is constructed. The relation \( X_w \subset X_w' \), where \( X_w \) is the superscheme closure of \( X_w \), gives an order in \( W \). In the present note, we define this order in an intrinsic manner.

The Schubert supervarieties \( X_w \) are interesting also in that they provide natural examples of supervarieties with singularities. In the classical geometry, results such as the proof of the rationality of singularities of the Schubert varieties, computation of the Chow ring of a flag space, the Demazure character formula can be obtained, relying on the construction of the Bott-Samelson scheme, giving a resolution of the singularities of a Schubert variety (see Bernshtein-Gel'fand-Gel'fand [2] and [3]). Here we propose a construction of a resolution of singularities of \( X_w \), following basically the Bott-Samelson plan. In addition, our construction is essentially different from the classical one, since not all Borel...
subgroups of G are conjugate, the group W does not have a system of representatives in G, and the root system of the supergroup G is, in general, not an abstract root system.

1. Flag Supervarieties. The sequence \( I = (\delta_1, \ldots, \delta_r) \), where \( \delta_i(I) = \dim \mathcal{F}_i = \dim \mathcal{F}_i - 1 \), is called the type of the complete flag \( 0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_r \). The supervariety of complete G-flags is, by definition, the disconnected union \( GF = \coprod GF(I) \) of the connected (for \( G = OSp(2r, 2s) \)) spaces \( GF(I) \) of complete G-flags of type I (in the case \( G = OSp(2r, 2s) \) each \( GF(I) \) consists of two connected components). The supervarieties \( GF(I) \) are orbits (in the supersense) of the supergroup G, and the decomposition of the flag space \( GF \) into connected components corresponds to the decomposition of the set of Boral subgroups of G into conjugacy classes.

2. Weyl Supergrupps. Let \( GI \) be the set of all types of complete G-flags in \( T \). Let us set \( SLW = S_{\mathfrak{sl}n} \) with the action on \( SLI \) by permutations, \( GW = \{ w \in SLW \mid w(GI) \subset GI \} \) for \( G = OSp \) or \( HSp \), and \( QW = Sm \).

3. Superlength. For \( G = SL \) we define the basic reflections in the group \( GW \) as \( \sigma_i = (i, i+1) \); see [1] for remaining G. We define the superlength \( l_{IJ}(w) = (\ell_0(w), \ell_1(w)) \in \mathbb{Z} \times \mathbb{Z} \) of an element \( w \in GW \) for \( I, J \in GI \), and \( w(I) = J \) as \( \sum \ell_{i+1} \ell_i(w_i) \), where \( w = w_s \ldots w_1 \) is the reduced decomposition of \( w \) in a product of basic reflections, \( I_0 = I \), and \( I_i = w_i(I_{i-1}) \). In addition, in the case \( G = SL \) we have \( l_{IJ}(\sigma_i) = 1 \) if \( J = \sigma_i(I) = I \), and \( 0 \) if \( J = \sigma_i(I) \neq I \) (for other G, \( l_{IJ}(\sigma_i) \) are equal to \( 0, 1 \), or \( 1 \); see [1] for the precise definitions).

4. Schubert Superells. Let \( (\mathcal{F}_i) \) be a tautological flag on \( GF \times GF \). The elements \( Y_{w,IJ} \) of the flattening decomposition [4] of the space \( GF(I) \times GF(J) \) for the system of sheaves \( \{ \mathcal{F}_{ij} \} \) are given uniquely by the matrix \( (d_{ij,w}) \). Let us define the Schubert supercells in \( GF(J) \) as \( X_{w,IJ} = Y_{w,IJ} \) for an arbitrary point \( x \in (GF(I))^{rd} \).

5. Order in a Weyl Superroup. We call \( w \in W \) a reflection if \( w \) is conjugate to a basic reflection in \( W \).

Definition (\( G = OSp(2r, 2s) \)). (i) Let \( I, J \in GI \), \( w_1, w_2 \in W_{IJ} \), and \( \sigma \in W_{JJ} \) be a reflection. Then \( w_1 \rightarrow w_2 \) means that \( \sigma w_1 = w_2 \) and \( \ell_\sigma(w_2) = \ell_\sigma(w_1) + 1 \).

(ii) Let us set \( w < w' \) if there exists a chain \( w = w_1 \rightarrow w_2 \rightarrow \ldots \rightarrow w_k = w' \).

Remark. Here and in the sequel, for \( G = OSp(2r, 2s) \) in place of \( W_{IJ} \) it is necessary to consider each of the two orbits in \( W_{IJ} \) of a certain subgroup of \( W_{JJ} \) of index 2. This subgroup is the image of the Weyl group \( W^0 \) of the connected component of the identity of the group \( Grd \) under the natural embedding in \( W_{JJ} \). We point out that this subgroup coincides with \( W_{JJ} \) for the remaining \( G \).

