The Representation Theory of Transporter Categories

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Chapter 1

Introduction

A key step in the development of the representation theory of algebras was the introduction in 1975 of almost split sequences, by Auslander and Reiten[2]. Since then, Auslander-Reiten theory has expanded and proved itself to be a great algebraic and combinatorial tool to study the module theory of Artin algebras. One of the innovations, developed by Happel in 1987 and described in his book [7], extends the work of Auslander and Reiten to derived categories of module categories of finite-dimensional algebras. Derived categories have been used as a tool in representation theory for decades. One of the features of derived categories is that they are a "natural" category in which to study homology and cohomology.

We will use the tools of Auslander-Reiten theory to study a class of Grothendieck constructions. The Grothendieck construction has appeared in several areas of representation theory and related fields. For starters, the semidirect product construction in group theory is an example of a Grothendieck construction. Indeed, one can think of the Grothendieck construction as a generalization of the semidirect product. In his thesis, Thomason showed, that given a functor $F : K \to \mathrm{SCat}$, the homotopy colimit of the nerve of K is homotopy equivalent to the nerve of the Grothendieck construction of F [14]. When studying skew group rings, we can realize the base ring as a category with a single object on which G acts, so the skew group ring can be regarded as the category algebra of a Grothendieck construction. The Grothendieck construction is also an example of an extension of categories, and it behaves like a split extension.

Our main objects of study are a class of Grothendieck constructions, called transporter categories. Given a group G and a G-poset $\mathcal{P}$ the transporter category $\mathcal{P} \rtimes G$ is a category which combines the structure of $\mathcal{P}$, G, and the information of the group action. As such, transporter categories are a generalization of both posets and groups.

We take inspiration from the work of Diveris, Purin, and Webb [5]. The authors study the representation theory of posets, i.e. the module theory of the category algebra $k\mathcal{P}$ where $\mathcal{P}$ is a poset (considered as a category) and k is a field. The category algebra of a poset may be identified as the opposite of the incidence algebra of the poset. The authors study the Auslander-Reiten quiver of the bounded derived category $D^b(k\mathcal{P}-\text{mod})$. There, the authors develop clamping theory, which relates the Auslander-Reiten quiver of $D^b(k\mathcal{P})$ with that of the bounded derived category of the category algebra of certain subintervals, called clamped intervals. More specifically, they showed that if $[a, b]$ is a clamped interval in a poset $\mathcal{P}$, a large portion of the Auslander-Reiten quiver of $D^b(k[a, b]-\text{mod})$ is copied into the Auslander-Reiten quiver of $D^b(k\mathcal{P}-\text{mod})$. The authors use this theory to quickly determine the shape of a component of the Auslander-Reiten quiver of $D^b(k\mathcal{P}-\text{mod})$ when $\mathcal{P}$ belongs to a class of posets, $\mathcal{IC}$, which stands for "iterated clamping."

In this document, we show that clamping theory for posets can be extended to transporter categories. As a first step, we consider the category algebra $k\mathcal{P} \rtimes G$ where k is a field. We develop the notion of a "clamped subcategory" $[a, b] \rtimes G_b$ in a transporter category $\mathcal{P} \rtimes G$. With this, we show that a large portion of the Auslander-Reiten quiver of the bounded derived category of $k[a, b] \rtimes \mathcal{P} \rtimes G$ is copied into that of $k\mathcal{P} \rtimes G$. We also introduce a class of transporter categories, $\mathcal{ICT}$. This class has the property that, given a transporter category $\mathcal{P} \rtimes G \in \mathcal{ICT}$, a large portion of the Auslander-Reiten quiver of $D^b(k\mathcal{P} \rtimes G)$ containing a slice of the quiver can be constructed iteratively from clamped subcategories.

In Chapter 2, we begin by covering preliminaries. This includes basic facts about representations of categories, Grothedieck constructions and transporter categories, descriptions of their simple and projective representations. We also cover basic facts about the bounded derived category of an abelian category, and the restriction, induction, and coinduction functors and their derived functors.

In Chapter 3, we define clamped subcategories in transporter categories. We will describe some of their basic properties. In addition, we show useful properties about complexes for clamped subcategories under induction and coinduction, followed by restriction. This culminates in the following theorem.

Theorem. *Let G be a group and $\mathcal{P}$ be a G-poset, and let $[a,b] \rtimes G_b$ be clamped in $\mathcal{P} \rtimes G$. Then the functors $\uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G}$ and $\Uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G}$ are naturally isomorphic on objects with homology supported on the open interval (a,b).*

In Chapters 4 and 5 discuss the relationship between clamped intervals and Auslander-Reiten triangles, culminating in Corollary 5.0.4.

Corollary. *Let k be a field. The regions of the Auslander-Reiten quiver of $D^b(k[a,b] \rtimes G_b)$ containing the meshes whose rightmost terms have homology supported on $[a,b)$ and whose leftmost terms have homology supported on $(a,b]$ are the restrictions of regions in the Auslander-Reiten quiver of $D^b(k\mathcal{P} \rtimes G)$. Said another way, the regions of the Auslander-Reiten quiver of $D^b(k[a,b] \rtimes G_b)$ with the property above are copied into the Auslander-Reiten quiver of $D^b(k\mathcal{P} \rtimes G)$ by extending the modules appearing in those complexes by 0 outside of ${}^G[a,b]$.*

In Chapter 6, we apply this corollary to a class of transporter categories, $\mathcal{ICT}$, to quickly construct the Auslander-Reiten quiver transporter categories in $\mathcal{ICT}$. Specifically, we prove the following theorem.

Theorem. *Let $\mathcal{P} \rtimes G$ be a transporter category in $\mathcal{ICT}$, and let k be a field with* $\mathrm{char}(k) \nmid |G|$. *Let α be the minimal element and ω the maximal element of $\mathcal{P}$. Define the following:*

- *n: The number of connected components, in the category-theoretic sense, of $(\alpha,\omega) \rtimes G$.*
- *$\alpha_1, \dots, \alpha_n$: A selection of minimal elements of $(\alpha,\omega) \rtimes G$, each from a different connected component.*

- *$\{W_1, \ldots, W_m\}$: A complete set of pairwise nonisomorphic simple kG-modules.*

- *m_i: The number of isomorphism classes of simple kG_{α_i}-modules.*

- *$\{V_{i,1}, \ldots, V_{i,m_i}\}$, with $1 \leq i \leq n$: A complete set of pairwise nonisomorphic simple kG_{α_i}-modules.*

- *$e_{j,i,k}$, where $1 \leq i \leq n$, $1 \leq j \leq m$, and $1 \leq k \leq m_i$: The multiplicity of $V_{i,k}$ in $W_j \downarrow^G_{G_{\alpha_i}}$.*

- *$\mathcal{T}$: The underlying directed graph of the slice $\mathcal{S}$ identified in Hypothesis 6.2.6 of the component of the Auslander-Reiten quiver of $D^b(k\mathcal{P} \rtimes G)$ containing the projective modules of the form $P_{\alpha,W}$.*

- *$\mathcal{T}_i$, $1 \leq i \leq n$: The underlying directed graph of the slice identified in Hypothesis 6.2.6 of the components of the Auslander-Reiten quiver of $D^b(k[\alpha_i, \omega_i] \rtimes G_{\alpha_i})$ containing the projective modules of the form $P_{\alpha_i,V}$.*

Then we have

$$\mathcal{T} = \mathcal{T}_1 \sqcup \cdots \sqcup \mathcal{T}_n \sqcup \bigcup_{j=1}^{m} \{v_{1,j}, v_{2,j}\}$$

where $\bigcup_{j=1}^{m}\{v_{1,j}, v_{2,j}\}$ is a set of $2m$ labelled vertices. For each j, we add an edge between $v_{1,j}$ and $v_{2,j}$, and for each i, k with $1 \leq i \leq n$ and $1 \leq k \leq m_i$, we add $e_{j,i,k}$ edges between $v_{2,j}$ and the vertex in $\mathcal{T}_i$ corresponding to the module $P_{\alpha_i,V_{i,k}}$.

We will show that these transporter category algebras are piecewise-hereditary, and more significantly, we can use the theory we develop to identify the specific path algebra to which a given transporter category in $\mathcal{ICT}$ is derived-equivalent. In Chapter 7, we list the transporter categories in $\mathcal{ICT}$ of finite representation type and identify the path algebra to which each is derived-equivalent. We end with an example of transporter category algebras $\mathbb{C}\mathcal{P} \rtimes G$ where a component of the Auslander-Reiten quiver of $D^b(\mathbb{C}\mathcal{P} \rtimes G)$ has a slice whose underlying undirected graph is closely related to Young's lattice of partitions.

Proposition. *Let S_n denote the symmetric group on n elements. There is a transporter category $\mathcal{P} \rtimes S_n$ where the underlying graph of a slice of the component of the Auslander-Reiten quiver of $D^b(\mathbb{C}\mathcal{P} \rtimes S_n)$ containing the projective-injective modules is a modification of Young's lattice of partitions. Starting with the Hasse diagram for Young's lattice, we eliminate rows in positions greater than n, and for rows greater than the first, we replace each vertex by two vertices joined by a new edge. The bottom row of this modified lattice (i.e. row $2n - 1$) corresponds to the projective-injective modules.*

Chapter 2

Preliminaries

2.1 Posets and transporter categories

Let k be a field, let G be a finite group with order invertible in k, and let $\mathcal{P}$ be a finite left G-poset, i.e. a poset with a left G action. We regard $\mathcal{P}$ as a category whose objects are the elements of $\mathcal{P}$ for any pair of objects $x, y \in \mathcal{P}$, we have $\#(\mathrm{Hom}(x, y)) = 1$ if $x \leq y$ and 0 otherwise. Note that this fully defines composition of morphisms in $\mathcal{P}$. We will use the representation-theoretic notation ${}^{g}x$ and ${}^{g}\alpha$ to denote the action of $g \in G$ on $x \in \mathrm{Ob}(\mathcal{P})$ and $\alpha \in \mathrm{Mor}(\mathcal{P})$. We will represent posets as Hasse diagrams. As opposed to typical conventions, we draw x above y if $x \leq y$ so that the minimal elements are at the top of the diagram.

We study the Grothendieck construction determined by the action of G on $\mathcal{P}$. We give the definition of the Grothendieck construction here.

Definition 2.1.1. Let C be a small category, and let $F : C \to$SCat be a functor from C to the category of small categories. The *Grothendieck construction*, denoted $F \rtimes C$, is a category whose objects are pairs

$$(x, c)\,,\ \text{where } c \in \mathrm{Ob}(C) \text{ and } x \in \mathrm{Ob}(F(c)),$$

and where $\mathrm{Hom}((x, c), (y, d))$ consists of pairs (α, f) where $f : c \to d$ is a morphism in

C and $\alpha : F(f)(x) \to y$ is a morphism in $F(d)$. Composition is given by

$$(\alpha, f) \circ (\beta, g) = (\alpha \circ F(f)(\beta), f \circ g).$$

One can think of a Grothendieck construction as the simplest category combining all of the information of the domain, the target, and the functor.

In representation theory, a frequently considered Grothendieck construction is one where C is a group, G, considered as a category with a single object, $*$. In this case, the functor F encodes the action of G on a small category $F(*)$. Observe that a Grothendieck construction of the form $C \rtimes G$ has objects $\{(c, *) \mid c \in C\}$, so the objects of $C \rtimes G$ biject with those of C. As such, we will write the objects of $C \rtimes G$ as those of C.

Definition 2.1.2. Let G be a group, and let $\mathcal{P}$ be a G-poset. The *transporter category* $\mathcal{P} \rtimes G$ is the Grothendieck construction of

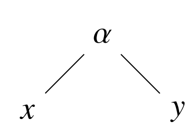
$$F : G \to \text{SCat}$$

where F encodes the action of G on $F(*) = \mathcal{P}$.

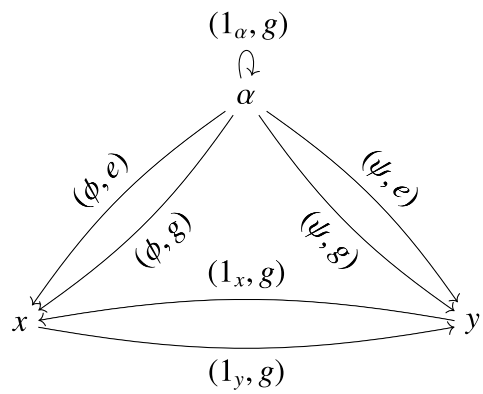
In this context, the poset $\mathcal{P}$ is called the *base poset* of $\mathcal{P} \rtimes G$.

It is important to note that every transporter category $\mathcal{P} \rtimes G$ is an *EI-category*, i.e. a category in which every endomorphism is an isomorphism. For an object $x \in \mathcal{P} \rtimes G$, we have $\text{End}_{\mathcal{P} \rtimes G}(x) \cong G_x$ where G_x is the stabilizer of x in $\mathcal{P}$.

Example 2.1.3. Let $\mathcal{P}$ be the poset



The object α is the unique minimal object of $\mathcal{P}$. As a category, there are two non-identity morphisms in $\mathcal{P}$, namely $\phi = (\alpha \leq x)$ and $\psi = (\alpha \leq y)$. Let $G = C_2 = \{e, g\}$ act on $\mathcal{P}$ by interchanging x and y. The category $\mathcal{P} \rtimes G$ can be represented by the following diagram in which the non-identity morphisms are shown:



2.2 Representations of Categories

We are interested in finite-dimensional modules for these categories. We begin by discussing these in two different, but equivalent, ways, summarizing from [16] .

Definition 2.2.1. Let C be a small category, let k be a commutative ring with a 1, and let k-mod be the category of k-modules. A *representation* of C over k is a functor

$$M : C \to k\text{−mod}.$$

A morphism $f : M \to N$ of representations is a natural transformation between the functors M and N.

Definition 2.2.2. Let C be a small category, and let k be a commutative ring with a 1. The *category algebra* kC is a free k-module with basis the morphisms of C. For any two morphisms $\phi, \psi \in C$, multiplication in kC given by

$$\phi \cdot \psi = \begin{cases} \phi \circ \psi \text{ if } \phi \text{ and } \psi \text{ are composable,} \\ 0 \text{ otherwise.} \end{cases}$$

We extend multiplication to arbitrary elements of C using bilinearity.

In the case where C is a group, i.e. a category with a single object in which all morphisms are invertible, representations of C over k are the same as representations of the group over k, and the category algebra kC is isomorphic to the corresponding group algebra. It is well-known that representations of a group G over k are equivalent to kG-modules. We have a similar situation in the case of categories.

Proposition 2.2.3. *(Mitchell, Theorem 7.1 [11]) Let k be a commutative ring with a 1, let $(k - \text{mod})^C$ denote the category of representations of C, and let kC-mod denote the category of left kC-modules. Then we have functors $r : (k{-}\text{mod})^C \to kC$-mod and $s : kC{-}\text{mod} \to (k{-}\text{mod})^C$ with the following properties*

1. *$sr \cong 1_{(k-\text{mod})^C}$.*
2. *r embeds $(k{-}\text{mod})^C$ as a full subcategory of $kC{-}\text{mod}$, and if C has finitely many objects, then $rs \cong 1_{kC-\text{mod}}$.*

The effect of this result is that representations of a category may be regarded as modules for the category algebra kC and vice-versa. We will be working with transporter categories with finitely many objects. Thus, we will refer to a module M of the category algebra $k\mathcal{P} \rtimes G$ as both a module and as a functor $M : \mathcal{P} \rtimes G \to k$-mod, when appropriate. Furthermore, the evaluation $M(x)$ of M at an object x can also be written algebraically as $1_x M$ and it has the structure of a kG_x-module. To shorten the notation we also write $M(x)$ as M_x.

We may now describe representations of $\mathcal{P} \rtimes G$ diagrammatically. We can write a poset as a Hasse diagram where (contrary to usual practice) the least elements are above the greater elements. For each object $x \in \mathcal{P}$ and $k\mathcal{P} \rtimes G$-module M we label the vertex x in the Hasse diagram with the kG_x-module M_x. This summarizes key information about the structure of M, e.g. its dimension and its composition factors.

Example 2.2.4. Consider the poset

$$\mathcal{P} = \begin{array}{ccccc} & & \alpha & & \\ & / & & \backslash & \\ x & & & & x' \\ | & & & & | \\ y & & & & y' \\ & \backslash & & / & \\ & & \omega & & \end{array}$$

We write a $k\mathcal{P}$-module M as

$$\begin{array}{cc} \multicolumn{2}{c}{M_\alpha} \\ M_x & M_{x'} \\ M_y & M_{y'} \\ \multicolumn{2}{c}{M_\omega} \end{array}$$

In this case, each term $M_{(-)}$ is a vector space over k because each object has a trivial endomorphism group.

We may consider the same poset with $G = C_2$ acting by switching the two chains. In this case, the terms x and x' are isomorphic, as are y and y', so $M_x \cong M_{x'}$ and $M_y \cong M_{y'}$ for any $k\mathcal{P} \rtimes G$-module M. Thus, to simplify the diagram, we can omit $M_{x'}$ and $M_{y'}$ and write M as

$$\begin{array}{c} M_\alpha \\ M_x \\ M_y \\ M_\omega \end{array}$$

where M_α and M_ω are kG-modules and M_x and M_y are k-vector spaces.

2.3 Simple and projective modules

Let C be an EI-category, i.e. a category in which every endomorphism is an isomorphism, and let k be a commutative ring. By a proposition due to Lück (see [4, Proposition 4.3]), the simple kC-modules bijects with pairs (x, W) where x is an object of the category, chosen up to isomorphism, and W is a simple $k\mathrm{End}_C(x)$-module. The value of the simple functor at y is nonzero if and only if $y \cong x$. When $C = \mathcal{P} \rtimes G$ is a transporter category, we may write $\mathrm{End}(x) = G_x$, where G_x is the stabilizer of $x \in \mathcal{P}$. For each $y \in \mathcal{P}$ with $y \cong x$ pick an element $g \in G$ satisfying ${}^g x = y$. Then $G_y = gG_xg^{-1}$, and we set the value of the simple module y as the kG_y-module ${}^g W = g \otimes_k W$. From now on, we denote the simple $k\mathcal{C}$-module corresponding to the pair (x, W) as $S_{x,W}$. The procedure above defines $S_{x,W}$ up to isomorphism.

We aim to describe the indecomposable projective $k\mathcal{P} \rtimes G$-modules, where $\mathcal{P} \rtimes G$ is a finite transporter category and k is a field in which $|G|$ is invertible. In particular, we will describe these modules. By standard facts about finite-dimensional algebras over fields, the indecomposable projective modules biject with the simple modules by sending the simple module to its projective cover. Let $P_{x,W}$ denote the projective cover of $S_{x,W}$. The $k\mathcal{P} \rtimes G$-module $P_{x,W}$ is formed in the following way: let $e_W \in kG_x$ be a primitive idempotent satisfying $e_W(W) \neq 0$. Then we can write $P_{x,W}$ as a $k\mathcal{P} \rtimes G$-module

$$P_{x,W} = k\mathcal{P} \rtimes G \cdot (1_x, e_W).$$

The fact that $P_{x,W}$ can be so written is a consequence of the description of indecomposable projectives modules for EI-categories given in [4, Prop. 11.29]. The top radical layer is easy to describe, but the other composition factors require some investigation. The reason for doing this is because we will be approximating modules and complexes of modules with their projective resolutions, and we will be particularly interested in inducing these projective resolutions from a clamped subcategory. We are guaranteed to have finite global dimension because $|G|$ is invertible in k. This allows for the use of many more tools from the bounded derived category.

The top radical layer of $P_{x,W}$ is $S_{x,W}$, but its other radical layers are more difficult to describe. In order to start working with examples, we need to be able to describe the radical layers precisely. In what follows, we will be referring to representations, or complexes of representations, of a transporter subcategory, $\mathcal{Q} \rtimes H$. In these instances when there may be some ambiguity, we will specify the transporter category for the said module or complex. For example, we may write $S^{\mathcal{Q} \rtimes H}_{x,W}$ for the simple $k\mathcal{Q} \rtimes H$-module generated at x with W as the corresponding simple kH_x-module.

Lemma 2.3.1. *Let G be a finite group, let $\mathcal{P}$ be a finite G-poset, and let k be a field with $|G|$ invertible in k. Let $x \in \mathcal{P} \rtimes G$ be an object with $G_x = G$. Then for any $y \in \mathcal{P} \rtimes G$ with $x \leq y$, we have*

$$1_y \cdot P_{x,W} \cong W \downarrow^G_{G_y}$$

as left kG_y-modules

Proof. Let $e_W \in k\,\mathrm{End}_{\mathcal{P}\rtimes G}(x) \cong G$ be a primitive idempotent associated to W so that $k\,\mathrm{End}_{\mathcal{P}\rtimes G}(x)e_W \cong W$. By direct calculation, we have

$$1_y P_{x,W} = 1_y(k\mathcal{P} \rtimes G(1_x e_W))$$

$$= k\{1_y \alpha g 1_x e_W \mid \alpha \in \mathcal{P} \text{ and } g \in G\}$$

Now for all $g \in G$, we have $g1_x = {}^g 1_x g = 1_x g$ because x is fixed by G. So

$$k\{1_y \alpha g 1_x e_W \mid \alpha \in \mathcal{P} \text{ and } g \in G\}$$

$$= k\{1_y \alpha 1_x g e_W \mid \alpha \in \mathcal{P} \text{ and } g \in G\}$$

$$= k\{(x \leq y) g e_W \mid g \in G\}$$

$$= k(x \leq y) G e_W.$$

We obtain an action of G on e_W by transporting the action of G through the isomorphism $G \cong \mathrm{End}_{\mathcal{P}\rtimes G}(x)$. Through this, we have an isomorphism of kG-modules $kGe_W \cong W$ because kG is semisimple. If $h \in G_y$, we have

$$h \cdot (x \leq y) g e_W = ({}^h x \leq {}^h y) h g e_W = (x \leq y) h g e_W.$$

The final equality holds because h fixes both x and y. This implies that there is an isomorphism of kG_y-modules given by

$$k(x \leq y) G e_w \to kGe_W,$$

$$(x \leq y)hge_W \mapsto hge_W$$

on a spanning set, and extended linearly. Noting that the kG_y-module kGe_W is precisely $W \downarrow^G_{G_y}$, we conclude that $k(x \leq y)Ge_W \cong W \downarrow^G_{G_y}$ as left kG_y-modules.

□

We have the following as an immediate corollary:

Corollary 2.3.2. *Let G be a finite group, let $\mathcal{P}$ be a G-poset, and let k be a field in which $|G|$ is invertible. Let $x \in \mathcal{P} \rtimes G$ be any object, and let W be a kG_x-module. Then,*

$$1_y \cdot P^{\mathcal{P} \rtimes G_x}_{x,W} \cong \begin{cases} W \downarrow^{G_x}_{G_x \cap G_y} & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

as left $k(G_x \cap G_y)$-modules.

Proof. We apply the previous lemma to the transporter subcategory $\mathcal{P} \rtimes G_x$. Note that in $\mathcal{P} \rtimes G_x$, the endomorphism group at the object y is precisely $G_x \cap G_y$. □

We now extend these results to the structure of projective $k\mathcal{P} \rtimes G$-modules. In what follows, we make use of the induction operation. Let C be a small category, and let $\mathcal{D} \subset C$ be a subcategory. If N is a $k\mathcal{D}$-module, then $N \uparrow^C_{\mathcal{D}}$ is the kC-module defined by

$$N \uparrow^C_{\mathcal{D}} := kC \otimes_{k\mathcal{D}} N.$$

We start by using the results for the structure of projective $k\mathcal{P} \rtimes H$-modules generated by their values on objects fixed by H, then inducing up to $\mathcal{P} \rtimes G$. We first need a lemma.

Lemma 2.3.3. *Let G be a finite group, let k be a field, and let $\mathcal{P}$ be a G-poset. Then for any object $x \in \mathcal{P} \rtimes G$ and for any simple kG_x-module W, we have*

$$P^{\mathcal{P} \rtimes G}_{x,W} \cong P^{\{x\} \rtimes G_x}_{x,W} \uparrow^{\mathcal{P} \rtimes G}_{\{x\} \rtimes G_x},$$

and

$$P^{\mathcal{P} \rtimes G}_{x,W} \cong P^{\mathcal{P} \rtimes G_x}_{x,W} \uparrow^{\mathcal{P} \rtimes G}_{\mathcal{P} \rtimes G_x},$$

Proof. For the first isomorphism, let $e_W \in k\,\mathrm{End}_{\{x\}\rtimes G_x}(x) \cong G_x$ be a primitive idempotent associated to W. Note that $\{x\} \rtimes G_x$ is the category with a single element and whose endomorphism monoid is isomorphic to G_x, so modules for $k\{x\} \rtimes G_x$ are the same as modules for kG_x. Thus we have $P^{\{x\}\rtimes G_x}_{x,W} \cong k(\{x\} \rtimes G_x)e_W$. Using the definition of induction, we have

$$\begin{aligned} P^{\{x\}\rtimes G_x}_{x,W} \uparrow^{\mathcal{P}\rtimes G}_{\{x\}\rtimes G_x} &= k\mathcal{P} \rtimes G \otimes_{k(\{x\}\rtimes G_x)} k(\{x\} \rtimes G_x)e_W \\ &\cong k(\mathcal{P} \rtimes G)e_W \\ &\cong P^{\mathcal{P}\rtimes G}_{x,W}. \end{aligned}$$

In the second line, we consider e_W to lie in $k\,\mathrm{End}_{\mathcal{P}\rtimes G}(x)$ because $\{x\} \rtimes G_x$ is a full subcategory of $\mathcal{P} \rtimes G$. The last isomorphism follows from the discussion at the beginning of this section.

The second isomorphism of the lemma follows from the first by transitivity of induction. Indeed, by the first isomorphism of the lemma, we have

$$P_{x,W} \cong P^{\{x\}\rtimes G_x}_{x,W} \uparrow^{\mathcal{P}\rtimes G}_{\{x\}\rtimes G_x} \cong P^{\{x\}\rtimes G_x}_{x,W} \uparrow^{\mathcal{P}\rtimes G_x}_{\{x\}\rtimes G_x} \uparrow^{\mathcal{P}\rtimes G}_{\mathcal{P}\rtimes G_x}.$$

Because $\{x\}\rtimes G_x$ is also a full subcategory of $\mathcal{P}\rtimes G_x$, it follows that $P^{\mathcal{P}\rtimes G_x}_{x,W} \cong P^{\{x\}\rtimes G_x}_{x,W} \uparrow^{\mathcal{P}\rtimes G_x}_{\{x\}\rtimes G_x}$. Thus we have

$$P^{\{x\}\rtimes G_x}_{x,W} \uparrow^{\mathcal{P}\rtimes G_x}_{\{x\}\rtimes G_x} \uparrow^{\mathcal{P}\rtimes G}_{\mathcal{P}\rtimes G_x} \cong P^{\mathcal{P}\rtimes G_x}_{x,W} \uparrow^{\mathcal{P}\rtimes G}_{\mathcal{P}\rtimes G_x},$$

and the result follows.

□

We now identify the value of arbitrary projective $k\mathcal{P} \rtimes G$-modules on any object. These values are given by Mackey-like formulas expressed as a direct sum over a subset of double coset representatives.

Proposition 2.3.4. *Let G be a finite group, let k be a field with* $\mathrm{char}(k) \nmid |G|$*, and let* $\mathcal{P}$

be a G-poset. If $x \leq y$ in $\mathcal{P}$, and W is a simple kG_x-module, then

$$1_y \cdot P^{\mathcal{P}\rtimes G}_{x,W} \cong \bigoplus_{\substack{g\in[G_y\backslash G/G_x] \\ {}^g x\leq y}} {}^g(W \downarrow^{G_x}_{G^g_y\cap G_x}) \uparrow^{G_y}_{G_y\cap {}^g G_x}$$

as kG_y-modules.

Remark 2.3.5. By this formula, if ${}^g x \leq y$ for every $g \in G$, then the double sum becomes the Mackey formula for $W \uparrow^G_{G_x} \downarrow^G_{G_y}$.

Proof. Note that $k\mathcal{P} \rtimes G$ is a skew group algebra with subalgebra $k\mathcal{P} \rtimes G_x$. By an argument similar to one for group algebras (See [15] Prop 4.3.1), we have a vector space isomorphism

$$P^{\mathcal{P}\rtimes G_x}_{x,W} \uparrow^{\mathcal{P}\rtimes G}_{\mathcal{P}\rtimes G_x} \cong \bigoplus_{h\in[G/G_x]} h \otimes P^{\mathcal{P}\rtimes G_x}_{x,W},$$

where the direct sum is a sum of vector spaces over a fixed set of coset representatives. The tensor is over $k(\mathcal{P} \rtimes G_x)$, but we suppress this throughout the proof. By lemma 2.3.3 we have

$$\begin{aligned} 1_y \cdot P^{\mathcal{P}\rtimes G}_{x,W} &= 1_y \cdot (P^{\mathcal{P}\rtimes G_x}_{x,W}) \uparrow^{\mathcal{P}\rtimes G}_{\mathcal{P}\rtimes G_x} \\ &= 1_y \cdot \bigoplus_{h\in[G/G_x]} h \otimes P^{\mathcal{P}\rtimes G_x}_{x,W} \\ &= \bigoplus_{h\in[G/G_x]} h \otimes 1_{h^{-1}y} \cdot P^{\mathcal{P}\rtimes G_x}_{x,W}. \end{aligned}$$

By Corollary 2.3.2, we have

$$1_{h^{-1}y} \cdot P^{\mathcal{P}\rtimes G_x}_{x,W} \cong \begin{cases} W \downarrow^{G_x}_{G^h_y\cap G_x} & \text{if } {}^h x \leq y \\ 0 \text{ otherwise} \end{cases}$$

as $kG_y{}^h \cap G_x$-modules. Thus we can rewrite the direct sum:

$$\bigoplus_{h\in[G/G_x]} h \otimes 1_{{}^{h^{-1}}y} \cdot P_{x,W}^{\mathcal{P}\rtimes G_x} \cong \bigoplus_{\substack{h\in[G/G_x] \\ {}^h x\leq y}} h \otimes W \downarrow^{G_x}_{G_y^h\cap G_x}$$

as kG_y-modules. We now decompose this direct sum in terms of double cosets.

$$\bigoplus_{\substack{h\in[G/G_x] \\ {}^h x\leq y}} h \otimes W \downarrow^{G_x}_{G_y^h\cap G_x} = \bigoplus_{g\in[G_y\backslash G/G_x]} \bigoplus_{\substack{h\in[G/G_x] \\ {}^h x\leq y \\ h\in G_y g G_x}} h \otimes W \downarrow^{G_x}_{G_y^h\cap G_x} .$$

In this double direct sum, we may choose the g so that they are a subset of the coset representatives, h, chosen earlier. Note that if H is a subgroup of G, V is an H-module, and $g \in G$, then $g \otimes V \cong {}^gV$ where ${}^gV$ is the ${}^gH$-module which is defined by the composition of homomorphisms

$${}^gH \xrightarrow{c_{g^{-1}}} H \to GL(V).$$

We claim that if any element of $r \in G_y g G_x$ satisfies ${}^r x \leq y$, then every element $s \in G_y g G_x$ satisfies ${}^s x \leq y$. Indeed, set $r = g_1 g g_2$ where $g_1 \in G_y$ and $g_2 \in G_x$, and set $s = h_1 g h_2$ where $h_1 \in G_y$ and $h_2 \in G_x$. Then $x = {}^{g_2^{-1}h_2}x$ and ${}^{h_1 g_1^{-1}}y = y$, so

$$\begin{aligned}
{}^{g_1 g g_2}x &\leq y, \\
{}^{g_1 g g_2}\left({}^{g_2^{-1}h_2}x\right) &\leq y, \\
{}^{g_1 g h_2}x &\leq y, \\
{}^{h_1 g_1^{-1}}\left({}^{g_1 g h_2}x\right) &\leq {}^{h_1 g_1^{-1}}y, \\
{}^{h_1 g h_2}x &\leq y.
\end{aligned}$$

This result allows us to rewrite the direct sum:

$$\bigoplus_{g\in[G_y\backslash G/G_x]} \bigoplus_{\substack{h\in[G/G_x] \\ {}^h x\leq y \\ h\in G_y g G_x}} h\otimes W\downarrow^{G_x}_{G_y^h\cap G_x} \cong \bigoplus_{\substack{g\in[G_y\backslash G/G_x] \\ {}^g x\leq y}} \bigoplus_{\substack{h\in[G/G_x] \\ h\in G_y g G_x}} {}^h(W\downarrow^{G_x}_{G_y^h\cap G_x})$$

We now show that for a fixed $g\in G$, we have

$$\bigoplus_{\substack{h\in[G/G_x] \\ h\in G_y g G_x}} {}^h(W\downarrow^{G_x}_{G_y^h\cap G_x}) \cong {}^g(W\downarrow^{G_x}_{G_y^g\cap G_x})\uparrow^{G_y}_{G_y\cap {}^g G_x}.$$

This follows from the proof of Mackey's decomposition theorem [15, Theorem 5.2.1]. Putting everything together yields

$$1_y\cdot P^{\mathcal{P}\rtimes G}_{x,W} \cong \bigoplus_{\substack{g\in[G_y\backslash G/G_x] \\ {}^g x\leq y}} {}^g(W\downarrow^{G_x}_{G_y^g\cap G_x})\uparrow^{G_y}_{G_y\cap {}^g G_x}$$

□

2.4 The bounded derived category

We will study the representation theory of transporter category algebras by first considering their bounded derived categories and later deducing results about their module categories. The rationale behind this slightly unusual approach is that certain properties of the bounded derived category may be easier to approach than the corresponding properties of the module category. Accordingly, we proceed with a description of the bounded derived category of an abelian category and its most salient properties. We will be summarizing basic results from [7, Chapter I].

Definition 2.4.1. Let $\mathcal{A}$ be an abelian category. We define the following categories:

- $\mathrm{Ch}(\mathcal{A})$: The category of chain complexes with terms in $\mathcal{A}$, denoted

$$C_\bullet = \cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} C_{-1} \xrightarrow{d_{-1}} C_{-2} \to \cdots$$

- $K^b(\mathcal{A})$: The homotopy category of chain complexes with zero homology in all but finitely many degrees.

- $D^b(\mathcal{A})$: The category $K^b(\mathcal{A})$ in which we include all "formal inverses" to the morphisms

$$f : C_\bullet \to D_\bullet$$

  for which $H_n(f)$ is an isomorphism for all n. When the category is a module category of an algebra $\mathcal{A} = A$-mod, we will write $D^b(A)$ to denote the bounded derived category $D^b(A\text{-mod})$.

The bounded derived category is equipped with a collection of automorphisms called shift functors. For each $n \in \mathbb{Z}$, we have the functor

$$[n] : D^b(\mathcal{A}) \to D^b(\mathcal{A}),$$

$$C_\bullet \to C_\bullet[n],$$

where $(C_\bullet[n])_i = C_{i-n}$ and the differential on $C_\bullet[n]$ is $d_{C_\bullet[n]} = (-1)^n d_{C_\bullet}$. The shift functor [1] makes $D^b(\mathcal{A})$ a triangulated category. Given a morphism $f : X_\bullet \to Y_\bullet$ in $D^b(\mathcal{A})$, the distinguished triangle starting with $f : X_\bullet \to Y_\bullet$ is

$$X_\bullet \xrightarrow{f} Y_\bullet \xrightarrow{i} \text{cone} f \xrightarrow{\pi} X_\bullet[1]$$

where $\text{cone} f$ denotes the mapping cone of f, i is the inclusion map, and π is projection.

The mapping cone $\text{cone} f$ is the complex with terms

$$(\text{cone} f)_i = X[1]_i \oplus Y_i$$

and differential

$$d_{\text{cone} f} = \begin{pmatrix} d_{X[1]} & f \\ 0 & d_Y \end{pmatrix}.$$

For an abelian category $\mathcal{A}$, let ${}_{\mathcal{A}}P$ denote the full subcategory of $\mathcal{A}$ containing the projective objects, and let ${}_{\mathcal{A}}I$ denote the full subcategory of $\mathcal{A}$ containing the injective objects. When $\mathcal{A}$ has enough projective (resp. injective) objects, we can always write a complex $C_\bullet \in D^b(\mathcal{A})$ as a complex of projective (resp. injective) objects. Moreover, two complexes of projective (resp. injective) objects are isomorphic in $D^b(\mathcal{A})$ if and only if the complexes are homotopy equivalent. This implies that there are triangle equivalences

$$K^b({}_{\mathcal{A}}P) \to D^b(\mathcal{A}) \to K^b({}_{\mathcal{A}}I).$$

In the context of this document, A is a finite dimensional algebra over a field and $\mathcal{A} = A$-mod. If $M \in D^b(A)$ is the shift of an A-module, i.e. a complex with nonzero homology only in one degree, then M is isomorphic to a projective or injective resolution of M, shifted appropriately. Maps between two such complexes are just chain maps, taken up to homotopy. Because of this, we will often use a projective or injective resolution of a module M in order to make calculations in $D^b(A)$.

2.5 Induction, coinduction, and restriction

In this section, we define induction, coinduction, and restriction between an algebra and a subalgebra, and we extend these notions to bounded derived categories. We then present results in the specific case when the algebras are transporter category algebras and the subalgebra comes from a clamped subcategory. We provide a sufficient condition for when a complex M in $D^b(k\mathcal{P} \rtimes G)$ is induced or coinduced from $k[a,b] \rtimes G_b$, and we determine a large class of complexes for which induction and coinduction coincide. In Chapter 6, these results will be key in determining which portions of the Auslander-Reiten quiver of $D^b(k[a,b] \rtimes G_b)$ are copied into that of $D^b(k\mathcal{P} \rtimes G)$.

Definition 2.5.1. Let k be a commutative ring with a 1, let Λ and Γ be k-algebras with $\Lambda \subset \Gamma$ where the inclusion is not necessarily unital. The *restriction* functor

$$(-)\downarrow^{\Gamma}_{\Lambda} : \Gamma\text{-mod} \to \Lambda\text{-mod}$$

is the functor defined by $M \downarrow^{\Gamma}_{\Lambda} := 1_\Lambda M$. Notice that we did not require the identity 1_Γ to be equal to 1_Λ. The left adjoint of restriction is the *induction* functor

$$(-)\uparrow_\Lambda^\Gamma \colon \Lambda\text{-mod} \to \Gamma\text{-mod},$$

defined by $N\uparrow_\Lambda^\Gamma := \Gamma \otimes_\Lambda N$. The right adjoint of restriction is the *coinduction* functor

$$(-)\Uparrow_\Lambda^\Gamma \colon \Lambda\text{-mod} \to \Gamma\text{-mod},$$

defined by $N\Uparrow_\Lambda^\Gamma := \mathrm{Hom}_\Lambda(\Gamma, N)$.

If $\mathcal{C}$ is a small category and $\mathcal{D}$ is a full subcategory, the inclusion functor $\mathcal{D} \hookrightarrow \mathcal{C}$ induces an inclusion of category algebras $k\mathcal{D} \hookrightarrow k\mathcal{C}$. We simplify the notation somewhat and write $\downarrow_\mathcal{D}^\mathcal{C}$, $\uparrow_\mathcal{D}^\mathcal{C}$, and $\Uparrow_\mathcal{D}^\mathcal{C}$ to denote restriction, induction, and coinduction between $k\mathcal{D}$ and $k\mathcal{C}$.

When working over bounded derived categories, we must use the total derived functors of induction, coinduction and restriction. Importantly, the total left derived functor of induction, and the total right derived functor of coinduction remain the left and right adjoints of the total derived functor of restriction. To apply the total left (resp. right) derived functor of induction (resp. coinduction) on a complex M, we replace M with a complex of projective (resp. injective) modules, then apply the ordinary induction (resp. coinduction) functor on each projective (resp. injective) term.

Under the right circumstances, projective and injective modules are very well-behaved under induction and coinduction. If $\mathcal{E}$ is a set of objects in a category $\mathcal{C}$, we say that a representation M of $\mathcal{C}$ is *generated* by its values on $\mathcal{E}$ if whenever $S \subset M$ is a subrepresentation with $S(x) = M(x)$ for all $x \in \mathcal{E}$, then $S = M$. Dually, we say that M is *cogenerated* by its values on $\mathcal{E}$ if, whenever S is a quotient of M with $S(x) \cong M(x)$ for all $x \in \mathcal{E}$, then $S = M$. We quote the most salient results from [16].

Proposition 2.5.2. *Let $\mathcal{C}$ be a small category, and let $\mathcal{D}$ be a full subcategory with finitely many morphisms. Let M be a representation of $\mathcal{D}$.*

1. *Induction $\uparrow_\mathcal{D}^\mathcal{C}$ sends projective modules to projective modules. Moreover, if $\mathcal{E}$ is a set of objects in $\mathcal{D}$ and M is a $\mathcal{D}$-module generated by its values on $\mathcal{E}$, then $M\uparrow_\mathcal{D}^\mathcal{C}$*

is also generated by its values on $\mathcal{E}$.

2. *Coinduction $\Uparrow_{\mathcal{D}}^{\mathcal{C}}$ sends injective modules to injective modules. Moreover, if $\mathcal{E}$ is a set of objects in $\mathcal{D}$ and M is a $\mathcal{D}$-module cogenerated by its values on $\mathcal{E}$, then $M \Uparrow_{\mathcal{D}}^{\mathcal{C}}$ is also cogenerated by its values on $\mathcal{E}$.*

Proof. The proof of 1 is [16, Proposition (3.2)], and 2 is dual to 1.

□

Chapter 3

Clamped subcategories

We now develop a theory of clamped subcategories of transporter categories. A closely related concept was employed by Diveris, Purin, and Webb in [5] to study the representation theory, particularly the Auslander-Reiten theory, of category algebras of posets over a field. To begin this chapter, we define clamped subcategories $[a,b] \rtimes G_b$ of a transporter category $\mathcal{P} \rtimes G$ and outline some of its basic properties. From there, we use these properties to study the effects of induction and coinduction on a subset of complexes in $D^b(k[a,b] \rtimes G_b)$. We begin with the concept of a clamped interval in a poset.

Definition 3.0.1 (Diveris, Purin, Webb [5])**.** Let $\mathcal{P}$ be a poset. A closed interval $[a,b]$ is called *clamped* if $y \leq b$ implies $y \leq a$ or $a \leq y$, and $a \leq y$ implies $y \leq b$ or $b \leq y$.

It was shown in [5] that a clamped interval $[a,b] \subset \mathcal{P}$ has the property that portions of the Auslander-Reiten quiver of $D^b(k[a,b])$ are copied into the Auslander-Reiten quiver of $D^b(k\mathcal{P})$. We define a notion of a clamped subcategory in a transporter category $\mathcal{P} \rtimes G$ which, as we will later show, has this type of copying property.

Definition 3.0.2. Let G be a finite group and let $\mathcal{P}$ be a finite G-poset. A subcategory of $\mathcal{P} \rtimes G$ of the form $[a,b] \rtimes G_b$, where G_b denotes the stabilizer of b, is called *clamped* if $[a,b] \subset \mathcal{P}$ is clamped in the sense of Definition 3.0.1.

This definition may appear a bit misleading, as it is not immediately obvious that G_b preserves the interval $[a, b]$. However, this is indeed the case.

Lemma 3.0.3. *Let G be a finite group and let $\mathcal{P}$ be a finite G-poset. Let $[a, b]$ be a clamped interval in $\mathcal{P}$. Then $G_b = G_a$, so G_b preserves the interval $[a, b]$.*

Proof. Let $g \in G_b$. Because we have $a \leq b$, it follows that ${}^g a \leq {}^g b = b$. By the clamping property, we have $a \leq {}^g a$ or ${}^g a \leq a$. Without loss of generality, assume $a \leq {}^g a$. Then because G is finite, we have $a \leq {}^g a \leq {}^{g^2} a \leq \cdots \leq a$. This forces equality, $a = {}^g a$, so $G_b \leq G_a$. A similar argument shows that $G_a \leq G_b$, and thus $G_b = G_a$. □

We prove some preliminary properties about clamped subcategories.

Lemma 3.0.4. *If $\mathcal{P} \rtimes G$ is a transporter category and the G-orbits of objects are finite, then clamped subcategories of $\mathcal{P} \rtimes G$ are full.*

Proof. Let $x, y \in [a, b] \rtimes G_b$ where $[a, b] \rtimes G_b$ is clamped in $\mathcal{P} \rtimes G$. We must show that if ${}^g x \leq y$ for some $g \in G$, then $g \in G_b$. Note that ${}^g x \in {}^g[a, b]$, and ${}^g[a, b]$ is clamped in $\mathcal{P}$ if and only if $[a, b]$ is clamped in $\mathcal{P}$. Then because ${}^g x \leq y$, we have either $y \leq {}^g b$ or ${}^g b \leq y$. In either case we have that $b \leq {}^g b$ or ${}^g b \leq b$. Without loss of generality, assume that $b \leq {}^g b$. Then because there are finitely many G-orbits of b, we have $b \leq {}^g b \leq {}^{g^2} b \leq \cdots \leq b$, which implies $b = {}^g b$. Thus $g \in G_b$. □

In the same spirit, we also have this key lemma.

Lemma 3.0.5. *Let G be a finite group and let $\mathcal{P}$ be a G-poset. Let $[a, b]$ be a clamped interval in $\mathcal{P}$ and let $x \in [a, b]$. Then if ${}^h b \leq y$ for some $h \in G$, then for any $g \in G$ we have ${}^g x \leq y$ if and only if ${}^g b \leq y$.*

Note that the condition ${}^h b \leq y$ for some $h \in G$ is required because it is possible that we could choose $y \in [a, b]$ or any conjugate of $[a, b]$. For such a y, it could be that ${}^g x \leq y$ but $y \leq {}^g b$.

Proof. If ${}^g b \leq y$, then ${}^g x \leq y$. Now suppose ${}^g x \leq y$. Because ${}^g[a, b]$ is clamped, we have either ${}^g b \leq y$ or $y \leq {}^g b$. If ${}^g b \leq y$, then we are done, so assume $y \leq {}^g b$. By assumption, we have ${}^{h}b \leq y \leq {}^{g}b$. Then $b \leq {}^{h^{-1}g}b$, and by applying powers of $h^{-1}g$ repeatedly, we get $b = {}^{h^{-1}g}b$ and thus ${}^{h}b = {}^{g}b$. This implies ${}^{g}b = {}^{h}b \leq y$, and we are done. □

We can express this result as a statement about representable functors when restricted to a certain class of objects. We will use the following notation: if C is an EI-category, and y is an object of C, then $C_{\geq y}$ is the full subcategory of C generated by the objects $\{x \mid \mathrm{Hom}_C(y,x) \neq \varnothing\}$. Similarly, $C_{\leq y}$ is the full subcategory of C with objects $\{x \mid \mathrm{Hom}_C(x,y) \neq \varnothing\}$.

Corollary 3.0.6. *Let $[a,b]$ be clamped in $\mathcal{P}$ and let $x \in [a,b]$. Let R be any ring. There is a natural isomorphism of functors*

$$R\,\mathrm{Hom}_{\mathcal{P}\rtimes G}(b,-)\downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}} \cong R\,\mathrm{Hom}_{\mathcal{P}\rtimes G}(x,-)\downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}} .$$

Proof. The map is given by sending α to $\alpha \circ ({}^{e}x \leq b, e)$ where $e \in G$ is the identity. This sends a morphism $({}^{g}b \leq y, g)$ in $\mathrm{Hom}(b,y)\downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}}$ to $({}^{g}x \leq y, g)$. By 3.0.5, this map is a bijection for any $y \in (\mathcal{P}\rtimes G)_{\geq b}$, so this is an isomorphism at each object. Because this map is given by precomposition, the isomorphism is natural. □

Not only does this tell us about representable functors, but this also tells us about all projective modules.

Corollary 3.0.7. *If $[a,b]$ is clamped in a G-poset $\mathcal{P}$, $x \in [a,b]$, k is a field in which $|G|$ is invertible, and W is a simple kG_x-module, then $P_{x,W}\downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}} \cong P_{b,V}\downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}}$, where $V \cong W\uparrow^{G_b}_{G_x}$.*

Proof. Because W is simple, we may write $k\,\mathrm{Hom}_{\mathcal{P}\rtimes G}(x,-) \cong P_{x,W} \oplus M$ for some $k\mathcal{P}\rtimes G$-module M. Restricting gives us

$$k\,\mathrm{Hom}_{\mathcal{P}\rtimes G}(x,-)\downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}} \cong P_{x,W}\downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}} \oplus M\downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}}$$

using the fact that restriction is an additive. By Corollary 3.0.6, we have

$$k\,\mathrm{Hom}_{\mathcal{P}\rtimes G}(b,-)\downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}} \cong k\,\mathrm{Hom}_{\mathcal{P}\rtimes G}(x,-)\downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}} \cong P_{x,W}\downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}} \oplus M\downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}} .$$

Because $(\mathcal{P} \rtimes G)_{\geq b}$ is the full subcategory whose objects are $\{y \mid \mathrm{Hom}(b, y) \neq \varnothing\}$, it follows that

$$k\,\mathrm{Hom}_{\mathcal{P}\rtimes G}(b, -) \downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}} \cong k\,\mathrm{Hom}_{(\mathcal{P}\rtimes G)_{\geq b}}(b, -).$$

This means that $P_{x,W} \downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}}$ is a summand of $k\,\mathrm{Hom}_{(\mathcal{P}\rtimes G)_{\geq b}}(b, -)$, so $P_{x,W} \downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}} \cong P^{(\mathcal{P}\rtimes G)_{\geq b}}_{b,V}$ for some V. To determine V, we evaluate $P_{x,W} \downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}} (b) = P_{x,W}(b) \cong W \uparrow^{G_b}_{G_x}$. Finally, we note that $P^{(\mathcal{P}\rtimes G)_{\geq b}}_{b,V} \cong P^{\mathcal{P}\rtimes G}_{b,V} \downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}}$ because both are summands of $k\,\mathrm{Hom}_{\mathcal{P}\rtimes G}(b, -) \downarrow^{\mathcal{P}\rtimes G}_{(\mathcal{P}\rtimes G)_{\geq b}} \cong k\,\mathrm{Hom}_{(\mathcal{P}\rtimes G)_{\geq b}}(b, -)$ and have the same evaluation at b. □

We are interested in values of these projectives outside of the G-orbits of $[a, b]$. To this end, we show that the values of representable functors generated on $[a, b] \rtimes G_b$ on elements greater than b are completely determined by the functor's value at b.

We will use the notation $F_x := k\,\mathrm{Hom}_{\mathcal{P}\rtimes G}(x, -)$.

Proposition 3.0.8. *Let $y \in \mathcal{P} \rtimes G$, and let $N(b, y)$ be a complete set of coset representatives $g \in [G/G_b]$ satisfying ${}^g b \leq y$. There is an isomorphism*

$$\alpha : \bigoplus_{g \in N(b,y)} F_b({}^g b) \to F_b(y)$$

where the component maps $\alpha_g : F_b({}^g b) \to F_b(y)$ are defined by post-composition with $({}^e({}^g b) \leq y, e)$.

Proof. First note that if $({}^h b \leq y, h) \in F_b(y)$, then $h = gk$ where $g \in N(b, y)$ and $k \in G_b$ and $({}^h b \leq y, h) = ({}^e({}^g b) \leq y, e) \circ ({}^h b \leq {}^g b, h)$. Because $({}^h b \leq {}^g b, h) \in F_b({}^g b)$, it follows that $({}^h b \leq y, h)$ is in the image of α, so α is surjective.

We now show that α is injective. Note that in $\bigoplus_{g \in N(b,y)} F_b({}^g b)$, we have $({}^{h_1} b \leq {}^{g_1} b, h_1) = ({}^{h_2} b \leq {}^{g_2} b, h_2)$ if and only if $h_1 = h_2$. Thus, $\alpha(({}^{h_1} b \leq {}^{g_1} b, h_1)) = \alpha(({}^{h_2} b \leq {}^{g_2} b, h_2))$ only if $h_1 = h_2$, so α is injective. □

It is important to note that if $\beta : \bigoplus_{g \in N(b,y)} F_b(-) \to \bigoplus_{g \in N(b,y)} F_b(-)$ is a direct sum of natural transformations (i.e. $k\mathcal{P} \rtimes G$-module homomorphisms) $F_b(-) \to F_b(-)$, then $\alpha \circ \beta = \beta \circ \alpha$ because the component maps of β are given by pre-composition and those of α are given by post-composition.