6. Theorem. The following conditions are equivalent for \( I, J \in GI \) and \( w, w' \in W_{IJ} \): a) \( w' \leq w \). b) \( d_{ij,w} \geq d_{ij,w'} \) for all i and j. c) \( X_{w',IJ} \subset \overline{X_{w,IJ}} \). d) \( (X_{w',IJ})^{rd} \subset (X_{w,IJ})^{rd} \).

Remark. The equivalence a) \( \Rightarrow \) d) shows that the order in \( W_{JJ} \) coincides with the standard order [2] in the Weyl group \( W^0 \) with respect to the embedding \( W^0 \rightarrow W_{JJ} \) (see Sec. 5). The equivalence c) \( \Rightarrow \) d) means that the decomposition into Schubert supercells is, in a natural sense, a supercellular complex.

7. Definition. Let \( X \) be a supervariety. A point \( x \in X_{rd} \) is said to be nonsingular if there exists a neighborhood \( U \subset X \) such that \( x \in U \) and \( U \) is a nonsingular supervariety. The set of singular points of the supervariety \( X \) is conjugate to the set of nonsingular points in \( X_{rd} \).

Since \( X_{w,IJ} \) are nonsingular, the Schubert supervarieties \( \overline{X_{w,IJ}} \) have singularities only at the points of \( (X_{w,IJ})^{rd} \setminus (X_{w,IJ})^{rd} \), and, a fortiori, have them there where the Schubert varieties \( (X_{w,IJ})^{rd} \) are singular.
8. The Bott-Samelson Superscheme. Let $I, J \subseteq G_I$, $w \subseteq W_{IJ}$, $w = s_k \ldots s_1$ be a reduced decomposition of $w$ into basic reflections, and $w_i = s_i \ldots s_1$, $1 \leq i \leq k$. Let us define the sequence of projective superfibers $\text{Spec} C = Z_0 \rightarrow Z_1 \rightarrow \ldots \rightarrow Z_k$ by the following inductive rule. (To economize space, we give the construction only for $G = SL$). The point $Z_0$ is the point $(\ell_j) = (I_{r_0}, I_0 = \cdots \subseteq \mathcal{P}_{r_0})$. Let $Z_j$ be already constructed; for each superscheme $S$ over $C$ each $S$-point $Z_j$ is a flag of type $I_j = w_j(1)$ in $T \otimes E$. Let $s_j = (i, i + 1)$. Then as $Z_{j+1}$ we take the relative projective superspace $P_z(\mathcal{F}_{j+1}/\mathcal{F}_j)$ of relative dimension $\ell_{I_j,I_{j+1}}(s_j)$, where $(\mathcal{F}_I)$ is tautological flag on $Z_j$. By construction, all $Z_j$ are nonsingular.

Definition. $Z_w = Z_k$ is called a Bott-Samelson superscheme (it clearly depends on the choice of the reduced decomposition of $w$).

9. Theorem. There exists a morphism $Z_w \rightarrow GF(J)$ that is a resolution of the singularities of $X_{w,IJ} \subseteq GF(J)$, i.e., a morphism $Z_w \rightarrow X_{w,IJ}$ is defined, is surjective, and induces an isomorphism of the open subsupervariety in $Z_w$ with $X_{w,IJ}$.

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LITERATURE CITED


SUPERAMENABILITY AND THE PROBLEM OF OCCURRENCE OF FREE SEMIGROUPS

R. I. Grigorchuk

In connection with studies concerning the Hausdorff-Banach-Tarski paradox, von Neumann [1] considered the following problem. Let $G$ be a group acting as a group of transformations on a set $S$ and let $Z \subseteq S$ be a nonempty subset. The question is whether there exists a normed, on $A$, finitely additive $G$-invariant measure $\mu$ defined on the sigma-algebra of all subsets of $S$ and assuming its values in the interval $[0, +\infty]$. Such measures are called invariant measures for the triple $(G, S, A)$. Groups $G$ for which invariant measures exist for the triple $(G, G, G)$ were classified by von Neumann as a special class $AG$, the class of amenable groups. Theory and applications of amenable groups have been studied in the monograph [2].

Even when a group $G$ is amenable, the problem of existence of an invariant measure for a triple $(G, S, A)$ is not always decidable. In [3] Rosenblatt introduced the notion of a superamenable group: a group $G$ is said to be superamenable if the problem of existence of an invariant measure is decidable for each triple $(G, S, A)$, where $A$ is not empty. Rosenblatt proved that a group of nonexponential growth is superamenable. He noted that if a group $G$ contains a free subsemigroup $L$ with two generators, then the triple $(G, G, L)$ has no invariant measure. So a group possessing a free subsemigroup with two generators is not superamenable. Rosenblatt proved that a solvable group is superamenable if and only if it contains no free subsemigroup with two generators. In the same article he proposed the following conjecture ([3], p. 41; also see [9]):

A group is superamenable if and only if it is amenable and contains no free subsemigroup with two generators.