We continue with results regarding induction and coinduction from clamped subcategories. The immediate goal is to prove that induction and coinduction coincide when applied to complexes in $D^b(k[a,b] \rtimes G_b)$ with homology supported on (a,b). This is a key result which we use later to obtain specific structural information about the Auslander-Reiten quiver of $k\mathcal{P} \rtimes G$.

In the following proposition, we will use the notation $\uparrow_{-}^{-}$ to mean both induction in the module category and the total left derived functor of induction in the homotopy category of chain complexes.

Proposition 3.0.9. *Let k be a field, let $\mathcal{P}$ be a G-poset, let $[a,b] \subset \mathcal{P}$, and let Π be a complex of projective $k[a,b] \rtimes G_b$-modules with homology supported on $[a,b)$. Then $\Pi \uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G}$ has homology supported on ${}^G[a,b)$, the G-orbits of $[a,b)$.*

Proof. The complex $\Pi \uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G}$ has terms of the form $P_{x,W} \uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G} \cong P_{x,W}^{\mathcal{P}\rtimes G}$ where $x \in [a,b]$ and W is a kG_x-module. A projective $k\mathcal{P} \rtimes G$-module P generated by its value at $x \in [a,b]$ takes nonzero values only on $(\mathcal{P} \rtimes G)_{\geq x}$. If ${}^g x \leq y$ then ${}^g b \leq y$ or $y \leq {}^g b$ because ${}^g[a,b]$ is clamped. If $y \leq {}^g b$, then $y \in {}^g[a,b]$ and $\Pi \uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G} (y)$ is exact if and only if $\Pi \uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G} ({}^{g^{-1}} y)$ is exact if and only if $\Pi({}^{g^{-1}} y)$ is exact. This shows that $\Pi \uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G} ({}^g b)$ is exact for all $g \in G$, and $\Pi \uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G}$ has zero homology outside of ${}^G[a,b) \cup (\mathcal{P} \rtimes G)_{\geq b}$.

Now suppose $y \in (\mathcal{P} \rtimes G)_{\geq b}$ and consider $\Pi \uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G} \downarrow_{(\mathcal{P}\rtimes G)_{\geq b}}^{\mathcal{P}\rtimes G}$. By 3.0.7, this complex is isomorphic to a complex $\tilde{\Pi}$ of projective $k(\mathcal{P} \rtimes G)_{\geq b}$-modules where each projective term is generated at b. Because $\Pi(b)$ is acyclic by assumption, so is $\tilde{\Pi}(b)$. We may replace $\tilde{\Pi}$ with a complex $F_\bullet$ of representable $k(\mathcal{P} \rtimes G)_{\geq b}$-modules where each term is generated by its value at b. By Proposition 3.0.8, there is an isomorphism of graded vector spaces

$$\tilde{\alpha} : \bigoplus_{g \in N(b,y)} F_\bullet({}^g b) \to F_\bullet(y).$$

where here, the grading is determined by the position in the complex. We wish to show that this is an isomorphism of complexes, which we proceed to do.

Note that the map α in 3.0.8 has the following property: If $\beta^{(k)} : \mathrm{Hom}(b, -) \to$

Hom$(b, -)$ is a set of natural transformations and

$$\overline{\beta} = \bigoplus_{g \in N(b,y)} \beta_g^{(k)}$$

is a direct sum with summands consisting of components of the $\beta^{(k)}$, then $\alpha \circ \overline{\beta} = \overline{\beta} \circ \alpha$. This is because the components of α are given by post-composition, while the natural transformations $\beta^{(k)}$ are given by precomposition by the Yoneda lemma, and these two operations commute by associativity of composition.

To finish the proof of this proposition, note that the morphisms in the chain complex $\bigoplus_{g \in N(b,y)} F_\bullet({}^g b)$ can be viewed as components of natural transformations between functors $F_b \to F_b$. By the comments in the previous paragraph, the map $\tilde{\alpha}$ commutes with these maps, so $\tilde{\alpha}$ is in fact an isomorphism of complexes. By assumption, the complex on the left is acyclic, so $F_\bullet(y)$ is acyclic. Thus, $\tilde{\Pi}(y)$ is acyclic for all y greater than any conjugate of b, and we are done. □

Proposition 3.0.9 is important in establishing that a certain class of complexes is induced from a clamped interval.

Proposition 3.0.10. *Let k be a field, let M be a complex of $k(\mathcal{P} \rtimes G)-$modules, and let $[a, b]$ be clamped in $\mathcal{P}$. Suppose M has homology supported on $[a, b) \rtimes G$. Then a minimal projective resolution Π_M of M consists only of projective terms of the form $P_{x,W}$ where $x \in [a, b]$. Moreover, M is induced from $[a, b] \rtimes G_b$ in $D^b(k\mathcal{P} \rtimes G)$, and the counit $M \downarrow^{\mathcal{P} \rtimes G}_{[a,b] \rtimes G_b} \uparrow^{\mathcal{P} \rtimes G}_{[a,b] \rtimes G_b} \to M$ is an isomorphism.*

Proof. Let $\Pi_M^{[a,b] \rtimes G_b}$ be a minimal projective resolution of $M \downarrow^{\mathcal{P} \rtimes G}_{[a,b] \rtimes G_b}$. Then $\Pi_M^{[a,b] \rtimes G_b}$ is a complex of projectives of the form $P_{x,W}^{[a,b] \rtimes G_b}$ where $x \in [a, b)$, being isomorphic to $M \downarrow^{\mathcal{P} \rtimes G}_{[a,b] \rtimes G_b}$ in $D^b(k[a, b] \rtimes G_b)$. By proposition 3.0.9, we have that $\Pi^{[a,b] \rtimes G_b} \uparrow^{\mathcal{P} \rtimes G}_{[a,b] \rtimes G_b}$ has homology supported on ${}^G[a, b)$ as well.

Now by basic properties of induction from full subcategory algebras, we have $\Pi^{[a,b] \rtimes G_b} \uparrow^{\mathcal{P} \rtimes G}_{[a,b] \rtimes G_b} \cong \Pi_M^{[a,b] \rtimes G_b}(x)$ for all $x \in [a, b]$, and by proposition 3.0.9, we have $\Pi^{[a,b] \rtimes G_b} \uparrow^{\mathcal{P} \rtimes G}_{[a,b] \rtimes G_b} (y) = M(y)$ is acyclic for all $y \notin {}^G[a, b]$. Note that $M \downarrow^{\mathcal{P} \rtimes G}_{[a,b] \rtimes G_b} \uparrow^{\mathcal{P} \rtimes G}_{[a,b] \rtimes G_b} \cong \Pi_M^{[a,b] \rtimes G_b} \uparrow^{\mathcal{P} \rtimes G}_{[a,b] \rtimes G_b}$.

By the last observation, we have a map $\varepsilon_M : \Pi_M^{[a,b]\rtimes G_b} \uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G} \to M$ which is the counit of the adjunction. This map is an isomorphism at each object $x \in {}^G[a,b]$, so ε_M must be an isomorphism because both $\Pi_M^{[a,b]\rtimes G_b} \uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G}$ and M are zero outside of ${}^G[a,b]$. □

We now turn our attention to coinduction. Coinduction between the module categories is defined by $M \Uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G} := \mathrm{Hom}_{k[a,b]\rtimes G_b}(k\mathcal{P} \rtimes G, M)$ and is the right adjoint to restriction. Between the bounded derived categories, coinduction refers to the total left derived functor of coinduction in the module categories. It is obtained by applying $\Uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G}$ to each term in an injective resolution of the complex in question. Our main goal is to prove that induction and coinduction coincide for a certain class of complexes in $D^b(k[a,b] \rtimes G_b)$. To this end, we must show the following:

Proposition 3.0.11. *Let $I_\bullet$ be a complex of injective $k[a,b] \rtimes G_b$-modules with homology supported on $(a,b]$. Further, assume that each module in the complex is a finite dimensional k-vector space. Then $I_\bullet \Uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G}$ has homology supported on ${}^G(a,b]$.*

This proposition can be considered the dual of proposition 3.0.9. We will include a proof of this statement here which uses the uniqueness of an adjoint. We begin with a lemma.

Lemma 3.0.12. *Let $\mathcal{P}$ be a G-poset. Then there is an isomorphism of categories $(\mathcal{P} \rtimes G)^{op} \cong \mathcal{P}^{op} \rtimes G^{op}$, where the action of $g \in G^{op}$ on $x \in \mathcal{P}^{op}$ is the same as the action of g^{-1} on $x \in \mathcal{P}$.*

Proof. Let $*$ denote multiplication in G^{op} and let $\cdot$ denote multiplication in G. We have

$$\begin{aligned}\mathrm{Hom}_{(\mathcal{P}\rtimes G)^{\mathrm{op}}}(x,y) =& \mathrm{Hom}_{\mathcal{P}\rtimes G}(y,x)\\ =&\{({}^g y \leq x, g)\\ =&\{({}^{g^{-1}} x \geq y, g)\}.\end{aligned}$$

This shows that the "correct" action of $g \in G^{op}$ on $x \in \mathcal{P}^{op}$ is that of g^{-1} on $x \in \mathcal{P}$.

Composition of morphisms in $(\mathcal{P} \rtimes G)^{op}$ is given by

$$\begin{aligned}({}^{h^{-1}}y \geq z, h) \underset{(\mathcal{P}\rtimes G)^{op}}{\circ} ({}^{g^{-1}}x \geq y, g) =& [({}^{g}y \leq x, g) \underset{\mathcal{P}\rtimes G}{\circ} ({}^{h}z \leq y, h)]^{op} \\ =& ({}^{g\cdot h}z \leq x, g \cdot h)^{op} \\ =& ({}^{h^{-1}\cdot g^{-1}}x \geq z, g \cdot h) \\ =& ({}^{h^{-1}\cdot g^{-1}}x \geq z, h * g).\end{aligned}$$

Letting G^{op} act on $\mathcal{P}^{op}$ as above, we see that the composition rule for $(\mathcal{P} \rtimes G)^{op}$ is the same as that of $\mathcal{P}^{op} \rtimes G^{op}$. This shows that $(\mathcal{P} \rtimes G)^{op} \cong \mathcal{P}^{op} \rtimes G^{op}$. □

We will now turn to the proof of proposition 3.0.13.

Proof. Let $I_\bullet$ be a complex of injective $k[a,b] \rtimes G_b$-modules with homology supported on $(a,b]$. We will start by proving that applying the vector space dual, inducing, and applying the dual again to $I_\bullet$ gives us a complex of injective $k\mathcal{P} \rtimes G$-modules with homology supported on ${}^{G}(a,b]$. We will then show that this procedure of dualizing, inducing, and dualizing again is the same as coinducing. For the sake of simplicity and to avoid clutter, let $C = \mathcal{P} \rtimes G$ and let $\mathcal{D} = [a,b] \rtimes G_b$. Note that $\mathcal{D}^{op} = [b,a] \rtimes G_a^{op}$, and it is convenient to use the fact that $G_a = G_b$ whenever $[a,b]$ is clamped. Thus $\mathcal{D}^{op}$ is a clamped subcategory of C^{op}.

To this end, consider the dual complex $I_\bullet^*$, where $(-)^*$ is the derived functor of $\mathrm{Hom}_k(-,k)$. Then by lemma 3.0.12, $I_\bullet^*$ is a complex of projective left $k\mathcal{D}^{op}$-modules.

Note that $I_\bullet^*$ has homology supported on $[b,a)$. By proposition 3.0.9, we have that $I_\bullet^* \uparrow_{\mathcal{D}^{op}}^{C^{op}}$ is a complex of projective kC^{op}-modules with homology supported on $[b,a)$. It follows that $I_\bullet^* \uparrow_{\mathcal{D}^{op}}^{C^{op}}{}^*$ is a complex of injective kC-modules with homology supported on $(a,b]$.

We now show that we have a natural isomorphism of functors $(-)^* \uparrow_{\mathcal{D}^{op}}^{C^{op}}{}^* \cong (-) \Uparrow_{\mathcal{D}}^{C}$ by showing that the functor on the left is a right adjoint of the restriction functor $(-) \downarrow_{\mathcal{D}}^{C}$. Let M be a $k\mathcal{P} \rtimes G$-module and let N be a $k[a,b] \rtimes G_b$-module. The proof below shows that this is indeed true if we replace kC with any algebra over a field and $k\mathcal{D}$ with any subalgebra.

By duality and the fact that induction is the left adjoint of restriction, we have

$$\begin{aligned}\mathrm{Hom}_{kC}(M, N^* \uparrow^{C^{op}}_{[\mathcal{D}^{op}} {}^*) &= \mathrm{Hom}_{kC^{op}}(N^* \uparrow^{C^{op}}_{\mathcal{D}^{op}}, M^*)\\ &= \mathrm{Hom}_{k\mathcal{D}^{op}}(N^*, M^* \downarrow^{C^{op}}_{\mathcal{D}^{op}})\\ &= \mathrm{Hom}_{k\mathcal{D}}(M^* \downarrow^{C^{op}}_{\mathcal{D}^{op}} {}^*, N).\end{aligned}$$

It remains to show that $M^* \downarrow^{C^{op}}_{\mathcal{D}^{op}} {}^* \cong M \downarrow^{C}_{\mathcal{D}}$. This follows from the fact that restricting and dualizing commute in the sense that $M^* \downarrow^{C^{op}}_{\mathcal{D}^{op}} \cong M \downarrow^{C}_{\mathcal{D}} {}^*$. From this, and the assumption that all modules in the complex M are finite dimensional k-vector spaces, we have

$$M^* \downarrow^{C^{op}}_{\mathcal{D}^{op}} {}^* \cong M \downarrow^{C}_{\mathcal{D}} {}^{**} \cong M \downarrow^{C}_{\mathcal{D}} .$$

We have shown that $(-)^* \uparrow^{C^{op}}_{\mathcal{D}^{op}} {}^*$ is the right adjoint of the restriction functor $(-) \downarrow^{C}_{\mathcal{D}}$. By the uniqueness of adjoints , we have $(-)^* \uparrow^{C^{op}}_{\mathcal{D}^{op}} {}^* \cong (-) \Uparrow^{C}_{\mathcal{D}}$. We conclude that $I_\bullet \Uparrow^{C}_{\mathcal{D}}$ is a complex of injective kC-modules with homology supported on $(a, b]$. □

Before the next result, we make a definition. Given an object, $x \in \mathcal{P} \rtimes G$ and a kG_x-module W, let $I_{x,W}$ denote the indecomposable injective module with simple socle $S_{x,W}$.

Proposition 3.0.13. *Let N be a complex of $k\mathcal{P} \rtimes G$-modules with homology supported on $(a, b]$. Then a minimal injective resolution I_N of N has terms of the form $I_{x,W}$ where $x \in [a, b]$ and N is coinduced from $[a, b] \rtimes G_b$. Moreover, the unit $N \to N \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b} \Uparrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}$ is a natural isomorphism.*

Proof. The proof of this statement is dual to proposition 3.0.10. □

Together, Proposition 3.0.9 and 3.0.13 imply the following key result.

Theorem 3.0.14. *Let G be a group and $\mathcal{P}$ be a G-poset, and let $[a, b] \rtimes G_b$ be clamped in $\mathcal{P} \rtimes G$. Then the functors $\uparrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}$ and $\Uparrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}$ are naturally isomorphic on objects with homology supported on the open interval (a, b).*

Proof. We need only establish that every object in $D^b(k[a,b] \rtimes G_b)$ with homology supported on $(a,b) \rtimes G_b$ is the restriction of a complex in $D^b(k\mathcal{P} \rtimes G)$. Let

$$M = \cdots M_{i+1} \to M_i \to M_{i-1} \to \cdots$$

be a complex in $D^b(k[a,b] \rtimes G_b)$ with homology supported on $(a,b) \rtimes G_b$. Define

$$\tilde{M} = \cdots \tilde{M}_{i+1} \to \tilde{M}_i \to \tilde{M}_{i-1} \to \cdots$$

to be the complex in $D^b(k\mathcal{P} \rtimes G)$ where $\tilde{M}_j$ is M_j with zeros extended to objects outside of ${}^G[a,b]$. Observe that the restriction of $\tilde{M}$ to $k[a,b] \rtimes G_b$ is M. Thus we have

$$M \uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G} \cong \tilde{M} \downarrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G} \uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G} \cong \tilde{M} \cong \tilde{M} \downarrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G} \Uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G} \cong M \Uparrow_{[a,b]\rtimes G_b}^{\mathcal{P}\rtimes G} .$$

□

Chapter 4

The left derived functor of ν and its properties

A key tool in analyzing the Auslander-Reiten quiver of $D^b(k\mathcal{P} \rtimes G)$ is the Nakayama functor ν and its left derived functor $\mathbb{L}\nu$. We will be summarizing from [8, Chapter 6.4] for properties of the Nakayama functor and the derived Nakayama functor. As opposed to most of the rest of this document, we do not assume that $|G|$ is invertible in k in this chapter.

Definition 4.0.1. Let A be any finite dimensional algebra over a field k. Let D denote vector space duality. The Nakayama functor, denoted $\nu : A\text{-mod} \to A\text{-mod}$, is the functor $D\operatorname{Hom}_A(-, {}_AA)$.

The functor ν is a covariant right exact functor, being the composition of a contravariant left exact functor and a contravariant exact functor. We will denote the left derived functor of ν by $\mathbb{L}\nu$, or simply by ν when it is clear from context which functor we intend to use.

The Nakayama functor ν has a quasi-inverse, denoted ν^{-1}, defined as $\operatorname{Hom}_A(D(-), {}_AA)$. Both ν and ν^{-1} enjoy several properties. In the module category, if we define P_S and I_S as the projective cover and injective envelope of the simple module S respectively, then $\nu(P_S) = I_S$ and $\nu^{-1}(I_S) = P_S$. In fact, this assignment induces an equivalence of categories

$$\nu : \mathrm{proj}(A) \to \mathrm{inj}(A),$$

where proj(A) is the full subcategory of A-mod generated by the projective A-modules, and inj(A) is the full subcategory generated by the injective modules.

When considering the entire derived category of A-mod, the functor $\mathbb{L}\nu$ shares similar properties. However, the convenient properties of ν are not always shared by $\mathbb{L}\nu$ when we restrict to the *bounded* derived category. Here are some complications:

1. The functor $\mathbb{L}\nu$ does not necessarily take complexes with bounded homology to complexes with bounded homology. This is guaranteed when $k\mathcal{P} \rtimes G$ has finite global dimension, but this occurs if and only if $\operatorname{char} k \nmid |G|$ by standard facts about skew group rings.

2. It is not immediately clear that perfect complexes, i.e. complexes which are isomorphic to a finite complex of projectives in $D^b(k\mathcal{P} \rtimes G)$, are mapped to perfect complexes by $\mathbb{L}\nu$.

To address point 1, we will restrict the examination to the Auslander-Reiten quiver of the full subcategory of $D^b(k\mathcal{P} \rtimes G)$ generated by the perfect complexes. To do this, we must first address point 2 by showing that $k\mathcal{P} \rtimes G$ is always Iwanaga-Gorenstein.

Definition 4.0.2. A Noetherian algebra A with a 1 is called *Iwanaga-Gorenstein* if each injective has finite projective dimension and each projective module has finite injective dimension.

We will only show that each injective module has finite projective dimension. The proof that projective modules have finite injective dimension is dual.

Lemma 4.0.3. *Let $I_{x,W}$ be an indecomposable injective $k\mathcal{P} \rtimes G$-module. Then for each $y \in \mathcal{P} \rtimes G$, we have that $I_{x,W}(y)$ is zero or a projective kG_y-module.*

Proof. If ${}^g y \not\leq x$ for all $g \in G$, then $Dk\operatorname{Hom}_{\mathcal{P}\rtimes G}(y, x) = 0$, and thus any injective of the form $I_{x,W}$, being a summand of $Dk\operatorname{Hom}_{\mathcal{P}\rtimes G}(-, x)$, is also 0 upon evaluation at y.

Suppose ${}^{g}y \le x$ for some $g \in G$. Then $Dk\,\mathrm{Hom}_{\mathcal{P}\rtimes G}(y, x)$ is free as a left kG_y-module always. Thus $I_{x,W}(y)$ is a summand of a free kG_y-module, so it is projective as a kG_y-module. □

For the next lemma, which will prove that injectives have finite projective dimension, we will use the notion of the support of a module.

Definition 4.0.4. Let M be a $k\mathcal{P} \rtimes G$-module. The support of M, denoted $\mathrm{supp}(M)$, is the set of objects $x \in \mathcal{P} \rtimes G$ such that $M(x) \neq 0$.

We now proceed with the proposition.

Proposition 4.0.5. *Let M be any $k\mathcal{P} \rtimes G$-module with the property that for all $x \in \mathcal{P} \rtimes G$, the term $M(x)$ is either zero or a projective kG_x-module. Then M has finite projective dimension.*

We will proceed by induction on $n = \#(\mathcal{P}_{\ge \mathrm{supp(M)}})$, the number of objects in $\mathcal{P}_{\ge \mathrm{supp(M)}}$. The base case is $n = 0$, which implies $M = 0$, and M has finite projective dimension.

Consider the projective cover $\varphi : P_M \to M$. We claim that the projective cover φ descends to an isomorphism $\varphi_x : P_M(x) \to M(x)$ on all minimal elements $x \in \mathrm{supp}(M)$. Such maps are surjective because φ is surjective. For injectivity, let $N_x = \ker \varphi_x$, and assume $N_x \neq 0$. Note that because x is minimal in $\mathrm{supp}(M)$, it follows that $M/\operatorname{Rad} M(x)$ is nonzero. Because $M(x)$ is a projective kG_x-module, it follows that $P_M(x) \cong M(x) \oplus N_x$. This means that the kernel of $P_M/\operatorname{Rad} P_M \to M/\operatorname{Rad} M$ is nonzero, as it contains the elements of $N_x/\operatorname{Rad} N_x$ regarded as elements of $\ker \varphi|_{P_M/\operatorname{Rad} P_M}$, and this contradicts the assumption that φ is a projective cover.

Now $\mathcal{P}_{\ge \mathrm{supp}(P_M)} = \mathcal{P}_{\ge \mathrm{supp}(M)}$, as these posets have the same set of minimal elements. Let $\Omega(M)$ denote the first syzygy of M, defined as the kernel of φ. The module $\Omega(M)$ has support precisely that of P_M, excluding the objects x for which φ_x is an isomorphism. This set of objects is nonempty, as it contains the minimal elements of $\mathcal{P}_{\ge \mathrm{supp}(M)}$. Thus we have

$$\#(\mathcal{P}_{\ge \mathrm{supp}(\Omega(M))}) < \#(\mathcal{P}_{\ge \mathrm{supp}(M)}).$$

Finally, observe that for all x, the value of $\Omega(M)(x)$ is the kernel of φ_x. This is always a projective kG_x-module or 0 because the image and domain of φ_x are 0 or projective. By the induction hypothesis, the module $\Omega(M)$ has finite projective dimension. If $P_\bullet$ denotes a finite projective resolution of $\Omega(M)$, then $\mathcal{P}_\bullet \to P_M$ is a finite projective resolution of M. This concludes the proof.

Remark 4.0.6. It is interesting to note that this proposition actually characterizes the $k\mathcal{P} \rtimes G$-modules of finite projective dimension. Indeed, if $M(x)$ is nonzero and not projective, then any projective resolution of M descends to a projective kG_x-resolution of $M(x)$, and it is well known that the only nonzero kG_x-modules with finite projective dimension are the projective kG_x-modules themselves.

Putting the proposition and the lemma together, we have that the injective $k\mathcal{P} \rtimes G$-modules have finite projective dimension. To get that the projective $k\mathcal{P} \rtimes G$ modules have finite injective dimension, note that each projective $k\mathcal{P} \rtimes G$-module is the dual of an injective $k\mathcal{P}^{op} \rtimes G^{op}$-module. We use the same argument to obtain a projective resolution of this injective $k\mathcal{P}^{op} \rtimes G^{op}$-module, then dualizing the complex yields an injective resolution of the original projective $k\mathcal{P} \rtimes G$-module.

This result can be used to show the following:

Theorem 4.0.7. *The set of perfect complexes in $D^b(k\mathcal{P} \rtimes G)$ is equal to the set of complexes isomorphic to a finite complex of injective modules.*

Proof. This follows from Theorem 6.4.6 and its proof in [8]. □

This result implies that $\mathbb{L}\nu$ takes perfect complexes to perfect complexes. It then makes sense to study the Auslander-Reiten quivers of $D^b(k\mathcal{P} \rtimes G)$, restricted to the perfect complexes. When $k\mathcal{P} \rtimes G$ has finite global dimension, this is the entirety of $D^b(k\mathcal{P} \rtimes G)$. Otherwise, the set of perfect complexes is strictly smaller.

4.1 The Serre functor

The perfect complexes form a triangulated subcategory of $D^b(k\mathcal{P} \rtimes G)$. When we restrict to the subcategory of perfect complexes, the functor $\mathbb{L}\nu[-1]$ becomes a Serre functor.

Definition 4.1.1. A Serre functor $S : C \to C$ on a triangulated category C is a self-equivalence of triangulated categories with the property that for all objects $A, B \in C$, we have a natural duality

$$\mathrm{Hom}_C(A, B) \cong \mathrm{Hom}_C(B, S(A))^*.$$

The functor $\mathbb{L}\nu$ does indeed have these properties, and in fact Serre functors, if they exist, are unique up to natural isomorphism. We are then justified in calling $\mathbb{L}\nu$ *the* Serre functor in this context.

The Serre functor and its quasi-inverse share a similar property: If $P_\bullet$ is a complex of projective modules where the projective in homological degree i is P_{M_i}, the projective cover of some module M_i, then $\nu(P_\bullet)$ has the module I_{M_i}, the injective envelope of M_i, in homological degree i.

With these properties in mind, we will investigate the Serre functor's relationship to induction, coinduction, and restriction.

Proposition 4.1.2. *Let G be a finite group and $\mathcal{P}$ be a finite G-poset. Suppose $[a, b]$ is clamped in $\mathcal{P}$, and let M be a complex in $D^b(k\mathcal{P} \rtimes G)$. Then*

1. *M has homology supported on ${}^G[a, b)$ if and only if νM has homology supported on ${}^G(a, b]$.*
2. *If M has homology supported on ${}^G[a, b)$, then*

$$(\nu M) \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b} \cong \nu(M \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}).$$

Dually, if N has homology supported on ${}^G(a, b]$, then

$$(\nu^{-1} N) \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b} \cong \nu^{-1}(N \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}).$$

Proof. Suppose that M has homology supported on ${}^G[a, b)$, and let Π_M be a projective resolution of M. By proposition 3.0.9, the complex Π_M has terms of the form $P_{x,W}$

where $x \in [a, b]$. We may replace some terms of Π_M to get an isomorphic complex with terms of the form $F_x := k\operatorname{Hom}_{\mathcal{P} \rtimes G}(x, -)$ with $x \in [a, b]$. We will also call this complex Π_M. Define $I_x := Dk\operatorname{Hom}_{\mathcal{P} \rtimes G}(-, x) = \nu F_x$, where D denotes vector space duality. Clearly, the complex $\nu\Pi_M$ consists of terms of the form I_x with $x \in [a, b]$. We want to show that $\Pi_M(b)$ is acyclic if and only if $\nu\Pi_M(a)$ is acyclic.

Note that a basis for $I_x(a)$ is $\{\delta_{({}^g a \leq x, g)} \mid g \in G_b\}$. We thus have an isomorphism of k-vector spaces

$$\phi_x : F_x(b) \cong (\nu F_x)(a) \cong I_x(a),$$

$$({}^g x \leq b, g) \mapsto \delta_{({}^{g^{-1}} a \leq x, g^{-1})}.$$

Our goal is to show that the isomorphisms ϕ_x assemble into a chain isomorphism $\Pi_M(b) \cong (\nu\Pi_M)(a)$. By the Yoneda lemma, any map $F_x \to F_y$ is induced by some element $\alpha \in k\operatorname{Hom}_{\mathcal{P} \rtimes G}(y, x)$. Explicitly, the map $\alpha^* : F_x \to F_y$ is given by $\beta \mapsto \beta \circ \alpha$, where here $\beta \circ \alpha$ is multiplication in the category algebra.

For simplicity, assume $\alpha = ({}^h y \leq x, h)$ in $\operatorname{Hom}_{\mathcal{P} \rtimes G}(y, x)$. We now show that the diagram

$$\begin{array}{ccc} F_x(b) & \xrightarrow{\alpha_b^*} & F_y(b) \\ \downarrow \phi_x & & \downarrow \phi_y \\ I_x(a) & \xrightarrow{(\nu\alpha^*)_a} & I_y(a) \end{array}$$

commutes. Note that

$$\nu\alpha^* = D\operatorname{Hom}_{k\mathcal{P} \rtimes G}(\alpha^*, k\mathcal{P} \rtimes G) = D\alpha_*,$$

where $\alpha_* : \operatorname{Hom}_{\mathcal{P} \rtimes G}(-, x) \to \operatorname{Hom}_{\mathcal{P} \rtimes G}(-, y)$ is the map $\gamma \mapsto \alpha \circ \gamma$.

Let's examine the map $(D\alpha_*)_a$ more carefully. It sends an element $\delta_{({}^g a \leq x, g)} \in I_x(a)$ to $\delta_{({}^g a \leq x, g)} \circ (\alpha_*)_a$, which is the k-linear map sending $({}^k a \leq y, k) \in k\operatorname{Hom}_{\mathcal{P} \rtimes G}(a, y)$ to

$$\delta_{({}^g a \leq x, g)}(({}^h y \leq x, h) \circ ({}^k a \leq y, k)) = \begin{cases} 1 \text{ if } k = h^{-1}g \\ 0 \text{ otherwise.} \end{cases}$$

We conclude from this that

$$(\nu\alpha^*)_a(\delta_{({}^g a\leq x,g)}) = (D\alpha_*)_a(\delta_{({}^g a\leq x,g)}) = \delta_{({}^{h^{-1}g}a\leq y,h^{-1}g)}.$$

On the other hand, we have

$$\begin{aligned}\phi_y\alpha_b^*\phi_x^{-1}(\delta_{({}^g a\leq x,g)}) &= \phi_y\alpha_b^*(({}^{g^{-1}}x\leq b,g)) \\ &= \phi_y(({}^{g^{-1}h}y\leq b,g^{-1}h)) \\ &= \delta_{({}^{h^{-1}g}y\leq b,h^{-1}g)}.\end{aligned}$$

It follows that $(\nu\alpha^*)_a = \phi_y\alpha_b^*\phi_x^{-1}$, so the diagram commutes in this case. When $\alpha \in k\,\mathrm{Hom}_{\mathcal{P}\rtimes G}(y,x)$ is arbitrary, then α is a k-linear combination of maps of the form $({}^h y \leq x, h)$. We then have $(\nu\alpha^*)_a = \phi_y\alpha_b^*\phi_x^{-1}$ in this case because ν, ϕ_y and ϕ_x are all k-linear. We conclude from all this that $\nu\Pi_M(a) \cong \Pi_M(b)$, and so $\nu\Pi_M(a)$ is acyclic because we assumed that $\Pi_M(b)$ was acyclic.

We now argue that for all $z \leq a$, the complex $\nu\Pi_M(z)$ is acyclic. This comes from a statement similar to proposition 5.7: let $M(z,a)$ be a set of coset representatives $g \in [G/G_a]$ satisfying $z \leq {}^g a$. There is an isomorphism

$$\beta : \bigoplus_{g\in M(z,a)} \mathrm{Hom}_{\mathcal{P}\rtimes G}({}^g a, a) \to \mathrm{Hom}_{\mathcal{P}\rtimes G}(z,a),$$

where the component maps $\beta_g : \mathrm{Hom}_{\mathcal{P}\rtimes G}({}^g a, a) \to \mathrm{Hom}\,\mathcal{P}\rtimes G(z,a)$ are given by pre-composition with $({}^e z \leq {}^g a, e)$. The proof of this statement is similar to the proof of proposition 5.7. The "only if" part of (a) is dual, and (a) is proved.

For (b), note that the projectives appearing in the minimal projective resolution of M have the same labels as those appearing in the minimal projective resolution of $M \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}$ because M is induced from $[a,b]\rtimes G_b$. Because the Nakayama functor takes $P_{x,W}$ to $I_{x,W}$ in both $k\mathcal{P}\rtimes G$-mod and $k[a,b]\rtimes G_b$-mod, and $P^{\mathcal{P}\rtimes G}_{x,W} \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b} \cong P^{[a,b]\rtimes G_b}_{x,W}$ and $I^{\mathcal{P}\rtimes G}_{x,W} \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b} \cong I^{[a,b]\rtimes G_b}_{x,W}$, it follows that restriction and the Serre functor commute. A similar argument shows that $(\nu^{-1}N) \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b} \cong \nu^{-1}(N \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b})$. □

Chapter 5

Auslander-Reiten triangles and Auslander-Reiten quivers

We now begin the process of calculating Auslander-Reiten quivers for the bounded derived category and the module category of some transporter category algebras containing clamped intervals. Recall that Auslander-Reiten triangles are completely determined by the first or third term. Thus, we refer to an Auslander-Reiten triangle

$$L \to M \to N \to L[1]$$

as the Auslander-Reiten triangle beginning at L or ending in N. We start by recalling the following, which shows that Auslander-Reiten triangles exist.

Proposition 5.0.1 (Happel [7]). *Let A be an algebra, and let $M \in D^b(A)$ be indecomposable. Then the Auslander-Reiten triangle beginning at M exists if and only if M is isomorphic to a finite complex of injective modules, and the Auslander-Reiten triangle ending at M exists if and only if M is isomorphic to a finite complex of projective modules.*

We now recall the structure of the Auslander-Reiten triangles.

Proposition 5.0.2. *Let A be a finite dimensional algebra over a field, and let $M \in D^b(A)$*

be indecomposable. Then the Auslander-Reiten triangle ending at M has the form

$$\nu M[-1] \to N \to M \to \nu M,$$

and the Auslander-Reiten triangle beginning at M has the form

$$M \to L \to \nu^{-1} M[1] \to M[1].$$

The term N in the first equation is obtained by taking the mapping cone of a morphism $M \to \nu M$ which lies in the socle of the $\mathrm{End}(M)$-module $\mathrm{Hom}(M, \nu M)$.

The morphisms $\nu M[-1] \to N$ and $N \to M$ are direct sums of irreducible morphisms, i.e. morphisms f which are neither sections nor retractions, and $f = g \circ h$ implies that h is a section or g is a retraction. We define the space $\mathrm{Irr}_C(X, Y)$ Moreover, up to reasonable equivalence, every irreducible morphism arises in an Auslander-Reiten triangle. The commutativity of ν with $\downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}$ allows us to make a concrete connection between the Auslander-Reiten triangles appearing in $k\mathcal{P} \rtimes G$ and those appearing in $k[a, b] \rtimes G_b$. For the following result, let $\tau = \nu[-1]$ denote the Auslander-Reiten translate. The Auslander-Reiten triangles in $D^b(k\mathcal{P} \rtimes G)$ are of the form $\tau N \to M \to N \to \tau N[1]$ where M and N are indecomposable complexes. We assemble the indecomposable complexes and irreducible morphisms between them into a quiver, forming the Auslander-Reiten quiver of the bounded derived category. In this case, the Auslander-Reiten translate τ makes this quiver into a translation quiver. The following corollary and the proofs of its parts extend Corollary 2.3 in Diveris, Purin, and Webb [5].

Corollary 5.0.3. *Let $[a, b] \rtimes G_b$ be a clamped subcategory in $\mathcal{P} \rtimes G$ and let M be a perfect complex in $D^b(k\mathcal{P} \rtimes G)$ with homology supported on ${}^G[a, b)$, the G-orbits of $[a, b)$.*

1. *For any $N \in D^b(k\mathcal{P} \rtimes G)$, we have*

$$\mathrm{Hom}_{D^b(k\mathcal{P}\rtimes G)(M,N)} \cong \mathrm{Hom}_{D^b(k[a,b]\rtimes G_b)}(M \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}, N \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b})$$

2. *There is a ring isomorphism*

$$\mathrm{End}_{D^b(k\mathcal{P}\rtimes G)}(M) \cong \mathrm{End}_{D^b(k[a,b]\rtimes G_b)}(M \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}).$$

Via this isomorphism, we have an isomorphism of $\mathrm{End}_{D^b(k\mathcal{P}\rtimes G)}(M)$*-modules*

$$\mathrm{Hom}_{D^b(k\mathcal{P}\rtimes G)}(M, \nu M) \cong \mathrm{Hom}_{D^b(k[a,b]\rtimes G_b)}(M \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}, \nu M \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}).$$

3. *We have* $\tau(M \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}) \cong (\tau M) \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}$.

4. *If M is indecomposable, then the Auslander-Reiten triangle ending at* $M \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}$ *is the restriction of the triangle ending at M. The same triangle in* $D^b(k[a,b] \rtimes G_b)$ *is also the restriction of the triangle starting at τM.*

Proof. 1. The complex M is induced from $[a,b] \rtimes G_b$, i.e. $M \cong M \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}\uparrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}$. Thus we have

$$\begin{aligned}\mathrm{Hom}_{D^b(k\mathcal{P}\rtimes G)}(M, N) &\cong \mathrm{Hom}_{D^b(k\mathcal{P}\rtimes G)}(M \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}\uparrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}, N)\\ &\cong \mathrm{Hom}_{D^b(k[a,b]\rtimes G_b)}(M \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}, N \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b})\end{aligned}$$

by the adjoint property.

2. Both statements follow from (1) and the fact that the isomorphism in the adjunction is natural in both variables.

3. The functor τ is the composite of ν and a shift, both of which commute with restriction.

4. The Auslander-Reiten triangle ending in M is computed by taking the mapping cone of a homomorphism $M \to \nu M$ which lies in the socle of $\mathrm{Hom}_{D^b(k\mathcal{P}\rtimes G)}(M, \nu M)$, regarded as an $\mathrm{End}_{D^b(k\mathcal{P}\rtimes G)}(M)$-module. By (2), this homomorphism restricts to a homomorphism in the socle of $\mathrm{Hom}_{D^b(k[a,b]\rtimes G_b)}(M \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}, \nu M \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b})$. Moreover, $\mathrm{End}_{k\mathcal{P}\rtimes G}(M)$ is local, so $\mathrm{End}_{k[a,b]\rtimes G_b}(M \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b})$ is local and thus $M \downarrow^{\mathcal{P}\rtimes G}_{[a,b]\rtimes G_b}$

is indecomposable. The mapping cone construction also commutes with restriction, so $\tau M \downarrow \to N \downarrow \to M \downarrow \to \tau M[1] \downarrow$ is an Auslander-Reiten triangle in $D^b(k[a,b] \rtimes G_b)$ and is obtained by restricting the triangle ending in M to $k[a,b] \rtimes G_b$. We note that Auslander-Reiten triangles are completely determined by their start or end, so the triangle starting at τM is the same as the one ending at M.

□

Corollary 5.0.3, particularly item (4) in that result, implies that some of the Auslander-Reiten triangles in $D^b(k[a,b] \rtimes G_b)$ are copied into the Auslander-Reiten quiver of $D^b(k\mathcal{P} \rtimes G)$.

Corollary 5.0.4. *The regions of the Auslander-Reiten quiver of $D^b(k[a,b] \rtimes G_b)$ containing the meshes whose rightmost terms have homology supported on $[a,b)$ and whose leftmost terms have homology supported on $(a,b]$ are the restrictions of regions in the Auslander-Reiten quiver of $D^b(k\mathcal{P} \rtimes G)$. Said another way, the regions of the Auslander-Reiten quiver of $D^b(k[a,b] \rtimes G_b)$ with the property above are copied into the Auslander-Reiten quiver of $D^b(k\mathcal{P} \rtimes G)$ by extending the modules appearing in those complexes by 0 outside of ${}^G[a,b]$.*

Before proceeding with an example, here is a useful lemma we will often use when calculating transposes of $k\mathcal{P} \rtimes G$-module homomorphisms, and its applications go into the theory of category algebras in general. While this is a straightforward application of the Yoneda lemma, we were unable to find a reference in the literature.

Lemma 5.0.5. *Let C be a category with finitely many morphisms, and let k be a field. Let $F_x = k\operatorname{Hom}_C(x,-)$ and $I_x = Dk\operatorname{Hom}_C(-,x)$ where D is the k-linear duality functor, and let $\nu = Dk\operatorname{Hom}_C(-,kC)$ denote the Nakayama functor. Then given a kC-module map $-\circ\varphi : F_x \to F_y$ where $\varphi \in k\operatorname{Hom}_C(y,x)$ is the element inducing the homomorphism, we have $\nu(-\circ\varphi) = D(\varphi\circ -)$ after making the identification $\nu(F_x) \cong I_x$.*

Proof. Set $\Phi = -\circ\varphi$. Applying the functor $\operatorname{Hom}_C(-,kC)$ to

$$F_x \xrightarrow{\Phi} F_y$$

yields the diagram

$$\mathrm{Hom}_C(F_y, kC) \xrightarrow{-\circ\Phi} \mathrm{Hom}_C(F_x, kC).$$

Now consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_C(F_y, kC) & \xrightarrow{-\circ\Phi} & \mathrm{Hom}_C(F_x, kC) \\ {\scriptstyle\Theta_y}\uparrow & & {\scriptstyle\Theta_x}\uparrow \\ DI_y & \longrightarrow & DI_x \end{array}$$

where the map Θ_y is the isomorphism in the Yoneda lemma mapping $\gamma \in k\,\mathrm{Hom}_C(-, y)$ to $- \circ \gamma$. To describe the bottom map, let $\gamma \in k\,\mathrm{Hom}_C(-, y) = DI_y$. We then have that the bottom map takes γ to

$$\Theta_x^{-1} \circ (- \circ \Phi) \circ \Theta_y(\gamma) = \Theta_x^{-1} \circ (- \circ \Phi)(- \circ \gamma) = \Theta_x^{-1}(- \circ (\varphi \circ \gamma)) = \varphi \circ \gamma.$$

We conclude that the bottom map is $\phi \circ -$, and the result follows after applying the duality functor, D, to the bottom map. □

We now proceed with several examples. In these examples, we will calculate a slice of a component of the Auslander-Reiten quiver.

Definition 5.0.6. Let Γ be a connected translation quiver with translate τ. A *slice* of Γ is a connected set of orbit representatives for the action of τ on Γ.

Thus a slice is a fundamental region in Γ for the action of τ.

We note that in some of the literature, e.g. in [9], there is the related concept of a *sectional path*. This is any path $x_0 \to \cdots \to x_n$ in the quiver such that $x_i \neq \tau x_{i+2}$ for all i with $0 \leq i \leq n-2$.

Example 5.0.7. Let k be a field with $\mathrm{char}(k) \neq 2$, and let $G = \langle g \rangle = C_2$ be the group of order 2. Let

$$\mathcal{P} = \begin{array}{ccc} & \alpha & \\ \diagup & & \diagdown \\ x & & x' \\ | & & | \\ y & & y' \\ \diagdown & & \diagup \\ & \omega & \end{array}$$

where the least element is α and the largest is ω. Let G act on $\mathcal{P}$ by permuting the two chains, and consider the transporter category $\mathcal{P} \rtimes G$. The category $\mathcal{P} \rtimes G$ is equivalent to the full subcategory generated by the objects $\{\alpha, x, y, \omega\}$, so the category algebra for the subcategory is Morita equivalent to the category alegbra $k\mathcal{P} \rtimes G$. Modules for this algebra will be written in the form

$$M = \begin{array}{c} M_\alpha \\ M_x \\ M_y \\ M_\omega \end{array}$$

where M_z is a $k\,\mathrm{End}(z)$-module for $z = \alpha, x, y, \omega$. We start by writing the indecomposable projective and the injective $\mathcal{P} \rtimes G$ modules. When appropriate, denote by k the trivial module for $kG = k\,\mathrm{End}(\alpha) = k\,\mathrm{End}(\omega)$, and let S denote the other simple kG-module where g acts by multiplication by -1.

We now describe the indecomposable projective modules. There are 6 of these, one for each simple module, and they are denoted $P_{\alpha,k}, P_{\alpha,S}, P_x, P_y, P_{\omega,k}$, and $P_{\omega,s}$. The projectives $P_{\alpha,k}$, and $P_{\alpha,S}$ are the indecomposable summands of the representable functor $F_\alpha := k\,\mathrm{Hom}(\alpha, -)$. The module $P_{\alpha,k}$ is acquired by composition on the right with $(\alpha \leq \alpha, e) + (\alpha \leq \alpha, g)$, and $P_{\alpha,S}$ is acquired by composition on the right with $(\alpha \leq \alpha, e) - (\alpha \leq \alpha, g)$. The values of these functors at x, y, and ω are all one dimensional. The value of $P_{\alpha,k}(\omega)$ is the k-vector space with basis $(\alpha \leq \omega, e) + (\alpha \leq \omega, g)$, on which $\mathrm{End}(\omega)$ acts trivially. The value of $P_{\alpha,k}(\omega)$ is the k-vector space with basis $(\alpha \leq \omega, e) - (\alpha \leq \omega, g)$, on which $(\omega \leq \omega, g) \in \mathrm{End}(\omega)$ acts as multiplication by -1.

We conclude that $P_{\alpha,k}$ and $P_{\alpha,S}$ have the following structures:

$$P_{\alpha,k} = \begin{matrix} k \\ k \\ k \\ k \end{matrix} \qquad P_{\alpha,S} = \begin{matrix} S \\ k \\ k \\ S \end{matrix}.$$

The projective $P_x = k\,\mathrm{Hom}(x, -)$ is representable; $P_x(x)$ and $P_x(y)$ are one dimensional, while $P_x(\omega) = \mathrm{span}\{(x \leq \omega, e), (x \leq \omega, g)\}$ is two dimensional and isomorphic to $k\,\mathrm{End}(\omega)$ as a left $k\,\mathrm{End}(\omega)$-module. Thus, $P_x(\omega) \cong k \oplus S$ as a $k\,\mathrm{End}(\omega)$-module. Similarly, the module $P_y(y)$ is one dimensional and $P_y(\omega) \cong k \oplus S$ as a left $k\,\mathrm{End}(\omega)$-module. Thus,

$$P_x = \begin{matrix} 0 \\ k \\ k \\ k \oplus S \end{matrix} \qquad P_y = \begin{matrix} 0 \\ 0 \\ k \\ k \oplus S \end{matrix}.$$

Finally, the projective modules $P_{\omega,k}$ and $P_{\omega,S}$ are the simple summands of $k\,\mathrm{Hom}(\omega, -)$, which is two dimensional, so they have the following structures:

$$P_{\omega,k} = \begin{matrix} 0 \\ 0 \\ 0 \\ k \end{matrix} \qquad P_{\omega,S} = \begin{matrix} 0 \\ 0 \\ 0 \\ S \end{matrix}.$$

To calculate the structure of the indecomposable injective modules, we repeat this process on the opposite category then take the vector space dual. The indecomposable injective modules are

$$I_{\alpha,k} = \begin{matrix} k \\ 0 \\ 0 \\ 0 \end{matrix} \qquad I_{\alpha,S} = \begin{matrix} S \\ 0 \\ 0 \\ 0 \end{matrix} \qquad I_x = \begin{matrix} k \oplus S \\ k \\ 0 \\ 0 \end{matrix}$$

$$I_y = \begin{matrix} k \oplus S \\ k \\ k \\ 0 \end{matrix} \qquad I_{\omega,k} = \begin{matrix} k \\ k \\ k \\ k \end{matrix} \qquad I_{\omega,S} = \begin{matrix} S \\ k \\ k \\ S \end{matrix}.$$

Note that the modules $P_{\alpha,k}$ and $P_{\alpha,S}$ are projective and injective. From the construction of Auslander-Reiten triangles described in [7], the triangles

$$I_{\alpha,k}[-1] \to \mathrm{Rad}(P_{\alpha,k}) \to P_{\alpha,k} \to I_{\alpha,k},$$

$$I_{\alpha,S}[-1] \to \mathrm{Rad}(P_{\alpha,S}) \to P_{\alpha,S} \to I_{\alpha,S},$$

$$P_{\alpha,k} \to P_{\alpha,k}/\operatorname{Soc}(P_{\alpha,k}) \to P_{\omega,k}[1] \to P_{\alpha,k}[1],$$

$$P_{\alpha,S} \to P_{\alpha,S}/\operatorname{Soc}(P_{\alpha,S}) \to P_{\omega,S}[1] \to P_{\alpha,S}[1]$$

are all Auslander-Reiten triangles. From the Auslander-Reiten theory for the module category, there are almost-split sequences

$$0 \to \mathrm{Rad}(P_{\alpha,k}) \to P_{\alpha,k} \oplus \mathcal{H}(P_{\alpha,k}) \to P_{\alpha,k}/\operatorname{Soc}(P_{\alpha,k}) \to 0,$$

$$0 \to \mathrm{Rad}(P_{\alpha,S}) \to P_{\alpha,S} \oplus \mathcal{H}(P_{\alpha,S}) \to P_{\alpha,S}/\operatorname{Soc}(P_{\alpha,S}) \to 0,$$

where $\mathcal{H}(M) := \mathrm{Rad}(M)/\operatorname{Soc}(M)$ denotes the heart of M. Set $M_k = \mathcal{H}(P_{\alpha,k})$, and note that

$$\mathcal{H}(P_{\alpha,k}) = \mathcal{H}(P_{\alpha,S}) = \begin{matrix} 0 \\ k \\ k \\ 0 \end{matrix}.$$

Because the socle quotients of $P_{\alpha,k}$ and $P_{\alpha,S}$ have projective dimension 1 and the radicals have injective dimension 1, it follows from 4.7 in [7] that we have Auslander-Reiten triangles

$$\mathrm{Rad}(P_{\alpha,k}) \to P_{\alpha,k} \oplus M_k \to P_{\alpha,k} / \mathrm{Soc}(P_{\alpha,k}) \to \mathrm{Rad}(P_{\alpha,k})[1],$$

$$\mathrm{Rad}(P_{\alpha,S}) \to P_{\alpha,S} \oplus M_k \to P_{\alpha,S} / \mathrm{Soc}(P_{\alpha,S}) \to \mathrm{Rad}(P_{\alpha,S})[1].$$

To get another Auslander-Reiten triangle, we will use the clamped interval $[x, y]$. Note that in the Auslander-Reiten triangle

$$\begin{matrix} 0 \\ k \end{matrix} \to \begin{matrix} k \\ k \end{matrix} \to \begin{matrix} k \\ 0 \end{matrix} \to \begin{matrix} 0 \\ k \end{matrix}[1]$$

in $D^b(k[x, y])$, the left term has homology supported on $[x, y)$ and the right term has homology supported on $(x, y]$. By Corollary 5.0.4, this triangle lifts to the Auslander-Reiten triangle

$$\begin{matrix} 0 \\ 0 \\ k \\ 0 \end{matrix} \to M_k \to \begin{matrix} 0 \\ k \\ 0 \\ 0 \end{matrix} \to \begin{matrix} 0 \\ 0 \\ k \\ 0 \end{matrix}[1]$$

in $D^b(k\mathcal{P} \rtimes G)$.

We claim to have a complete column of modules in this component of the Auslander-Reiten quiver of $D^b(k\mathcal{P} \rtimes G)$, consisting of $P_{\alpha,k}$, $P_{\alpha,S}$, and M_k. We have already calculated the meshes ending in $P_{\alpha,k}$ and $P_{\alpha,S}$ and the meshes beginning with $\mathrm{Rad}\, P_{\alpha,k}$ and $\mathrm{Rad}\, P_{\alpha,S}$. It remains to show that the mesh ending with M_k has middle term

$S_y \oplus \operatorname{Rad} P_{\alpha,k} \oplus \operatorname{Rad} P_{\alpha,S}$. By direct calculation, we have

$$\begin{aligned} \nu M_k \cong& \nu((P_{\omega,k} \oplus P_{\omega,S}) \to P_x) \\ =& (I_{\omega,k} \oplus I_{\omega,S}) \to I_x \end{aligned}$$

The map in the complex $(P_{\omega,k} \oplus P_{\omega,S} \to P_x)$ is injective, so using the fact that $P_{\omega,k} \oplus P_{\omega,S} = k \operatorname{Hom}(\omega, -)$ is representable, we can assume that the map is $- \circ (x \leq \omega, e)$. By Lemma 5.0.5, applying ν to this map gives $D((x \leq \omega, e) \circ -)$, which is surjective. Thus, $\tau(M_k)$ is a module with structure

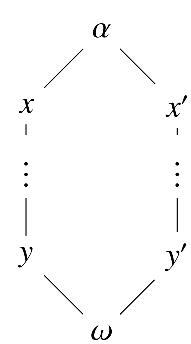
$$\tau(M_k) = \begin{array}{c} 0 \\ k \\ k^2 \\ k \oplus S \end{array} .$$

By additivity, the terms $S_y, \operatorname{Rad} P_{\alpha,k}$, and $\operatorname{Rad} P_{\alpha,S}$ are the only ones in the mapping cone of $M_k \to \nu M_k$. This implies that the column consisting of the terms $P_{\alpha,k}$, $P_{\alpha,S}$, and M_k is complete.

We conclude that the portion of the quiver consisting of $P_{\alpha,k}, \operatorname{Rad} P_{\alpha,k}, P_{\alpha,S}, \operatorname{Rad} P_{\alpha,S}, M_k$, and S_y is a slice for the quiver. It is a Dynkin quiver of type E_6, so we conclude by [13] Theorem 4.15 that this quiver has one component.

Example 5.0.8. We now generalize this example by examining the case where we extend the clamped intervals $[x, y]$ and $[x', y']$ to chains of length n:

$\mathcal{P} =$



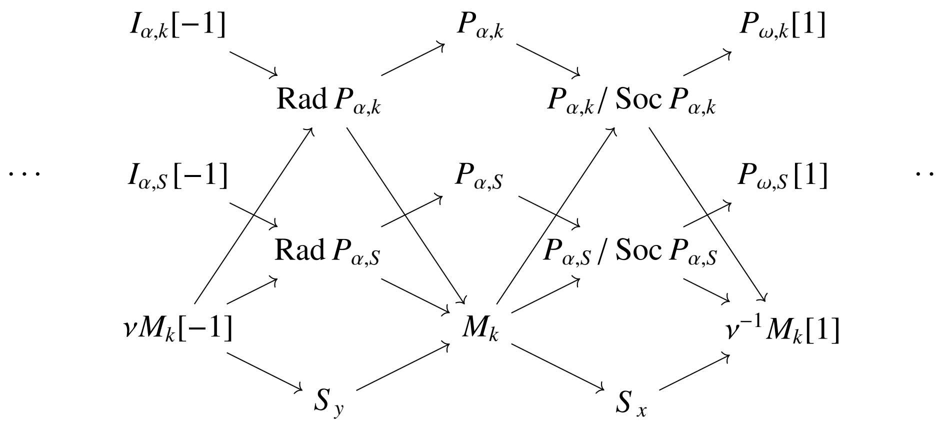


Figure 5.1: A portion of the Auslander-Reiten quiver in Example 5.0.7.

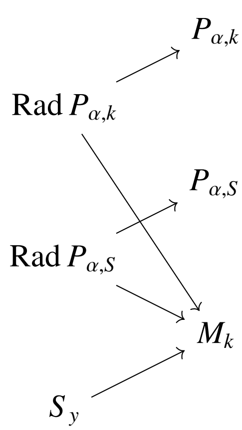


Figure 5.2: A slice of the Auslander-Reiten quiver. It is of type E_6.

Here, $G = C_2 = \langle g \rangle$ acts by permuting the two chains, as before. There are $n+4$ indecomposable projective modules, up to isomorphism, namely $P_{\alpha,k}, P_{\alpha,S}, P_{\omega,k}, P_{\omega,S}$, and P_z for each $z \in [x, y]$. Likewise, there are $n+4$ isomorphism classes of indecomposable injective modules, labelled by the same symbols. Note that in this example, $P_{\alpha,k} = I_{\omega,k}$ and $P_{\alpha,S} = I_{\omega,S}$ are projective-injective.

The calculation for the Auslander-Reiten triangles

$$I_{\alpha,k}[-1] \to \mathrm{Rad}(P_{\alpha,k}) \to P_{\alpha,k} \to I_{\alpha,k},$$

$$I_{\alpha,S}[-1] \to \mathrm{Rad}(P_{\alpha,S}) \to P_{\alpha,S} \to I_{\alpha,S},$$

$$P_{\alpha,k} \to P_{\alpha,k}/\operatorname{Soc}(P_{\alpha,k}) \to P_{\omega,k}[1] \to P_{\alpha,k}[1],$$

$$P_{\alpha,S} \to P_{\alpha,S}/\operatorname{Soc}(P_{\alpha,S}) \to P_{\omega,S}[1] \to P_{\alpha,S}[1]$$

is essentially the same as in the previous example. Similarly,

$$0 \to \mathrm{Rad}(P_{\alpha,k}) \to P_{\alpha,k} \oplus \mathcal{H}(P_{\alpha,k}) \to P_{\alpha,k}/\operatorname{Soc}(P_{\alpha,k}) \to 0,$$

$$0 \to \mathrm{Rad}(P_{\alpha,S}) \to P_{\alpha,S} \oplus \mathcal{H}(P_{\alpha,S}) \to P_{\alpha,S}/\operatorname{Soc}(P_{\alpha,S}) \to 0,$$

are Auslander-Reiten sequences in the module category. Note that $\mathcal{H}(P_{\alpha,k}) = \mathcal{H}(P_{\alpha,S}) = k[x, y]$. By 4.7 in [7], these sequences lift to Auslander-Reiten triangles

$$\mathrm{Rad}(P_{\alpha,k}) \to P_{\alpha,k} \oplus k[x, y] \to P_{\alpha,k}/\operatorname{Soc}(P_{\alpha,k}) \to \mathrm{Rad}(P_{\alpha,k})[1],$$

$$\mathrm{Rad}(P_{\alpha,S}) \to P_{\alpha,S} \oplus k[x, y] \to P_{\alpha,S}/\operatorname{Soc}(P_{\alpha,S}) \to \mathrm{Rad}(P_{\alpha,S})[1].$$

The interval $[x, y]$ is clamped, and the algebra $k[x, y]$ is hereditary because it is isomorphic to the path algebra of a quiver of type A_n. Because of this, every $k[x, y]$-module has projective dimension ≤ 1 and injective dimension ≤ 1. Therefore by 4.7 in [7], every Auslander-Reiten sequence in the module category $k[x, y]-\mathrm{mod}$ lifts to an Auslander-Reiten triangle in $D^b(k[x, y])$. Also, every Auslander-Reiten sequence begins with a module supported on $(x, y]$ and ends with a module supported on $[x, y)$,

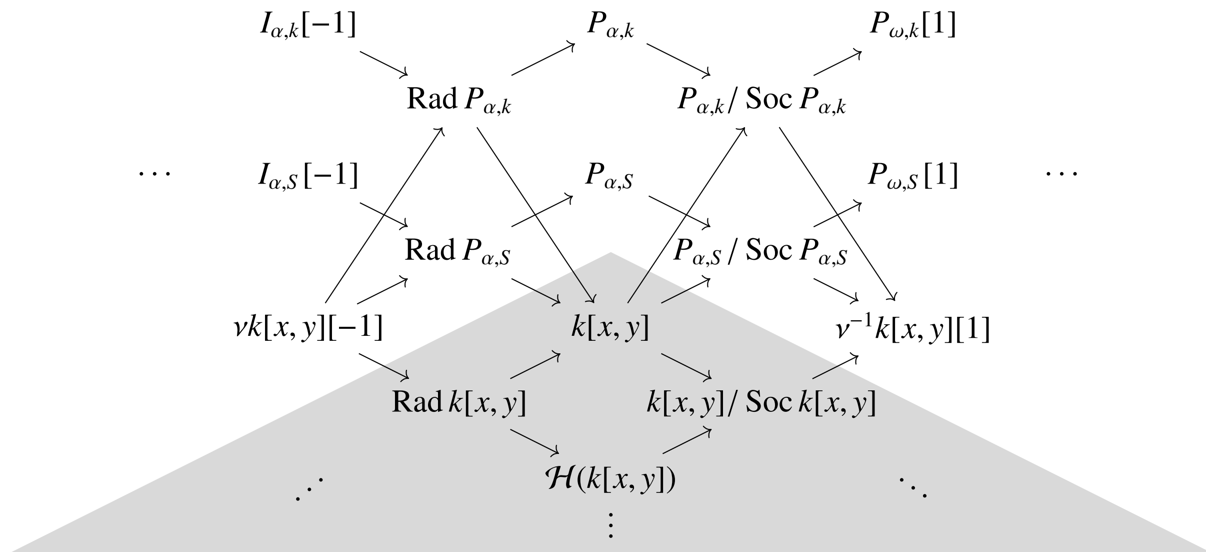
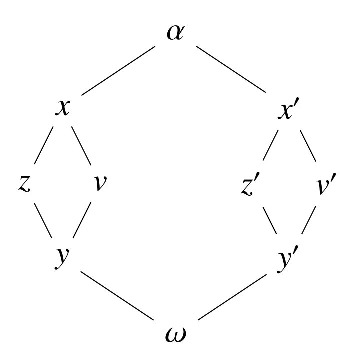


Figure 5.3: A portion of the Auslander-Reiten quiver in example 9.5. The triangle consists of the complexes in the Auslander-Reiten quiver for $D^b(k[x,y])$.

so by Corollary 5.0.4 the entire Auslander-Reiten quiver of $k[x,y]$-mod is copied into the Auslander-Reiten quiver of $D^b(k\mathcal{P} \rtimes G)$.

The Auslander-Reiten triangles at the bottom of $D^b(k[x,y])$ are complete, so they are also complete in $D^b(k\mathcal{P} \rtimes G)$. When $n = 2$, the tree class is Dynkin of type E_6. When $n \geq 3$ the tree class is not of Dynkin type.

Example 5.0.9. For this example, let k and G be as before, and let $\mathcal{P}$ be the poset



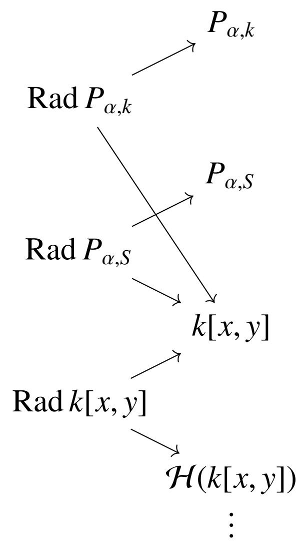


Figure 5.4: A slice of the AR quiver in example 9.5. For $n \geq 3$ the slice is not of Dynkin type.

where G acts on $\mathcal{P}$ by switching the two diamonds. We will calculate the Auslander-Reiten quiver of $D^b(k[x,y])$ explicitly. We have the Auslander-Reiten triangles

$$S_x[-1] \to \operatorname{Rad} k[x,y] \to k[x,y] \to S_x,$$

$$k[x,y] \to k[x,y]/S_y \to S_y[1] \to k[x,y][1],$$

$$\operatorname{Rad} k[x,y] \to k[x,y] \oplus S_z \oplus S_v \to k[x,y]/S_y.$$

The portion of the quiver consisting of the modules $k[x,y], \operatorname{Rad} k[x,y], S_z$ and S_v form a slice for the Auslander-Reiten quiver of $D^b(k[x,y])$, so we can conclude that the Auslander-Reiten quiver of $k[x,y]$ has tree class D_4.

The Auslander-Reiten triangles above get lifted to Auslander-Reiten triangles in $D^b(k\mathcal{P} \rtimes G)$. When we add the modules $P_{\alpha,k}, \operatorname{Rad} P_{\alpha,k}, P_{\alpha,S}$, and $\operatorname{Rad} P_{\alpha,S}$, the result is a slice for this component of the quiver. Note that the number of arrows between any pair of modules is at most one.

Example 5.0.10. For this example, let k be a field of characteristic 0, let G be the

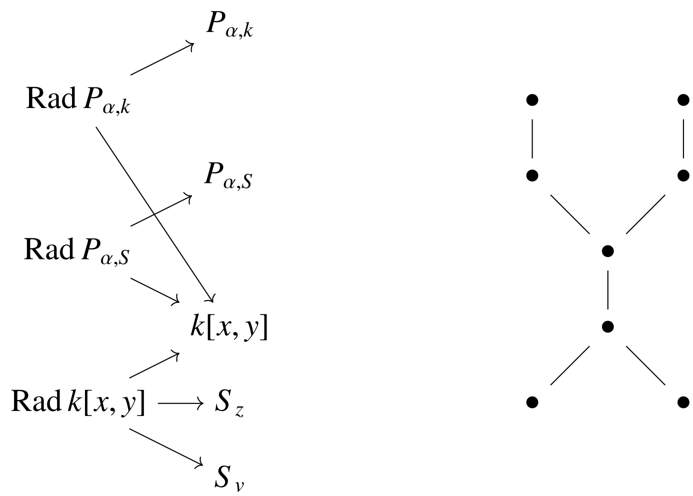
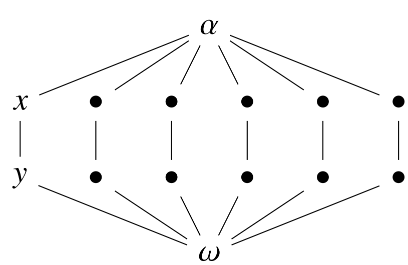


Figure 5.5: A slice of a component of the Auslander-Reiten quiver in Example 5.0.10 and a drawing of the underlying graph of a slice.

symmetric group on three elements, and let $\mathcal{P}$ be the poset



where G acts on the six chains regularly. Let k denote the trivial kG-module, let S be the sign module, and let W be the two-dimensional simple kG-module. The tree class of the Auslander-Reiten quiver of $D^b(k[x,y])$ is A_2, and a slice consists of the modules S_x and $k[x,y]$.

To see how this slice contributes to a slice for the Auslander-Reiten quiver for $D^b(k\mathcal{P} \rtimes G)$, note that we have three projective-injective $k\mathcal{P} \rtimes G$-modules, namely $P_{\alpha,k}$, $P_{\alpha,S}$, and $P_{\alpha,W}$. We highlight that

$$P_{\alpha,W} = \begin{matrix} W \\ k^2 \\ k^2 \\ W \end{matrix}.$$

As in previous examples, we have the following Auslander-Reiten triangles:

$$I_{\alpha,*}[-1] \to \mathrm{Rad}(P_{\alpha,*}) \to P_{\alpha,*} \to I_{\alpha,*},$$

$$P_{\alpha,*} \to P_{\alpha,*}/\,\mathrm{Soc}(P_{\alpha,*}) \to P_{\omega,*}[1] \to P_{\alpha,*}[1],$$

where $*$ is k, S, or W. We also have the triangles

$$\mathrm{Rad}(P_{\alpha,k}) \to P_{\alpha,k} \oplus k[x,y] \to P_{\alpha,k}/\,\mathrm{Soc}(P_{\alpha,k}) \to \mathrm{Rad}(P_{\alpha,k})[1],$$

$$\mathrm{Rad}(P_{\alpha,S}) \to P_{\alpha,S} \oplus k[x,y] \to P_{\alpha,S}/\,\mathrm{Soc}(P_{\alpha,S}) \to \mathrm{Rad}(P_{\alpha,S})[1],$$

$$\mathrm{Rad}(P_{\alpha,W}) \to P_{\alpha,W} \oplus (k[x,y])^2 \to P_{\alpha,W}/\,\mathrm{Soc}(P_{\alpha,W}) \to \mathrm{Rad}(P_{\alpha,W})[1].$$

Notice that $\dim_k(\mathrm{Irr}(\mathrm{Rad}(P_{\alpha,W}), k[x,y])) = 2$, so that a slice for the Auslander-Reiten quiver has a double edge.

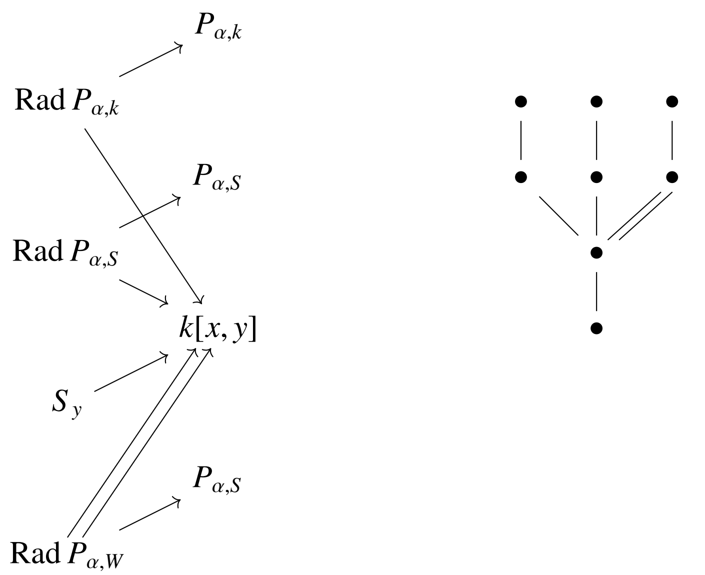


Figure 5.6: A slice of a component of the Auslander-Reiten quiver in example 5.0.11 and a drawing of the tree class.

Chapter 6

The class $\mathcal{ICT}$

As in [5], we define a class of transporter category for which we my iteratively construct the tree class of its bounded derived category. We start by recalling the definition of the class $\mathcal{IC}$ of posets. To avoid confusion later on, we will rename this class $\mathcal{ICP}$.

Definition 6.0.1. Let $\mathcal{ICP}_0$ be the class containing only the singleton poset, $\bullet$. We now define $\mathcal{ICP}_n$, $n \geq 1$ as the class of posets containing a unique maximal element ω and a unique minimal element α such that the open interval (α, ω) is a disjoint union of finitely many (and possibly zero) posets from the class $\mathcal{ICP}_{n-1}$. We then define

$$\mathcal{ICP} = \bigcup_{n=0}^{\infty} \mathcal{ICP}_n$$

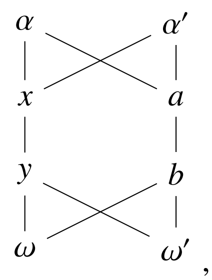
We can adjust the definition to form a class $\mathcal{ICT}$ of transporter categories. Recall that the *base* poset of the transporter category $\mathcal{P} \rtimes G$ is the poset $\mathcal{P}$

Definition 6.0.2. Let $\mathcal{ICT}_0$ be the class containing the transporter categories with a single object, $\bullet \rtimes G$, i.e. the class of finite groups. We now define $\mathcal{ICT}_n$, $n \geq 1$ as the class of transporter categories $\mathcal{P} \rtimes G$ whose base poset $\mathcal{P}$ contains a unique maximal element ω and a unique minimal element α such that the transporter subcategory

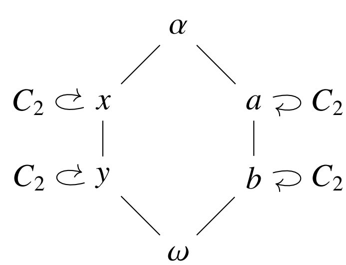
$(\alpha, \omega) \rtimes G$ is equivalent to a disjoint union of finitely many (and possibly zero) transporter categories from the class $\mathcal{ICT}_{n-1}$. We then define

$$\mathcal{ICT} = \bigcup_{n=0}^{\infty} \mathcal{ICT}_n.$$

We can always construct a poset in $\mathcal{ICP}$ by taking a disjoint union of posets in $\mathcal{ICP}$ and adding a minimal element and a maximal element. The same is not true for transporter categories in $\mathcal{ICT}$. For example, we can take the poset



and let $G = C_2$ act on this poset by switching α and α', switching ω and ω', and fixing the other objects. The corresponding transporter category is equivalent to its skeleton:



This is *not* in the class $\mathcal{ICT}$. Indeed, this is not equivalent to any transporter category with unique minimal and maximal elements because the stabilizers of x and y are not subgroups of the stabilizers of α and ω, as this next lemma shows.

Lemma 6.0.3. *Let T be an EI-category with a unique minimal element α and a unique maximal element ω with respect to the preorder $\leq$ where $x \leq y$ if and only if* $\mathrm{Hom}_T(x, y) \neq$

$\varnothing$. *Suppose that T is equivalent to a finite transporter category $P \rtimes G$ where P has a unique minimal element α' and a unique maximal element ω'. Then* $\mathrm{End}_T(\alpha) \cong \mathrm{End}_T(\omega) \cong G$, *and for all $x \in T$, we have that* $\mathrm{End}_T(x)$ *is isomorphic to a subgroup of* $\mathrm{End}_T(\alpha)$.

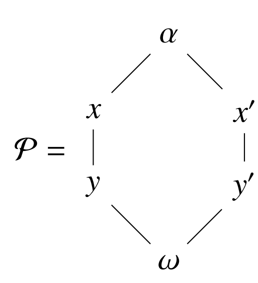
Proof. We first show that this statement holds when $T = \mathcal{P} \rtimes G$. Because every automorphism of $\mathcal{P}$ fixes α' and ω', we have $G_{\alpha'} = G_{\omega'} = G$, so $\mathrm{End}_{\mathcal{P}\rtimes G}(\alpha') \cong \mathrm{End}_{\mathcal{P}\rtimes G}(\omega') \cong G$ as groups. If $x \in \mathcal{P} \rtimes G$, then $\mathrm{End}_{\mathcal{P}\rtimes G}(x) \cong G_x \leq G \cong \mathrm{End}_{\mathcal{P}\rtimes G}(\alpha')$.

Now let's suppose T is any EI category satisfying the conditions of the lemma. We first show that any equivalence $f : T \to \mathcal{P} \rtimes G$ satisfies $f(\alpha) = \alpha'$ and $f(\omega) = \omega'$. Because f is a full and faithful functor, it follows that for any $x \in T$ with $x \neq \alpha$, we have $\mathrm{Hom}_{\mathcal{P}\rtimes G}(f(x), f(\alpha)) \cong \mathrm{Hom}_{\mathcal{P}\rtimes G}(x, \alpha) = \varnothing$. Because f is essentially surjective, we have $\mathrm{Hom}_{\mathcal{P}\rtimes G}(y, f(\alpha)) = \varnothing$, and it follows that $f(\alpha)$ is a minimal element, so we must have $f(\alpha) = \alpha'$. Similarly, we must have $f(\omega) = \omega'$.

We now have $\mathrm{End}_T(\alpha) \cong \mathrm{End}_{\mathcal{P}\rtimes G}(\alpha') \cong G$, and $\mathrm{End}_T(\omega) \cong \mathrm{End}_{\mathcal{P}\rtimes G}(\omega') \cong G$. Moreover, for any $x \in T$, we have $\mathrm{End}_T(x) \cong \mathrm{End}_{\mathcal{P}\rtimes G}(f(x)) \cong G_{f(x)} \leq G$. This proves the lemma. □

6.1 Transporter categories up to equivalence

We aim to show that the transporter categories in $\mathcal{ICT}$ can be easily identified by their base poset. However, the definition of a transporter category in $\mathcal{ICT}_n$ requires only that the interval (α, ω) be *equivalent* to a transporter category in $\mathcal{ICT}_{n-1}$. Indeed, let's revisit the transporter category whose base poset is the C_2-poset



where the nonidentity element in C_2 acts by switching the two chains. The transporter subcategory $(\alpha, \omega) \rtimes G$ is *not* in $\mathcal{ICT}$ because the base poset has two maximal and minimal elements. However, it is equivalent to the transporter category $[x, y] \rtimes \{e\}$, which *is* in $\mathcal{ICT}$. As we saw earlier, this poset has the clamping property we want, so we would like to include it.

We will start by analyzing transporter categories of the form $\mathcal{P} \rtimes G$ where $\mathcal{P}$ has a unique minimal element, α. We will show that the base poset $\mathcal{P}$ can be recovered from the information contained in a skeletal subcategory. We could just as easily restrict to transporter categories whose base poset has a unique maximal element ω and get a similar result.

Lemma 6.1.1. *Let $\mathcal{P} \rtimes G$ be a finite transporter category where $\mathcal{P}$ has a unique minimal element, α. Let C be a skeletal subcategory of $\mathcal{P} \rtimes G$.*

1. *The set*
$$Q = \{(x, [\phi]) \mid x \in \mathrm{Ob}C, \phi \in [\,\mathrm{End}_C(x) \setminus \mathrm{Hom}_C(\alpha, x)]\}$$
where $[\phi]$ denotes the orbit class of $\phi \in \mathrm{Hom}_C(\alpha, x)$ under the left action of $\mathrm{End}_C(x)$ induced by composition, forms a poset where $(x, [\phi]) \leq (y, [\psi])$ if and only if $\psi = \gamma \circ \phi$ for some $\gamma \in \mathrm{Hom}_C(x, y)$.

2. *The map*
$$Q \to \mathcal{P},$$
$$(x, [(\alpha \leq x, g)]) \mapsto {}^{g^{-1}}x$$
is a G-poset isomorphism where the action on Q is
$${}^{g}(x, [(\alpha \leq x, h)]) = (x, [(\alpha \leq x, hg^{-1})]).$$

Proof. 1. We show that the relation on Q is reflexive, antisymmetric, and transitive. If $(x, [\phi]) = (x, [\psi])$, then there is an element $\gamma \in \mathrm{End}_C(x)$ such that $\psi = \gamma \circ \phi$. This is the condition $(x, [\phi]) \leq (x, [\psi])$.

To show antisymmetry, suppose that $(x, [\phi]) \leq (y, [\psi])$ and $(y, [\psi]) \leq (x, [\phi])$. Then for some $\gamma_1 \in \mathrm{Hom}_C(x, y)$ and $\gamma_2 \in \mathrm{Hom}_C(y, x)$ we have $\psi = \gamma_1 \circ \phi$ and $\phi = \gamma_2 \circ \psi$. Then $\gamma_2 \circ \gamma_1 \in \mathrm{End}_C(x)$ and $\gamma_1 \circ \gamma_2 \in \mathrm{End}_C(y)$, so they are isomorphisms because C is an EI-category. This implies that γ_1 and γ_2 are isomorphisms, so $x = y$ because C is skeletal. This implies that ϕ and ψ differ by an element in $\mathrm{End}_C(x)$, so $[\phi] = [\psi]$.

For transitivity, suppose $(x, [\phi]) \leq (y, [\psi])$ and $(y, [\psi]) \leq (z, [\theta])$. Then there exists $\gamma_1 \in \mathrm{Hom}_C(x, y)$ such that $\psi = \gamma_1 \circ \phi$, and there exists $\gamma_2 \in \mathrm{Hom}_C(y, z)$ such that $\theta = \gamma_2 \circ \psi$. Then $\theta = (\gamma_2 \circ \gamma_1) \circ \phi$, so $(x, [\phi]) \leq (z, [\theta])$.

2. For this part of the proof, we will write each morphism originating at α as $(\alpha \leq -, g)$ for some $g \in G$. We first show that the map

$$f : \mathcal{Q} \to \mathcal{P},$$

$$(x, [(\alpha \leq x, g)]) \mapsto {}^{g^{-1}}x$$

is well-defined. We have $(x, [(\alpha \leq x, g)]) = (x, [(\alpha \leq x, h)])$ if and only if $h = kg$ for some $k \in G_x$. Then $f((x, [(\alpha \leq x, g)])) = {}^{g^{-1}}x = {}^{g^{-1}k^{-1}}x = f((x, [(\alpha \leq x, h)]))$, so f is well-defined.

We now show that f is a G-poset homomorphism. Suppose $(x, [(\alpha \leq x, g)]) \leq (y, [(\alpha \leq y, h)])$. Then $(\alpha \leq y, h) = ({}^k x \leq y, k) \circ (\alpha \leq x, g)$ for some $k \in G$, so ${}^k x \leq y$ and $h = kg$. Thus, ${}^{g^{-1}}x = {}^{h^{-1}k}x \leq {}^{h^{-1}}y$, i.e. $f((x, [(\alpha \leq x, g)])) \leq f((y, [(\alpha \leq y, h)]))$. Moreover

$$f({}^g(x, [(\alpha \leq x, h)])) = f((x, [(\alpha \leq x, hg^{-1})])) = {}^{gh^{-1}}x = {}^g f((x, [(\alpha \leq x, h)])),$$

so f is a G-poset homomorphism.

We now show that f is an isomorphism. Suppose $f((x, [(\alpha \leq x, g)])) = f((y, [(\alpha \leq y, h)]))$. Then ${}^{g^{-1}}x = {}^{h^{-1}}y$, so x and y are in the same G-orbit in $\mathcal{P}$. This means that $x = y$ because there is only one G-orbit representative for each orbit in $\mathrm{Ob}C$.

Now ${}^{g^{-1}}x = {}^{h^{-1}}x$ implies that $h^{-1} = g^{-1}k$ for some $k \in G_x$. Thus $(x, [(\alpha \leq x, g)]) = (y, [(\alpha \leq y, h)])$, so f is one-to-one.

Surjectivity comes from the fact that each object of $\mathcal{P}$ is a conjugate of an object in C. Thus, f is an isomorphism.

□

Notice that in the previous lemma, we can also interpret the G-action on Q as an $\mathrm{End}_C(\alpha)$-action via

$$^{\theta}(x, [\gamma]) = (x, [\gamma \circ \theta^{-1}]).$$

Thus, the structure of Q as a G-poset only relies on the structure of C. This fact is important for the following proposition.

Proposition 6.1.2. *Let $\mathcal{P}$ be a finite G-poset and let $\mathcal{R}$ be finite H-poset, each with a unique minimal element, $\alpha \in \mathcal{P}$ and $\alpha' \in \mathcal{R}$. Then $\mathcal{P} \rtimes G \simeq \mathcal{R} \rtimes H$ implies that $G \cong H$ and $\mathcal{P} \cong \mathcal{R}$ as G-posets where G acts on $\mathcal{R}$ via the isomorphism with H. Furthermore, this implies $\mathcal{P} \rtimes G \cong \mathcal{R} \rtimes H$.*

Proof. Observe that $\mathcal{P} \rtimes G$ and $\mathcal{R} \rtimes H$ both have unique 'minimal elements', namely objects x such that $\mathrm{Hom}(y, x) \neq \emptyset$ implies $y \cong x$. With our conventions of notation, these 'minimal elements' are α and α'. Any equivalence sends a minimal element in this sense to a minimal element, and we have $\mathrm{End}_{\mathcal{P}\rtimes G}(\alpha) \cong G$ and $\mathrm{End}_{\mathcal{R}\rtimes H}(\alpha') \cong H$. The equivalence then induces an isomorphism $\mathrm{End}_{\mathcal{P}\rtimes G}(\alpha) \cong \mathrm{End}_{\mathcal{R}\rtimes H}(\alpha')$, so $G \cong H$.

Because the two transporter categories are equivalent, they are equivalent to a skeletal category C, which is unique up to isomorphism. We can then construct the G-poset Q as we did in part 1 in the previous lemma, and this relies only on the structure of C. Part 2 of that lemma implies that $\mathcal{P} \cong Q \cong \mathcal{R}$ as G-posets, and this implies $\mathcal{P} \rtimes G \cong Q \rtimes H$. □

The next step of this process is to analyze transporter categories whose base posets have multiple connected components and are equivalent to a transporter category whose base poset has only one connected component. We start with some easy lemmas about groups acting on posets with multiple connected components.

Lemma 6.1.3. *Let $\mathcal{P} = \mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_n$ be a finite poset with n connected components. Let ϕ be an automorphism of $\mathcal{P}$ such that there exists an element $x \in \mathcal{P}_i$ where $\phi(x) \in \mathcal{P}_j$ with $j \neq i$. Then $\phi(\mathcal{P}_i) \subseteq \mathcal{P}_j$, and $\phi|_{\mathcal{P}_i} : \mathcal{P}_i \to \mathcal{P}_j$ is an isomorphism.*

Proof. Any automorphism of $\mathcal{P}$ permutes the connected components, so $\phi(\mathcal{P}_i) \subseteq \mathcal{P}_j$. The map $\phi|_{\mathcal{P}_i}$ has inverse $\phi^{-1}|_{\mathcal{P}_j}$, so $\phi|_{\mathcal{P}_i}$ is an isomorphism. □

Lemma 6.1.4. *Let $\mathcal{P} = \mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_n$ be a finite G-poset with n connected components with G permuting the components transitively. Let G_i, $i = 1, 2, \ldots, n$ denote the setwise stabilizer of $\mathcal{P}_i$. Then $\mathcal{P}_i \rtimes G_i \cong \mathcal{P}_j \rtimes G_j$ for all i and j. Moreover, $\mathcal{P} \rtimes G \simeq \mathcal{P}_i \rtimes G_i$ for any i.*

Proof. Because G permutes the connected components transitively, this means that there are automorphisms of $\mathcal{P}$ which send each component to any other. By the previous lemma, this implies that $\mathcal{P}_i \cong \mathcal{P}_j$ for all i, j. Moreover, if $g \in G$ maps $\mathcal{P}_i$ to $\mathcal{P}_j$, then $G_j = {}^g G_i$. This induces an isomorphism of transporter categories

$$f : \mathcal{P}_i \rtimes G_i \to \mathcal{P}_j \rtimes G_j,$$

$$(x \leq y, h) \mapsto ({}^g x \leq {}^g y, {}^g h).$$

The equivalence $\mathcal{P} \rtimes G \simeq \mathcal{P}_i \rtimes G_i$ for any i comes from the fact that $\mathcal{P}_i \rtimes G_i$ is a full subcategory of $\mathcal{P} \rtimes G$ containing a representative of every isomorphism class of objects in $\mathcal{P} \rtimes G$. Any such subcategory is equivalent to the whole category because the inclusion functor $\mathcal{P}_i \rtimes G_i \hookrightarrow \mathcal{P} \rtimes G$ is full, faithful, and essentially surjective. □

Proposition 6.1.5. *Let $\mathcal{P} \rtimes G$ be a finite transporter category whose base poset has n components, and suppose $\mathcal{P} \rtimes G \simeq \mathcal{Q} \rtimes H$ where $\mathcal{Q}$ is a poset with a unique minimal element α. Then each component of $\mathcal{P}$ is isomorphic to $\mathcal{Q}$, and the setwise stabilizers of each component are isomorphic to H.*

Proof. Under an equivalence of categories $f : \mathcal{P} \rtimes G \to \mathcal{Q} \rtimes H$, the minimal elements of $\mathcal{P} \rtimes G$ are all sent to the unique minimal element of $\mathcal{Q} \rtimes H$, and $f(x) = f(y)$ implies $x \cong y$. This implies that the minimal elements of $\mathcal{P} \rtimes G$ are isomorphic to each other.

Objects x, y of $\mathcal{P} \rtimes G$ are isomorphic if and only if there exists $g \in G$ satisfying ${}^g x = y$. We deduce that all of the minimal elements of $\mathcal{P}$ are in the same G orbit.

We deduce that the minimal elements in each connected component of $\mathcal{P}$ are in the same G orbit and that G permutes the connected components of $\mathcal{P}$ transitively, which are thus all isomorphic, say to some component $\mathcal{R}$. By Lemma 6.1.4 $\mathcal{P} \rtimes G \simeq \mathcal{R} \rtimes G_{\mathcal{R}}$, where $G_{\mathcal{R}}$ denotes the setwise stabilizer of $\mathcal{R}$. Now we have an equivalence of categories $f : \mathcal{R} \rtimes G_{\mathcal{R}} \to Q \rtimes H$ where Q has a unique minimal element.

We now argue that $\mathcal{R}$ also has a unique minimal element. Suppose for the sake of contradiction that $\alpha_1, \ldots, \alpha_k$ are the distinct minimal elements of $\mathcal{R}$ with $k > 1$. Because these minimal elements are in the same connected component, there exists an element $x \in \mathcal{R}$ such that $\alpha_i \leq x$ and $\alpha_j \leq x$ with $i \neq j$. The elements α_i and α_j are in the same $G_{\mathcal{R}}$-orbit, so in $\mathcal{R} \rtimes G_{\mathcal{R}}$, the set $\mathrm{Hom}_{\mathcal{R} \rtimes G_{\mathcal{R}}}(\alpha_i, x)$ has size at least $2|\mathrm{End}(\alpha_i)| = 2|H|$. However, $|\mathrm{Hom}_{Q \rtimes H}(f(\alpha_i), f(x)| = |H|$, so the functor f cannot be faithful, contradicting the fact that f is an equivalence. It follows that $\mathcal{R}$ has a unique minimal element.

We now have an equivalence of transporter categories $\mathcal{R} \rtimes G_{\mathcal{R}} \simeq Q \rtimes H$ where both $\mathcal{R}$ and Q have a unique minimal element. By Proposition 6.1.2, this implies that $\mathcal{R} \cong Q$ as posets and $G_{\mathcal{R}} \cong H$. In particular, each component of $\mathcal{P}$ is isomorphic to Q. □

6.2 An alternative formulation of $\mathcal{ICT}$

We can now identify transporter categories in the class $\mathcal{ICT}$ as follows.

Proposition 6.2.1. *The transporter categories of type $\mathcal{ICT}$ are precisely those transporter categories whose base poset is in the class $\mathcal{ICP}$.*

Proof. We prove this claim by induction, showing, for all n, that the transporter categories of type $\mathcal{ICT}_n$ are those transporter categories whose base poset is in the class $\mathcal{ICP}_n$. The class $\mathcal{ICT}_0$ is evidently the same as the class of transporter categories whose base poset is a point. Now let $\mathcal{P} \rtimes G$ be a transporter category in $\mathcal{ICT}_n$. Then $\mathcal{P} \rtimes G$ has a unique minimal element α, a unique maximal element ω, and the interval (α, ω) is a disjoint union of transporter categories equivalent to transporter categories in $\mathcal{ICT}_{n-1}$. Write $(\alpha, \omega) = \bigcup_{i=1}^{k} \mathcal{P}_i \rtimes G_i$ where G_i is the setwise stabilizer of $\mathcal{P}_i$. By Proposition 6.1.5, we for each i that the base poset $\mathcal{P}_i$ consists of a disjoint union of copies of some poset Q_i where $Q_i \rtimes G_{Q_i}$ is in $\mathcal{ICT}_{n-1}$. By the induction hypothesis, the poset Q_i is in $\mathcal{ICP}_{n-1}$. We conclude that $\mathcal{P}$ is in the class $\mathcal{ICP}$. □

We now want to establish the existence of various Auslander-Reiten triangles in $D^b(k\mathcal{P} \rtimes G)$. We will assume throughout this section that $\mathcal{P}$ has a unique minimal element α and a unique maximal element ω. To start, we identify some projective-injective objects in $k\mathcal{P} \rtimes G$-mod. Recall that an indecomposable projective $k\mathcal{P} \rtimes G$-module can be written as $P_{x,W}$ where $x \in \mathcal{P}$ and W is a simple $k\,\mathrm{End}(x)$-module. This is the projective cover of the module $S_{x,W}$ whose support is x and $S_{x,W}(x) \cong W$ as a $k\,\mathrm{End}(x)$-module. Similarly, the indecomposable injective modules are of the form $I_{x,W}$, which is the injective envelope of $S_{x,W}$. We start with a lemma.

Lemma 6.2.2. *Let G be a finite group and let $\mathcal{P}$ be a finite G-poset. Let k be any field. Then the functors $\uparrow_{\mathcal{P}}^{\mathcal{P}\rtimes G}$ and $\Uparrow_{\mathcal{P}}^{\mathcal{P}\rtimes G}$ are naturally isomorphic.*

Using this lemma, we can show the following:

Proposition 6.2.3. *Suppose $\mathcal{P}$ is a G-poset as above, and suppose that $\mathcal{P}$ has a unique minimal element α and a unique maximal element ω. Then for all simple kG-modules W, we have an isomorphism of $k\mathcal{P} \rtimes G$-modules $P_{\alpha,W} \cong I_{\omega,W}$.*

Proof. We first show that there is an isomorphism of functors $k\,\mathrm{Hom}_{\mathcal{P}\rtimes G}(\alpha, -) \cong Dk\,\mathrm{Hom}_{\mathcal{P}\rtimes G}(-, \omega)$, where D denotes the vector space duality. First note that there is an isomorphism of $k\mathcal{P}$-modules $P_\alpha \cong I_\omega$. Moreover,

$$P_\alpha \uparrow_{\mathcal{P}}^{\mathcal{P}\rtimes G} = k\mathcal{P}1_\alpha \otimes_{k\mathcal{P}} (k\mathcal{P} \rtimes G) = k\mathcal{P} \otimes_{k\mathcal{P}} 1_\alpha k\mathcal{P} \rtimes G \cong k\,\mathrm{Hom}_{\mathcal{P}\rtimes G}(\alpha, -).$$

By the lemma and the fact that $P_\alpha^{\mathcal{P}} \cong I_\omega^{\mathcal{P}}$ we also have

$$P_\alpha \uparrow_{\mathcal{P}}^{\mathcal{P}\rtimes G} \cong I_\omega \Uparrow_{\mathcal{P}}^{\mathcal{P}\rtimes G} = I_\omega^{k\{\omega\}} \Uparrow_{\{\omega\}}^{\mathcal{P}} \Uparrow_{\mathcal{P}}^{\mathcal{P}\rtimes G} \cong I_\omega^{k\{\omega\}} \Uparrow_{\{\omega\}}^{\{\omega\}\rtimes G} \Uparrow_{\{\omega\}\rtimes G}^{\mathcal{P}\rtimes G} \cong I_\omega^{\{\omega\}\rtimes G} \Uparrow_{\{\omega\}\rtimes G}^{\mathcal{P}\rtimes G}.$$

The term $I_\omega^{\{\omega\}\rtimes G}\Uparrow_{\{\omega\}\rtimes G}^{\mathcal{P}\rtimes G}$ is an injective module coinduced from a full subcategory, so it must be an injective $k\mathcal{P}\rtimes G$-module. It is of the form $I_{\omega,W}$ because it is coinduced from a module of that form. It takes the value kG on ω since $k\operatorname{Hom}_{\mathcal{P}\rtimes G}(\alpha,\omega)\cong kG$ and $I_\omega^{\{\omega\}\rtimes G}\Uparrow_{\{\omega\}\rtimes G}^{\mathcal{P}\rtimes G}\cong k\operatorname{Hom}_{\mathcal{P}\rtimes G}(\alpha,\omega)$, so we conclude that $I_\omega^{\{\omega\}\rtimes G}\Uparrow_{\{\omega\}\rtimes G}^{\mathcal{P}\rtimes G}\cong Dk\operatorname{Hom}_{\mathcal{P}\rtimes G}(-,\omega)$. Putting it all together, we have $k\operatorname{Hom}_{\mathcal{P}\rtimes G}(\alpha,\omega)\cong Dk\operatorname{Hom}_{\mathcal{P}\rtimes G}(-,\omega)$.

We now show that this isomorphism is additive. Let $e_W\in kG$ be a primitive idempotent corresponding to the simple module W, and identify e_W with the element of $k\operatorname{End}(\alpha)$. In this case, we have $k(\mathcal{P}\rtimes G)e_W = k\operatorname{Hom}_{\mathcal{P}\rtimes G}(\alpha,-)e_W = P_{\alpha,W}$. Define $\overline{e_W}$ to be the image of e_W under the isomorphism

$$\varphi:\operatorname{End}_{\mathcal{P}\rtimes G}(\alpha)\to\operatorname{End}_{\mathcal{P}\rtimes G}(\omega),$$

$$({}^g\alpha\leq\alpha,g)\mapsto({}^{g^{-1}}\omega\leq\omega,g^{-1}).$$

Then we have

$$\begin{aligned}P_{\alpha,W} &= (k\operatorname{Hom}_{\mathcal{P}\rtimes G}(\alpha,-))e_W\\ &\cong (Dk\operatorname{Hom}_{\mathcal{P}\rtimes G}(-,\omega))e_W\\ &= D(\overline{e_W}(k\operatorname{Hom}_{\mathcal{P}\rtimes G}(-,\omega)))\\ &= I_{\omega,W}.\end{aligned}$$

□

Proposition 6.2.3 identifies a collection of projective-injective modules in $k\mathcal{P}\rtimes G$. Over any field, this means that every projective generated at the minimal element is injective.

We now prove that certain Auslander-Reiten triangles always exist, and some others exist in the case where we have a clamped interval.

Proposition 6.2.4. *Let $\mathcal{P}\rtimes G$ be a transporter category such that $\mathcal{P}$ has a unique minimal element α and a unique maximal element ω. Let k be a field of characteristic 0, and let W be a simple kG-module. Then in $D^b(k\mathcal{P}\rtimes G)$, there are Auslander-Reiten* *triangles*

$$I_{\alpha,W}[-1] \to \operatorname{Rad}(P_{\alpha,W}) \to P_{\alpha,W} \to I_{\alpha,W},$$

$$P_{\alpha,W} \to P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W}) \to P_{\omega,W}[1] \to P_{\alpha,W}[1],$$

and

$$\operatorname{Rad}(P_{\alpha,W}) \to P_{\alpha,W} \oplus \mathcal{H}(P_{\alpha,W}) \to P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W}) \to \operatorname{Rad}(P_{\alpha,W})[1],$$

where $\mathcal{H}(M) := \operatorname{Rad}(M)/\operatorname{Soc}(M)$ *denotes the heart of* M.

Proof. We compute the first triangle by taking a nonzero morphism lying in the socle of $\operatorname{Hom}(P_{\alpha,W}, I_{\alpha,W})$ and computing its mapping cone. The morphism $P_{\alpha,W} \to I_{\alpha,W}$ taking the simple top isomorphically to the simple socle (i.e. all of $I_{\alpha,W}$) fulfills this requirement. The mapping cone is isomorphic to $\operatorname{Rad}(P_{\alpha,W})[1]$ in $D^b(k\mathcal{P} \rtimes G)$, so the middle term of this triangle is $\operatorname{Rad}(P_{\alpha,W})$. Note that the radical is indeed indecomposable because it has a simple socle.

For the second triangle, we must compute the mapping cone of the map $P_{\omega,S}[1] \to P_{\alpha,W}[1]$ which is an injection into the simple socle of $P_{\alpha,W}[1]$. The mapping cone is isomorphic to $P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W})[1]$ in $D^b(k\mathcal{P} \rtimes G)$, so the middle term of the triangle is $P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W})$.

For the third triangle, note that the short exact sequence

$$0 \to \operatorname{Rad}(P_{\alpha,W}) \to P_{\alpha,W} \oplus \mathcal{H}(P_{\alpha,W}) \to P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W}) \to 0$$

is an Auslander-Reiten sequence in A-mod. Moreover, the end term, $P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W})$ has projective dimension 1 and the beginning term $\operatorname{Rad}(P_{\alpha,W})$ has injective dimension 1. Thus, by 4.7 in [7], the triangle

$$\operatorname{Rad}(P_{\alpha,W}) \to P_{\alpha,W} \oplus \mathcal{H}(P_{\alpha,W}) \to P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W}) \to \operatorname{Rad}(P_{\alpha,W})[1]$$

is an Auslander-Reiten triangle. □

The next result occurs in the case where we have a clamped interval in the poset.

Proposition 6.2.5. *Assume that we have the same conditions as in the previous proposition, and assume that* $[a, z]$ *is a clamped interval such that* a *covers* α *and* ω *covers* z*. For a simple* kG_z*-module* V*, define the* $k\mathcal{P} \rtimes G$*-module* $M_V := P_{a,V}/\operatorname{Soc}(P_{a,V})$*, and let* $N_V = \nu M_V$ *and* $L_V = \nu^{-1} M_V$*. Then in* $D^b(k\mathcal{P} \rtimes G)$ *we have Auslander-Reiten triangles*

$$N_V[-1] \to \operatorname{Rad}(P_{\alpha, V\uparrow_{G_z}^G}) \oplus \operatorname{Rad} M_V \to M_V \to N_V,$$

and

$$M_V \to M_V/\operatorname{Soc}(M_V) \oplus P_{\alpha, V\uparrow_{G_z}^G}/\operatorname{Soc}(P_{\alpha, V\uparrow_{G_z}^G}) \to L_V[1] \to M_V[1].$$

Note that the term $P_{\alpha, V\uparrow_{G_z}^G}$ decomposes as $P_{\alpha, V\uparrow_{G_z}^G} = \bigoplus P_{\alpha,W}^{n_V}$ where the sum ranges over the simple kG-modules W appearing as a summand of $V \uparrow_{G_z}^G$, and n_W is the multiplicity of W in $V \uparrow_{G_z}^G$.

Proof. We start by calculating N_V as a complex of injective modules. The term M_V has projective cover $P_{a,V} \to M_V$, and the kernel of this cover is $\operatorname{Soc}(P_{a,V})$. The socle of $P_{a,V}$ is supported only on ω, so it is projective, and $\operatorname{Soc}(P_{a,V})(\omega) \cong V \uparrow_{G_z}^G$ as a kG-module. We conclude that M_V is isomorphic to the complex $P_{\omega, V\uparrow_{G_z}^G} \to P_{a,V}$ in $D^b(k\mathcal{P} \rtimes G)$, where the rightmost term is in homological degree 0. Let γ be the map appearing between the terms in this complex.

Applying the Nakayama functor, we see that $N_V \cong (I_{\omega, V\uparrow_{G_z}^G} \to I_{a,V})$, where the rightmost term is in homological degree 0. The map in this complex is $\nu(\gamma) = D \operatorname{Hom}_{k\mathcal{P}\rtimes G}(\gamma, k\mathcal{P} \rtimes G)$ where D denotes the vector space dual. Because $\operatorname{coker}(\gamma) \cong M_V$ is 0 on ω, it follows that $D \operatorname{Hom}_{k\mathcal{P}\rtimes G}(\operatorname{coker}(\gamma), k\mathcal{P} \rtimes G) = 0$ because any nonzero map must be nonzero on $\operatorname{Soc}(k\mathcal{P} \rtimes G)$, which is supported only on ω. this implies that applying ν to the short exact sequence

$$0 \to P_{\omega, V\uparrow_{G_z}^G} \xrightarrow{\gamma} P_{a,V} \to M_V \to 0$$

yields the exact sequence

$$I_{\omega, V\uparrow_{G_z}^G} \xrightarrow{\nu(\gamma)} I_{a,V} \to 0.$$

We conclude from this that $\nu(\gamma)$ is surjective, so $N_V \cong \ker(\nu(\gamma))[1]$.

Because γ is injective and $\nu(\gamma)$ is surjective, it follows that every map of complexes

$$\begin{array}{ccc} (P_{\omega,V\uparrow_{G_b}^G} & \xrightarrow{\gamma} & P_{a,V}) \\ \downarrow{\phi_0} & & \downarrow{\phi_1} \\ (I_{\omega,V\uparrow_{G_b}^G} & \xrightarrow{\nu(\gamma)} & I_{a,V}) \end{array}$$

yields a mapping cone concentrated in degree 1 with the same composition factors as any other mapping cone. The mapping cone would then have the same composition factors as $N_V[-1] \oplus M_V$. Denote by C_ϕ the mapping cone of ϕ where ϕ lies in the socle of the Hom-space $\mathrm{Hom}(N_V[-1], M_V)$ under the action of $\mathrm{End}_{D^b(k\mathcal{P} \rtimes G)}(M_V)$.

We now identify the summands of $H_1(C_\phi)$. There is an irreducible morphism $\mathrm{Rad}(M_V) \to M_V$ in $D^b(k\mathcal{P} \rtimes G)$ because this morphism lies in the Auslander-Reiten quiver of $D^b(k[a, z] \rtimes G_z)$ where the left hand term has homology supported on $(a, z]$ and this region is copies into the Auslander-Reiten quiver of $D^b(k\mathcal{P} \rtimes G)$ by Corollary 5.0.4. Thus $\mathrm{Rad}(M_V)$ is a summand of $H_1(C_\phi)$.

By Propostion 6.2.4, there is an Auslander-Reiten triangle

$$\mathrm{Rad}(P_{\alpha,W}) \to P_{\alpha,W} \oplus \mathcal{H}(P_{\alpha,W}) \to P_{\alpha,W}/\,\mathrm{Soc}(P_{\alpha,W}) \to \mathrm{Rad}(P_{\alpha,W})[1]$$

for each simple kG-module W. We claim that if the kG_z-module S is a summand of $W \downarrow_{G_z}^G$ with multiplicity n, then M_S is a summand of $\mathcal{H}(P_{\alpha,W})$ with multiplicity n. First we consider $k\,\mathrm{Hom}(a, -)/\,\mathrm{Soc}(k\,\mathrm{Hom}(a, -))$ and $\mathcal{H}(k\,\mathrm{Hom}(\alpha, -))$. Let $g_1, \ldots, g_n$ be a right transversal of G_z in G. Then for each $i \in \{1, \ldots n\}$, there is a split monomorphism of $k\mathcal{P} \rtimes G_z$-modules

$$\phi_i : k\,\mathrm{Hom}(a, -)/\,\mathrm{Soc}(k\,\mathrm{Hom}(a, -)) \hookrightarrow \mathcal{H}(k\,\mathrm{Hom}(\alpha, -)) \downarrow_{\mathcal{P} \rtimes G_z}^{\mathcal{P} \rtimes G},$$

$$(a \leq x, g) \mapsto (\alpha \leq x, gg_i).$$

These split monomorphisms have images intersecting in 0, and $\bigoplus_i \phi_i$ is surjective. Now let $e_S = \sum_{g \in G_z} c_g g \in kG_z$ be an idempotent corresponding to S, and if $\gamma \in \mathcal{P}$, then let $(\gamma, e_S) = \sum_{g \in G_z} (\gamma, c_g g)$. Then these homomorphisms preserve right multiplication by $(1_a, e_S)$.

Now suppose S is a summand of $W \downarrow^G_{G_z}$ occurring with multiplicity n. If e_W is an idempotent in kG corresponding to W, then we have $e_W = \sum_{i=1}^n e_{S_i} + f$ where f is an idempotent and $kG_z e_{S_i} \cong S$ as a kG_z-module for each i. Thus,

$$\mathcal{H}(k\operatorname{Hom}(\alpha, -)) \downarrow^{\mathcal{P}\rtimes G}_{\mathcal{P}\rtimes G_z} (\alpha \leq \alpha, e_V) \cong P_{\alpha,W} \downarrow^{\mathcal{P}\rtimes G}_{\mathcal{P}\rtimes G_z}$$

contains $P_{a,S}$ with multiplicity n.

Now this implies that there are n irreducible morphisms from $\operatorname{Rad}(P_{\alpha,W})$ to M_S whenever S is a summand of $W \downarrow^G_{G_z}$. Thus, the term $\operatorname{Rad}(P_{\alpha,W'})$ is a summand of $H_1(C_\phi)$, where

$$W' = \bigoplus_W W^{\dim k\operatorname{Hom}_{kG_z}(S, W\downarrow^G_{G_z})}$$

and the sum is taken over those simple kG-modules for which S is a summand of $W \downarrow^G_{G_z}$. By Frobenius reciprocity, there is an equality $\dim k\operatorname{Hom}_{kG_z}(S, W \downarrow^G_{G_z}) = \dim k\operatorname{Hom}_{kG}(S \uparrow^G_{G_z}, W)$, which counts the multiplicity of W in $S \uparrow^G_{G_z}$ because W is simple. Then the term W' becomes

$$W' = \bigoplus_W W^{\dim k\operatorname{Hom}_{kG}(W\uparrow^G_{G_z}, S)}$$

where the sum is taken over those simple kG-modules which appear as a summand of $S \uparrow^G_{G_z}$. From this we see that $W' = S \uparrow^G_{G_z}$, so $\operatorname{Rad}(P_{\alpha,V\uparrow^G_{G_z}})$ is a summand of $H_1(C_\phi)$.

Finally, we claim that $\operatorname{Rad}(P_{\alpha,V\uparrow^G_{G_z}})$ and $\operatorname{Rad} M_V$ are the only nonzero summands of $H_1(C_\phi)$. The value of $H_1(C_\phi)$ at an object x is the same as the value of $N_V \oplus M_V$ at x. By Proposition 2.3.4, these values are

$$N_V \oplus M_V(x) = \begin{cases} 0 \text{ if } x = \alpha, \\ V \uparrow^G_{G_z}\downarrow^G_{G_z} \text{ if } x = a, \\ V \uparrow^G_{G_z}\downarrow^G_{G_x} \oplus V \downarrow^{G_z}_{G_x} \text{ if } a < x \leq z, \\ V \uparrow^G_{G_z} \text{ if } x = \omega. \end{cases}$$

These are precisely the values that we find in $\mathrm{Rad}(P_{\alpha,V\uparrow_{G_z}^G}) \oplus \mathrm{Rad}(M_V)$, so we can conclude that this term equals $H_1(C_\phi)$. This concludes the calculation of the first Auslander-Reiten triangle.

For the second triangle

$$M_V \to M_V/\operatorname{Soc}(M_V) \oplus P_{\alpha,V\uparrow_{G_z}^G}/\operatorname{Soc}(P_{\alpha,V\uparrow_{G_z}^G}) \to L_V[1] \to M_V[1],$$

we can get the middle term from the previous triangle by applying τ. Let $\theta \in \mathrm{Hom}(L_V, M_V)$ be a homomorphism in the socle of the Hom-space, and let C_θ be the mapping cone of θ. By properties of irreducible morphisms and the Auslander-Reiten translate, we have $\dim_k \mathrm{Irr}(M_V, I) = \dim_k \mathrm{Irr}(\nu I, M_V)$ for each indecomposable complex I in $D^b(k\mathcal{P} \rtimes G)$. Thus, the terms of C_θ are precisely those terms appearing in C_ϕ with the inverse Auslander-Reiten translate applied, i.e.

$$\begin{aligned} C_\theta =& \tau^{-1}(C_\phi) \\ =& \tau^{-1}((\mathrm{Rad}(P_{\alpha,V\uparrow_{G_z}^G}) \oplus \mathrm{Rad}(M_V))[-1]) \\ =& P_{\alpha,V\uparrow_{G_z}^G}/\operatorname{Soc}(P_{\alpha,V\uparrow_{G_z}^G}) \oplus M_V/\operatorname{Soc}(M_V)[-1]. \end{aligned}$$

After factoring in the shift functor, we get the second triangle.

□

The triangles in Propositions 6.2.4 and 6.2.5 show how M_V connects to the rest of the Auslander-Reiten quiver of $D^b(k\mathcal{P} \rtimes G)$. We will use this information to iteratively construct the tree class of a component of the Auslander-Reiten quiver of $D^b(k\mathcal{P} \rtimes G)$ where $\mathcal{P} \rtimes G \in \mathcal{ICT}$. First, we will introduce a technical hypothesis which can apply to arbitrary transporter categories. This hypothesis is analogous to Hypothesis 4.2 in [5].

Hypothesis 6.2.6. Suppose we are given a transporter category $\mathcal{P} \rtimes G$ and a field k. The base poset $\mathcal{P}$ of $\mathcal{P} \rtimes G$ has a unique minimal element α and a unique maximal element ω. Moreover, for each component of the Auslander-Reiten quiver of $D^b(k\mathcal{P} \rtimes G)$ containing a module of the form $P_{\alpha,W}$ where W is a simple kG-module, there is a slice

$\mathcal{S}$ of that component, and the terms of $\mathcal{S}$ consists only of modules. The modules in each slice $\mathcal{S}$ include all modules of the form $P_{\alpha,W}$ which lie in that component, where W is a simple kG-module. Apart from these, we suppose that none of the remaining modules in each slice $\mathcal{S}$ have ω in their support.

The following proposition can be used to show that all transporter categories in $\mathcal{ICT}$ satisfy Hypothesis 6.2.6.

Proposition 6.2.7. *Let $\mathcal{P} \rtimes G$ be a finite transporter category with a unique minimal object α and a unique maximal object ω, and let k be a field with char$(k) \nmid |G|$. Write $(\alpha, \omega) \rtimes G$ as a disjoint union of connected transporter subcategories*

$$(\alpha, \omega) \rtimes G = \bigsqcup_{i=1}^{n} \mathcal{P}_i \rtimes G.$$

Note that each $\mathcal{P}_i$ is a single G-orbit of objects of $\mathcal{P}$. Suppose further that each component $\mathcal{P}_i \rtimes G$ is equivalent to a transporter subcategory $[\alpha_i, \omega_i] \rtimes G_{\alpha_i}$ satisfying Hypothesis 6.2.6. Let $\mathcal{S}_i$ be the slice corresponding to $[\alpha_i, \omega_i] \rtimes G_{\alpha_i}$. Then $\mathcal{P} \rtimes G$ also satisfies Hypothesis 6.2.6 with slice

$$\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_n \cup \bigcup_{W} \{P_{\alpha,W}, P_{\alpha,W} / \operatorname{Soc}(P_{\alpha,W})\}$$

where the disjoint union is taken over all simple kG-modules up to isomorphism.

Proof. For a simple kG_{α_i}-module V, define $M_{i,V} := P_{\alpha_i,V} / \operatorname{Soc}(P_{\alpha_i,V})$. By Proposition 6.2.5, we have, for each simple kG-module W, an Auslander-Reiten triangle

$$\operatorname{Rad}(P_{\alpha,W}) \to P_{\alpha,W} \oplus \mathcal{H}(P_{\alpha,W}) \to P_{\alpha,W} / \operatorname{Soc}(P_{\alpha,W}) \to \operatorname{Rad}(P_{\alpha,W})[1].$$

We claim that

$$\mathcal{H}(P_{\alpha,W}) = \bigoplus_{i=1}^{n} \bigoplus_{\substack{\text{simple} \\ kG_{\alpha_i}\text{-modules} \\ V}} M_{i,V}^{\dim \operatorname{Hom}_{kG_{\alpha_i}}(W\downarrow^{G}_{G_{\alpha_i}}, V)}. \tag{6.1}$$

In the calculation for Proposition 6.2.5, we showed that if $[\alpha_i, \omega_i]$ is clamped in $\mathcal{P}$, where α_i covers α and ω covers ω_i, then $M_{i,V}$ is a summand of $\mathcal{H}(P_{\alpha,W})$ with multiplicity $\dim \mathrm{Hom}_{kG_{\alpha_i}}(W \downarrow^G_{G_{\alpha_i}}, V)$. We need only show that these are all of the summands of $\mathcal{H}(P_{\alpha,W})$. For $x \in \mathrm{Ob}(\mathcal{P} \rtimes G)$, we have

$$\mathcal{H}(P_{\alpha,W})(x) = \begin{cases} 0 \text{ if } x = \alpha \text{ or } x = \omega, \\ W \downarrow^G_{G_x} \text{ otherwise.} \end{cases}$$

On the other hand,

$$M_{i,V}(x) = \begin{cases} 0 \text{ if } x \notin \mathcal{P}_i \rtimes G \\ V \downarrow^{G_{\alpha_i}}_{G_x} \text{ otherwise.} \end{cases}$$

If $x = \alpha$ or $x = \omega$, then the right hand side of equation (1) evaluated at x is 0 because α and ω are not in $\mathcal{P}_i \rtimes G$ for any i, and $M_{i,V}$ is zero outside of $\mathcal{P}_i \rtimes G$. Now suppose $x \in (\alpha, \omega)$, i.e. $x \in \mathcal{P}_j \rtimes G$ for some j. Then we have

$$\begin{aligned}
&\bigoplus_{i=1}^{n} \bigoplus_{\substack{\text{simple} \\ kG_{\alpha_i}\text{-modules} \\ V}} M_{i,V}^{\dim \mathrm{Hom}_{kG_{\alpha_i}}(W\downarrow^G_{G_{\alpha_i}}, V)}(x) \\
&= \bigoplus_{\substack{\text{simple} \\ kG_{\alpha_j}\text{-modules} \\ V}} M_{j,V}^{\dim \mathrm{Hom}_{kG_{\alpha_j}}(W\downarrow^G_{G_{\alpha_j}}, V)}(x) \\
&\cong \bigoplus_{\substack{\text{simple} \\ kG_{\alpha_j}\text{-modules} \\ V}} (V \downarrow^{G_{\alpha_j}}_{G_x})^{\dim \mathrm{Hom}_{kG_{\alpha_j}}(W\downarrow^G_{G_{\alpha_j}}, V)} \\
&\cong \Big(\bigoplus_{\substack{\text{simple} \\ kG_{\alpha_j}\text{-modules} \\ V}} V^{\dim \mathrm{Hom}_{kG_{\alpha_j}}(W\downarrow^G_{G_{\alpha_j}}, V)} \Big) \downarrow^{G_{\alpha_j}}_{G_x} .
\end{aligned}$$

The second equality occurs because summands in the first line are zero except when $i = j$. Now the quantity $\dim \mathrm{Hom}_{kG_{\alpha_j}}(W \downarrow^G_{G_{\alpha_j}}, V)$ is equal to the multiplicity of V in

$W\downarrow^G_{G_{\alpha_j}}$. Thus we have

$$\Big(\bigoplus_{\substack{\text{simple}\\ kG_{\alpha_j}\text{-modules}\\ V}} V^{\dim \operatorname{Hom}_{kG_{\alpha_j}}(W\downarrow^G_{G_{\alpha_j}}, V)}\Big)\downarrow^{G_{\alpha_j}}_{G_x} = (W\downarrow^G_{G_{\alpha_j}})\downarrow^{G_{\alpha_j}}_{G_x} = W\downarrow^G_{G_x}.$$

Thus the right hand side of equation (1) accounts for all of the summands of $\mathcal{H}(P_{\alpha,W})$, so the two sides of equation (1) are equal.

By Hypothesis 6.2.6, we have for all i that the slice $\mathcal{S}_i$ includes the modules of the form $P^{[\alpha_i,\omega_i]\rtimes G_{\alpha_i}}_{\alpha_i,V}$ where V is a simple $k[\alpha_i,\omega_i]\rtimes G_{\alpha_i}$-module, and aside from these modules, no other terms in $\mathcal{S}_i$ have ω_i in their support. By Corollary 5.0.4, the Auslander-Reiten triangles in $D^b(k[\alpha_i,\omega_i]\rtimes G_{\alpha_i})$ containing these terms on the right remain Auslander-Reiten triangles in $D^b(k\mathcal{P}\rtimes G)$. Thus, the slices $\mathcal{S}_1,\dots,\mathcal{S}_n$ remain a part of a slice for the Auslander-Reiten quiver of $D^b(k\mathcal{P}\rtimes G)$. By Propositions 6.2.4 and 6.2.5, we can adjoin the modules of the form $P_{\alpha,W}$ and $P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W})$ in constructing such a slice.

We now argue that the modules

$$\mathcal{S} = \mathcal{S}_1\cup\cdots\cup\mathcal{S}_n\cup\bigcup_W\{P_{\alpha,W}, P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W})\}$$

do form a complete slice. Suppose $N\in D^b(k\mathcal{P}\rtimes G)$ is a term such that there is an irreducible morphism $N\to x$ or $x\to N$ with $x\in\mathcal{S}$. We will show that $N\in\mathcal{S}$, up to Auslander-Reiten translate. We consider several cases. If $x = P_{\alpha,W}$ for some W, then up to translate, we may assume there is an irreducible morphism $x\to N$. Then $N = P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W})$ by Proposition 6.2.4, so $N\in S$.

Now suppose $x = P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W})$ for some W. Then up to translate, we may assume there is an irreducible morphism $N\to x$. By Proposition 6.2.4 again, we have $N = P_{\alpha,W}$ or $N = M_{i,V}$ for some i and V, so $N\in\mathcal{S}$.

If $x = M_{i,V}$ for some i and V, we may assume that there is an irreducible morphism $x\to N$. Then by Proposition 6.2.5, we have either $N\in\mathcal{S}_i$ or $N = P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W})$ for some W. In both cases, we have $N\in\mathcal{S}$.

Finally, suppose $x\in S_i$ for some i, and $x\neq M_{i,V}$ for any V. We may assume there is an irreducible morphism $N \to x$. By assumption, x does not have ω_i in its support, so by Corollary 5.0.4, the triangle ending in $x \downarrow^{\mathcal{P} \rtimes G}_{[\alpha_i, \omega_i] \rtimes G_{\alpha_i}}$ is induced from $[\alpha_i, \omega_i] \rtimes G_{\alpha_i}$. This implies that N is induced, so either N, τN or $\tau^{-1} N$ is included in $\mathcal{S}_i$, so up to translate we have $N \in \mathcal{S}$. Thus $\mathcal{S}$ is a slice. □

Proposition 6.2.7 contains the inductive step which shows that a component of the Auslander-Reiten quiver of $D^b(k\mathcal{P} \rtimes G)$ where $\mathcal{P} \rtimes G \in \mathcal{ICT}$ and $\mathrm{char}(k) \nmid |G|$ has a tree class which can be described iteratively. The following theorem, which gives the process of constructing these trees, follows directly from previous propositions.

Theorem 6.2.8. *Let $\mathcal{P} \rtimes G$ be a transporter category in $\mathcal{ICT}$, and let k be a field with* $\mathrm{char}(k) \nmid |G|$. *Let α be the minimal element and ω the maximal element of $\mathcal{P}$. Define the following:*

- *n: The number of connected components, in the category-theoretic sense, of $(\alpha, \omega) \rtimes G$.*
- *$\alpha_1, \ldots, \alpha_n$: A selection of minimal elements of $(\alpha, \omega) \rtimes G$, each from a different connected component.*
- *$\{W_1, \ldots, W_m\}$: A complete set of pairwise nonisomorphic simple kG-modules.*
- *m_i: The number of isomorphism classes of simple kG_{α_i}-modules.*
- *$\{V_{i,1}, \ldots, V_{i,m_i}\}$, with $1 \leq i \leq n$: A complete set of pairwise nonisomorphic simple kG_{α_i}-modules.*
- *$e_{j,i,k}$, where $1 \leq i \leq n$, $1 \leq j \leq m$, and $1 \leq k \leq m_i$: The multiplicity of $V_{i,k}$ in $W_j \downarrow^G_{G_{\alpha_i}}$.*
- *$\mathcal{T}$: The underlying directed graph of the slice $\mathcal{S}$ identified in Hypothesis 6.2.6 of the component of the Auslander-Reiten quiver of $D^b(k\mathcal{P} \rtimes G)$ containing the projective modules of the form $P_{\alpha,W}$.*

- $\mathcal{T}_i$, $1 \leq i \leq n$*: The underlying directed graph of the slice identified in Hypothesis 6.2.6 of the components of the Auslander-Reiten quiver of* $D^b(k[\alpha_i, \omega_i] \rtimes G_{\alpha_i})$ *containing the projective modules of the form* $P_{\alpha_i, V}$.

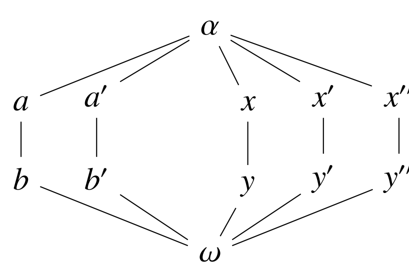
Then we have

$$\mathcal{T} = \mathcal{T}_1 \sqcup \cdots \sqcup \mathcal{T}_n \sqcup \bigcup_{j=1}^{m} \{v_{1,j}, v_{2,j}\}$$

where $\bigcup_{j=1}^{m} \{v_{1,j}, v_{2,j}\}$ *is a set of* $2m$ *labelled vertices. For each* j*, we add an edge between* $v_{1,j}$ *and* $v_{2,j}$*, and for each* i, k *with* $1 \leq i \leq n$ *and* $1 \leq k \leq m_i$*, we add* $e_{j,i,k}$ *edges between* $v_{2,j}$ *and the vertex in* $\mathcal{T}_i$ *corresponding to the module* $P_{\alpha_i, V_{i,k}}$.

We proceed with an example of this theorem.

Example 6.2.9. Let

$\mathcal{P} =$ 

and let $G = S_3$ act on $\mathcal{P}$ by permuting the first two chains with stabilizer $G_a = \langle (1\,2\,3) \rangle$ and by permuting the last three chains with stabilizer $G_x = \langle (2\,3) \rangle$. We will calculate the tree class of a component of $D^b(\mathbb{C}\mathcal{P} \rtimes G)$. The subcategory $(\alpha, \omega) \rtimes G$ has two connected components, where a is a minimal element of one component, and x is a minimal element of the other. Define the following modules:

- $W_1, V_{1,1}, V_{2,1}$: The trivial modules for $\mathbb{C}G$, $\mathbb{C}G_a$, and $\mathbb{C}G_x$ respectively.

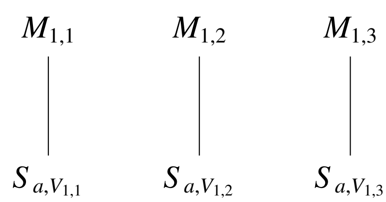
- W_2: The sign module for $\mathbb{C}G$.

- W_3: The two-dimensional simple $\mathbb{C}G$-module.

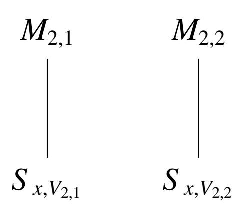
- $V_{1,2}$: The simple $\mathbb{C}G_a$-module on which $(1\,2\,3)$ acts as multiplication by $e^{\frac{2\pi}{3} i}$.

- $V_{1,3}$: The simple $\mathbb{C}G_a$-module on which $(1\,2\,3)$ acts as multiplication by $e^{\frac{4\pi}{3}i}$.
- $V_{2,2}$: The simple $\mathbb{C}G_x$-module on which $(2\,3)$ acts as multiplication by -1.

Note that if we set $a = a_1$ and $x = a_2$, then $V_{i,j}$ is the same as it is in Theorem 6.2.8. The two connected components are equivalent to $[a, b] \rtimes G_a$ and $[x, y] \rtimes G_x$. The stabilizers act trivially on these subposets, so the underlying graph of a slice of a component of the Auslander-Reiten quiver of $D^b(\mathbb{C}[a, b] \rtimes G_a)$ will consist of three disjoint copies of A_2, one copy for each simple $\mathbb{C}G_a$-module, shown below where each vertex is labelled with the module in the corresponding slice.



Similarly, the tree class of the Auslander-Reiten quiver of $D^b(\mathbb{C}[x, y] \rtimes G_x)$ will consist of two disjoint copies of A_2.



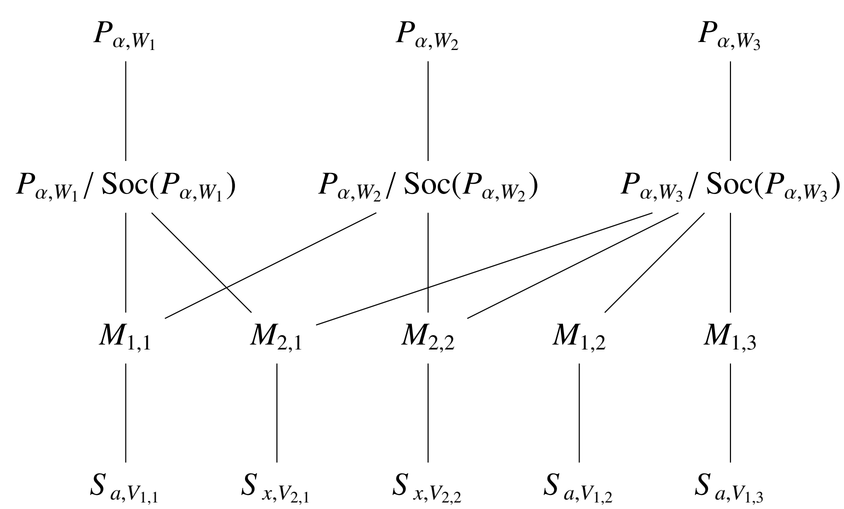
As in Theorem 6.2.8, we define $e_{j,i,k}$ to be the multiplicity of $V_{i,k}$ in $W_j \downarrow^G_{G_a}$ if $i = 1$, or in $W_j \downarrow^G_{G_x}$ if $i = 2$. We have

$$\begin{aligned} W_1 \downarrow^G_{G_a} &= V_{1,1}, & W_1 \downarrow^G_{G_x} &= V_{2,1} \\ W_2 \downarrow^G_{G_a} &= V_{1,1}, & W_2 \downarrow^G_{G_x} &= V_{2,2}, \\ W_3 \downarrow^G_{G_a} &= V_{1,2} \oplus V_{1,3}, & W_3 \downarrow^G_{G_x} &= V_{2,1} \oplus V_{2,2}. \end{aligned}$$

This gives us

$$e_{1,1,1} = e_{1,2,1} = e_{2,1,1} = e_{2,2,2} = e_{3,1,2} = e_{3,1,3} = e_{3,2,1} = e_{3,2,2} = 1,$$

and $e_{j,i,k} = 0$ for all other possible values of j, i, and k. By Theorem 6.2.8, the value of $e_{j,i,k}$ gives the number of edges between the vertex corresponding to $P_{\alpha,W_j}/\operatorname{Soc}(P_{\alpha,W_j})$ and the vertex corresponding to $P_{a,V_{i,k}}$ if $i = 1$, and $P_{x,V_{i,k}}$ if $i = 2$. For ease of notation, let $M_{i,j} = P_{z,V_{ij}}/\operatorname{Soc}(P_{z,V_{ij}})$, where $z = a$ if $i = 1$ and $z = x$ if $i = 2$. We have by Theorem 6.2.8 that the tree class $\mathcal{T}$ is given by the graph below. Each vertex is labelled with a module that comes from the corresponding slice.



The next proposition, which will be useful in constructing the Auslander-Reiten quiver for the module category $k\mathcal{P} \rtimes G$-mod, refers to the *wing* of the Auslander-Reiten quiver determined by a vertex x, which we denote by $\mathcal{W}(x)$. By this we mean the set of vertices in the quiver which can appear in a slice with x. This forms a connected region of the quiver. The rightmost bound of the wing consists of the vertices y for which there is a sequence of irreducible morphisms $x = x_0 \to x_1 \to \cdots \to x_n = y$ where $x_i \neq \tau x_{i+2}$ for all $i \leq n-2$. Similarly, the leftmost bound of the wing consists of the vertices y for which there is a sequence of irreducible morphisms $y = x_0 \to x_1 \to \cdots \to x_n = x$ where $x_i \neq \tau x_{i+2}$ for all $i \leq n-2$. We will need the following lemma, which appears as

Lemma 5.1 in [5].

Lemma 6.2.10. *Let Λ be a finite dimensional algebra and let*

$$L \to M \to N \to L[1]$$

be an Auslander-Reiten triangle in $D^b(\Lambda)$. Assume that N is not the shift of a projective module. Then the associated long exact homology sequence is the splice of short exact sequences

$$0 \to H_i(L) \to H_i(M) \to H_i(N) \to 0$$

with zero connecting homomorphisms.

We aim to show that transporter category algebras in $\mathcal{ICT}$ are piecewise hereditary. Indeed, these algebras are skew group algebras over the incidence algebra of the underlying poset $k\mathcal{P}$. In [5], the authors showed that these algebras are piecewise hereditary, and a theorem of Dionne, Lanzilotta, and Smith states that when $\operatorname{char}(k) \nmid |G|$, the resulting skew group algebra $k\mathcal{P}\#G$ is also piecewise hereditary. However, we would like to go further and identify the type of the piecewise hereditary algebra. To do this, we will identify a tilting complex whose endomorphism ring is hereditary.

Definition 6.2.11. Let Λ be a finite dimensional algebra over a field. A *tilting complex* in $D^b(\Lambda)$ is a complex U satisfying the following:

1. $\operatorname{Hom}_{D^b(\Lambda)}(U, U[i]) = 0$ if $i \neq 0$.
2. U is a perfect complex.
3. U generates $D^b(\Lambda)$ in the following sense: the full subcategory of $D^b(\Lambda)$ containing U which is closed under direct summands, direct sums, shifts, isomorphisms, and forming cones, is all of $D^b(\Lambda)$.

Proposition 6.2.12. *Let $\mathcal{P} \rtimes G \in \mathcal{ICT}$, and let k be a field with* $\operatorname{char}(k) \nmid |G|$. *If $\mathcal{S}$ denotes the union of all slices considered in Hypothesis 6.2.6, then the module*

$$U = \bigoplus_{M \in \mathcal{S}} M$$

is a tilting complex.

The following proof is modified from that of Proposition 4.3 (3) in [5].

Proof. Assume $\mathcal{P} \rtimes G \in \mathcal{ICT}_p$. We will proceed by induction on p. When $p = 0$, the algebra $k\mathcal{P} \rtimes G$ is a semisimple group algebra, and the modules in $\mathcal{S}$ are the complete set of simple modules, which are projective, so that U is a tilting complex.

Now consider the case where $p > 0$. Write

$$(\alpha, \omega) \rtimes G = \bigsqcup_{i=1}^{n} \mathcal{P}_i \rtimes G$$

and suppose that for each i, we have elements $\alpha_i, \omega_i \in \mathcal{P}_i$ satisfying $\mathcal{P}_i \rtimes G \simeq [\alpha_i, \omega_i] \rtimes G_{\alpha_i}$. Let $\mathcal{S}_i$ be the slice corresponding to $k[\alpha_i, \omega_i] \rtimes G_{\alpha_i}$. Recall that in this setup, Proposition 6.2.7 states

$$\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_m \cup \bigcup_{W} \{P_{\alpha,W}, P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W})\}.$$

We will show that if M and N are in $\mathcal{S}$, then $\operatorname{Hom}(M, N[i]) = 0$ if $i \neq 0$. We first consider the case where M is supported on $[\alpha_r, \omega_r) \rtimes G_{\alpha_r}$ for some r and N is in $\mathcal{S}_t$ for some t. Then M is induced from $[\alpha_r, \omega_r) \rtimes G_{\alpha_r}$, so

$$\operatorname{Hom}_{D^b(k\mathcal{P}\rtimes G)}(M, N[i]) = \operatorname{Hom}_{D^b(k[\alpha_r,\omega_r]\rtimes G_{\alpha_r})}(M \downarrow^{\mathcal{P}\rtimes G}_{[\alpha_r,\omega_r]\rtimes G_{\alpha_r}}, N[i] \downarrow^{\mathcal{P}\rtimes G}_{[\alpha_r,\omega_r]\rtimes G_{\alpha_r}}).$$

If $r \neq t$, then $N[i] \downarrow^{\mathcal{P}\rtimes G}_{[\alpha_r,\omega_r]\rtimes G_{\alpha_r}} = 0$ because N is 0 on $[\alpha_r, \omega_r] \rtimes G_{\alpha_r}$, so

$$\operatorname{Hom}_{D^b(k\mathcal{P}\rtimes G)}(M, N[i]) = 0.$$

If $r = t$, then by induction we have $\operatorname{Hom}_{D^b(k\mathcal{P}\rtimes G)}(M, N[i]) = 0$ if $i \neq 0$.

We next consider the case where M is supported on $[\alpha_r, \omega_r) \rtimes G_{\alpha_r}$ and N is of the form $P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W})$. Note that

$$P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W}) \downarrow^{\mathcal{P}\rtimes G}_{[\alpha_r,\omega_r]\rtimes G_{\alpha_r}} \cong \mathcal{H}(P_{\alpha,W}) \downarrow^{\mathcal{P}\rtimes G}_{[\alpha_r,\omega_r]\rtimes G_{\alpha_r}}.$$

The work done in the proof of Proposition 6.2.7 implies that $\mathcal{H}(P_{\alpha,W}) \downarrow^{\mathcal{P}\rtimes G}_{[\alpha_r,\omega_r]\rtimes G_{\alpha_r}}$ is a direct sum of modules of the form $M_{\alpha_r,V} := P_{\alpha_r,V}/\operatorname{Soc}(P_{\alpha_r,V})$. Such modules are in the slice $\mathcal{S}_r$, so $\operatorname{Hom}_{D^b(k\mathcal{P}\rtimes G)}(M, N[i]) = 0$ when $i \neq 0$ by hypothesis.

We next consider the case where $M = M_{\alpha_r,V}$ for some V. Then M is isomorphic to the complex $(P_{\omega,V\uparrow^G_{G_{\alpha_r}}} \to P_{\alpha_r,V})$. Thus for any module $N \in \mathcal{S}$, we have that $\operatorname{Hom}(M, N[i]) = 0$ if $i \neq 0, 1$. If $i = 1$, then there must be a nonzero homomorphism $P_{\omega,V\uparrow^G_{G_{\alpha_r}}} \to N$ so N is supported on ω. This implies $N = P_{\alpha,W}$ for some W, as these are the only modules in $\mathcal{S}$ with ω in their support. In this case, N is injective, so $\operatorname{Hom}(M, N[i]) = 0$ for $i \neq 0$.

Now consider the case where $M = P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W})$ for some W. Then M is isomorphic to the complex $(P_{\omega,W} \to P_{\alpha,W})$. This implies that for any module $N \in \mathcal{S}$, we have $\operatorname{Hom}(M, N[i]) = 0$ if $i \neq 0, 1$. As before, if $i = 1$, then there is a nonzero homomorphism $P_{\omega,W} \to N$, so N is supported on ω. This implies $N = P_{\alpha,W}$, so N is injective and $\operatorname{Hom}(M, N[i]) = 0$ for $i \neq 0$.

Finally, if $M = P_{\alpha,W}$, then M is projective and $\operatorname{Hom}(M, N[i]) = 0$ if $i \neq 0$. If $N = P_{\alpha,W}$, then N is injective and $\operatorname{Hom}(M, N[i]) = 0$ if $i \neq 0$. This exhausts all possibilities of M and N.

We now show that U generates $D^b(k\mathcal{P} \rtimes G)$. Because $D^b(k\mathcal{P} \rtimes G)$ is generated by the injective modules, it suffices to show that U generates the injective modules. Let $I_{x,S}$ be any injective $k\mathcal{P} \rtimes G$-module. If $x = \omega$, then $I_{x,S} = P_{\alpha,S}$ is included in the slice. If $x \neq \omega$, then there is a surjection

$$P_{\alpha,S\uparrow^G_{G_x}} / \operatorname{Soc}(P_{\alpha,S\uparrow^G_{G_x}}) \to I_{x,S}.$$

Interpreting this as a complex with the left term in degree 0, the complex is isomorphic to a module supported on $(\alpha, \omega) \rtimes G$, so it is a direct sum of modules supported on $[\alpha_i, \omega_i] \rtimes G_r$. The modules in S_i generate such modules by hypothesis, so $I_{x,S}$ is generated by the terms in $\mathcal{S}$.

Finally, $D^b(k\mathcal{P} \rtimes G)$ has finite global dimension, so U has finite projective dimension. □

Using this result, we can show that $k\mathcal{P} \rtimes G$ is piecewise hereditary in certain circumstances. We start with a lemma which gives a sufficient condition for an endomorphism ring to be isomorphic to the path algebra of a quiver.

Lemma 6.2.13. *Let k be a field, and let C be a k-linear category. Let $S = \{U_1, \ldots, U_n\}$ be a set of objects in C, and set $\mathcal{U} = \bigoplus_{i=1}^{n}(U_i)$ with canonical inclusion and projection maps*

$$\mathbf{i}_j : U_j \to \mathcal{U},$$

$$\pi_j : \mathcal{U} \to U_j.$$

Suppose that Q is a finite quiver with vertex set $\{v_1, \ldots, v_n\}$ and set of paths E_{ij} between vertices v_i and v_j. Then, if Q has no directed paths of length 2 and $\#E_{ij} =$ $\dim_k \operatorname{Hom}(U_i, U_j)$ for all i and j, there is an isomorphism $\operatorname{End}(\mathcal{U}) \cong kQ$.

Proof. Note that because Q has no paths of length 2, it follows that Q has no loops, and thus $\operatorname{End}(U_i) \cong k$ for each i, i.e. every endomorphism of U_i is a multiple of the identity endomorphism Id_{U_i}. Now let

$$f_{ij} : \operatorname{Hom}(U_i, U_j) \to kE_{ij} \leq kQ$$

be any collection of k-linear isomorphisms satisfying $f_{ii}(\operatorname{Id}_{U_i}) = 1_{v_i}$ for all i. If $\phi \in$ $\operatorname{Hom}(U_i, U_j)$, let $\bar{\phi} \in \operatorname{End}(\mathcal{U})$ denote the morphism

$$\bar{\phi}(u) = \begin{cases} \mathbf{i}_i \circ \phi \circ \pi_i(u) \text{ if } u \in U_i, \\ 0 \text{ otherwise.} \end{cases}$$

Note that the endomorphisms of the form $\bar{\phi}$ span $\operatorname{End}((U))$. We claim that the isomorphisms f_{ij} assemble into an algebra isomorphism

$$F : \operatorname{End}(\mathcal{U}) \to kQ,$$

where $F(\bar{\phi}) = f_{ij}(\phi)$ if $\phi \in \operatorname{Hom}(U_i, U_j)$. Note that by our assumptions, F is a bijection, so we just need to show that F is an algebra homomorphism.

Consider $F(\bar{\phi}\circ\bar{\psi})$ for $\phi \in \mathrm{Hom}(U_i, U_j)$ and $\psi \in \mathrm{Hom}(U_s, U_t)$. If $i \neq t$, then $\bar{\phi}\circ\bar{\psi} = 0$, so $F(\bar{\phi} \circ \bar{\psi}) = F(0) = 0 = F(\bar{\phi}) \circ F(\bar{\psi})$. If $i = t$, then because Q has no paths of length 2, this implies that either $i = j$ and $\phi = a \cdot \mathrm{Id}_{U_j}$ for some $a \in k$, or $t = j$ and $\psi = b \cdot \mathrm{Id}_{U_j}$ for some $b \in k$. We will handle the first case as the second is dealt with similarly. In this case, we have

$$F(\bar{\phi} \circ \bar{\psi}) = F(a \cdot \bar{\psi}) = a \cdot F(\bar{\psi}) = a \cdot F(\mathrm{I}_{U_j}) \cdot F(\bar{\psi}) = F(a \cdot \mathrm{I}_{U_j}) \cdot F(\bar{\psi}) = F(\bar{\phi}) \cdot F(\bar{\psi}).$$

This shows that F is an algebra homomorphism, so these algebras are isomorphic. □

Proposition 6.2.14. *Let $\mathcal{P} \rtimes G \in \mathcal{ICT}$, and let k be a splitting field for kG with* $\mathrm{char}(k) \nmid |G|$. *Let $\mathcal{S}, \mathcal{S}_1, \ldots, \mathcal{S}_n$, and U be as in Proposition 6.2.12. Then* $\mathrm{End}_{D^b(k\mathcal{P}\rtimes G)}(U)$ *is hereditary, so $k\mathcal{P} \rtimes G$ is piecewise hereditary.*

While this proof relies on the field being a splitting field for kG, it can be adapted to work for other fields. We use this assumption for simplicity.

Proof. We will show that $\mathrm{End}(U)$ is isomorphic to the path algebra of a quiver with no oriented cycles. To do this, we will show that each indecomposable summand of U has a trivial endomorphism ring, and we will analyze $\mathrm{Hom}(M, N)$ for each pair of different indecomposable summands M and N in U to identify the path algebra to which $\mathrm{End}(U)$ is isomorphic. Letting $\mathcal{P} \rtimes G \in \mathcal{ICT}_p$, we proceed by induction on p. We will start by analyzing $\mathrm{Hom}(M, N)$ and $\mathrm{Hom}(N, M)$ for different M and N. There are several cases to consider.

If $M, N \in \mathcal{S}_i$, for some i, then M, N are induced from $D^b(k[\alpha_i, \omega_i] \rtimes G_{\alpha_i})$ where $[\alpha_i, \omega_i] \rtimes G_{\alpha_i} \in \mathcal{ICT}_{p-1}$ by Corollary 5.0.3. By hypothesis, the endomorphism ring of the direct sum of these M over $D^b(k[\alpha_i, \omega_i])$ is hereditary of type Q_i for some quiver Q_i. This will be copied into the path algebra for the whole endomorphism ring.

If $M \in \mathcal{S}_i$ and $N \in \mathcal{S}_j$ with $i \neq j$, then these modules have disjoint support, so $\mathrm{Hom}(M, N) = \mathrm{Hom}(N, M) = 0$.

If $M = P_{\alpha,W}$ for some W and $N \in \mathcal{S}_i$ for some i, then $\mathrm{Hom}(M, N) = 0$ because any nonzero image of $f : P_{\alpha,W} \to N$ will have support on α, but N is 0 on α. Similarly, we have $\operatorname{Hom}(N, M) = 0$ because every nonzero submodule of $P_{\alpha,W}$ is supported on ω and N is 0 on ω.

If $M = P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W})$ for some W and $M \in \mathcal{S}_i$ for some i, then $\operatorname{Hom}(M, N) = 0$ because every nonzero map $f : P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W}) \to N$ has image supported on α, but N is not supported on α. We also have $\operatorname{Hom}(N, M) = 0$ whenever N is supported on $[\alpha_i, \omega_i) \rtimes G_i$. Otherwise, $N = M_{\alpha_i,V}$ for some V, and in this case we have that $\dim_k \operatorname{Hom}(N, M)$ is the multiplicity of V in $W \downarrow^G_{G_{\alpha_i}}$.

If $M = P_{\alpha,W}$ or $P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W})$ and $N = P_{\alpha,W'}$ or $P_{\alpha,W'}/\operatorname{Soc}(P_{\alpha,W'})$ for some $W' \ncong W$, then $\operatorname{Hom}(M, N) = \operatorname{Hom}(N, M) = 0$ because the image of any nonzero map would induce an isomorphism between $M(\alpha)$ and $N(\alpha)$, which is not possible.

Finally, if $M = P_{\alpha,W}$ and $N = P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W})$, then $\operatorname{Hom}(N, M) = 0$. Moreover, we have $\operatorname{Hom}(M, N) \cong \operatorname{Hom}_{kG}(M(\alpha), N(\alpha)) = \operatorname{End}_{kG}(W)$. This endomorphism ring is isomorphic to k because k is a splitting field for G, and the endomorphism ring of a simple module over an such a field is always isomorphic to k.

We now turn to the endomorphism rings of the summands of U. If $M \in \mathcal{S}_i$ for some i, then M is induced and thus $\operatorname{End}(M) \cong k$ by hypothesis. If $M = P_{\alpha,W}$ or $P_{\alpha,W}/\operatorname{Soc}(P_{\alpha,W})$, then $\operatorname{End}(M) \cong \operatorname{End}_{kG}(W) = k$. By Lemma 6.2.13 that $\operatorname{End}(U)$ is isomorphic to a path algebra kQ with no paths of length 2. To describe Q, we first define the following:

- n: The number of connected components, in the category-theoretic sense, of $(\alpha, \omega) \rtimes G$.
- $\alpha_1, \ldots, \alpha_n; \omega_1, \ldots, \omega_n$: A selection of minimal and maximal elements of $(\alpha, \omega) \rtimes G$, each from a different connected component.
- U_i, where $1 \leq i \leq n$: The direct sum of the $k[\alpha_i, \omega_i] \rtimes G_{\alpha_i}$-modules lying in the slice $\mathcal{S}_i$.
- $\{W_1, \ldots, W_m\}$: A complete set of pairwise nonisomorphic simple kG-modules.
- m_i: The number of isomorphism classes of simple kG_{α_i}-modules.

- $\{V_{i,1}, \ldots, V_{i,m_i}\}$, with $1 \le i \le n$: A complete set of pairwise nonisomorphic simple kG_{α_i}-modules.

- $M_{i,k}$, with $1 \le i \le n$ and $1 \le k \le m_i$: The $k\mathcal{P} \rtimes G$-module $P_{\alpha_i,V_k} / \operatorname{Soc}(P_{\alpha_i,V_k})$.

- $e_{j,i,k}$, where $1 \le i \le n$, $1 \le j \le m$, and $1 \le k \le m_i$: The multiplicity of $V_{i,k}$ in $W_j \downarrow^G_{G_{\alpha_i}}$.

- Q_i, $1 \le i \le n$: The quiver satisfying $\operatorname{End}(U_i) \cong kQ_i$ with m_i distinguished vertices labelled $M_{i,k}$ with $1 \le k \le m_i$.

Then we have

$$Q = Q_1 \sqcup \cdots \sqcup Q_n \sqcup \bigcup_{j=1}^{m} \{v_{1,j}, v_{2,j}\}$$

where $\bigcup_{j=1}^{m} \{v_{1,j}, v_{2,j}\}$ is a set of $2m$ labelled vertices corresponding to P_{α,W_j} and $P_{\alpha,W_j} / \operatorname{Soc}(P_{\alpha,W_j})$ respectively. For each j, we add an arrow from $v_{1,j}$ to $v_{2,j}$, and for each i, k with $1 \le i \le n$ and $1 \le k \le m_i$, we add $e_{j,i,k}$ arrows from $v_{2,j}$ to the vertex in $\mathcal{T}_i$ corresponding to the module $M_{i,j}$. $\square$

Chapter 7

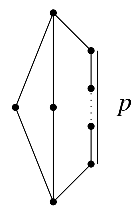
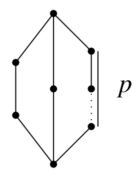
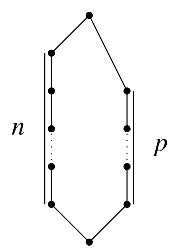
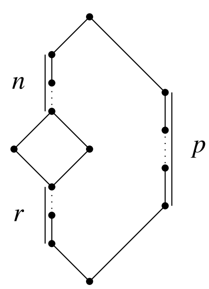
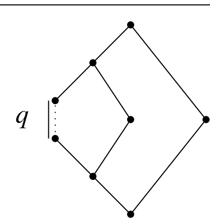
Transporter categories of finite representation type

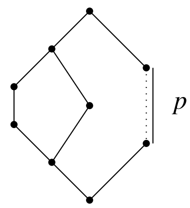
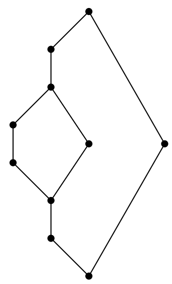
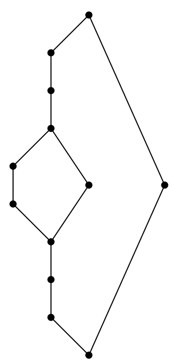
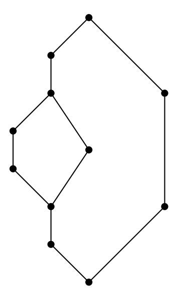
The classification of transporter categories in $\mathcal{ICT}$ of finite representation type can be determined using classical results. We remind the reader that a transporter category algebra $k\mathcal{P} \rtimes G$ is isomorphic to the skew group ring $k\mathcal{P}\#G$. We are able to deduce the next result immediately from work of Reiten and Riedtmann on skew group rings.

Theorem 7.0.1 (Reiten-Riedtmann [12])**.** *Let $\mathcal{P} \rtimes G$ be a finite transporter category, and let k be a field with* $\mathrm{char}(k) \nmid |G|$. *Then $k\mathcal{P} \rtimes G$ has finite representation type if and only if $k\mathcal{P}$ is of finite representation type. In particular, the transporter categories in $\mathcal{ICT}$ of finite representation type are those whose base poset is one listed in* [5].

Proof. Because $k\mathcal{P} \rtimes G$ is a skew group ring, this theorem follows immediately from [12] Theorems 1.1 and 1.3 and the fact that the clamped posets that arise in $\mathcal{ICT}$ are the same as the posets that arise in $\mathcal{ICP}$ in [5], by Proposition 6.2.1. □

Below is the table of the posets in $\mathcal{ICP}$ of finite representation type, as shown in [5]. The posets of finite representation type were first determined by Loupias in [10], and they were later categorized by Drozdowski and Simson in [6]. The Rys labels in the righthand column refer to the labels given in [6].

Poset	Rys Label
	28 $p \geq 1$
	28 $2 \leq p \leq 4$
	29 $n \geq 0$ $p \geq 0$
	30 $n \geq 1$ $r \geq 1$ $p \geq 1$
	30 $1 \leq q \leq 4$

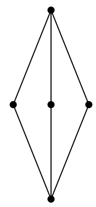
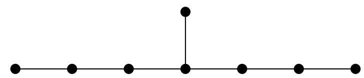
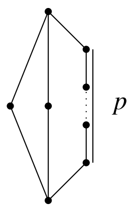
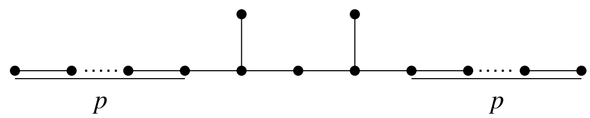
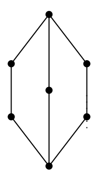
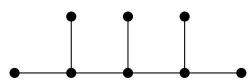
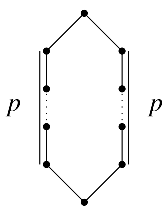
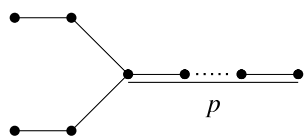
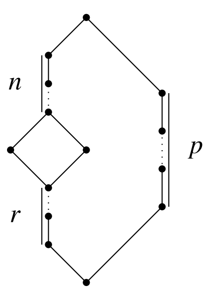
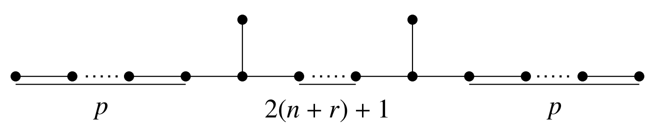
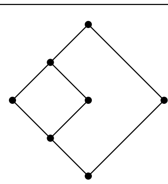
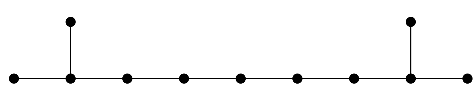
Poset	Rys Label
	30 $1 \le p \le 3$
	30
	30
	30

The following table shows the posets in the table above which have nontrivial automorphism groups, and thus admit a nontrivial group action. We then list the automorphism group and the slice of the component of the Auslander-Reiten quiver for $D^b(\mathcal{P} \rtimes G)$ containing the projective-injective modules. This gives us the following result.

Theorem 7.0.2. *Let $\mathcal{P} \rtimes G$ be a transporter category algebra with $\mathcal{P} \rtimes G \in \mathcal{ICT}$, and let k be a field with* $\mathrm{char}(k) \nmid |G|$. *Of the transporter category algebras $k\mathcal{P} \rtimes G$ of finite representation type where $k = \mathbb{C}$ and the action of the nontrivial group G on $\mathcal{P}$ is free, we have that $\mathbb{C}\mathcal{P} \rtimes G$ is derived equivalent to a quiver algebra of wild representation type, with the following exceptions:*

1. *The base poset and group is in row 4 of the following table and $p = 1$ or $p = 2$, in which case $\mathbb{C}\mathcal{P} \rtimes G$ is derived equivalent to a quiver algebra of finite representation type.*

2. *The poset and underlying group are one of the following rows of this table: row 1, row 1 with S_3 replaced with C_2 or C_3 (not shown), row 2 ($p = 1$), row 4 ($p = 3$), row 5 ($p = 1$), or row 6. In these cases $\mathbb{C}\mathcal{P} \rtimes G$ is derived equivalent to a quiver algebra of tame representation type.*

Poset $\mathcal{P}$	$G = Aut(\mathcal{P})$	Slice of AR quiver for $D^b(\mathbb{C}\mathcal{P} \rtimes G)$
	S_3	
	C_2	
	C_2	
	C_2	
	C_2	
	C_2	

Chapter 8

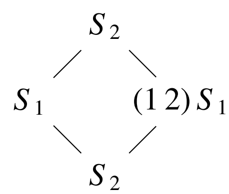
An example involving Young's lattice of partitions

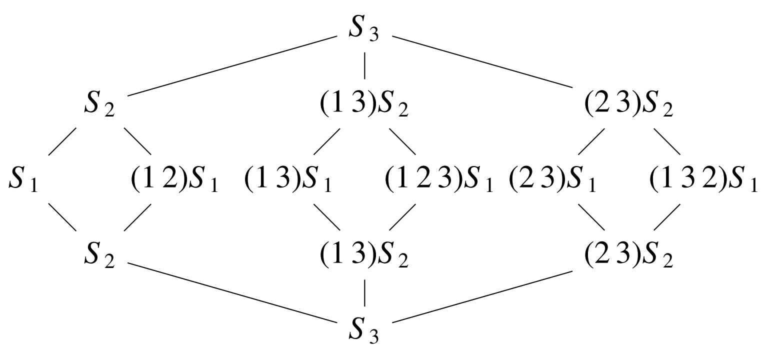
The following example illustrates how the structure of the slice associated to a transporter category in $\mathcal{ICT}$ can relate to other combinatorial structures.

Definition 8.0.1. Let G be a finite group, and let $H_1 \leq H_2 \leq \cdots \leq H_n = G$ be a sequence of nested subgroups of G. Define the G-poset $\mathcal{P}(H_1, \ldots, H_n)$ to be the ranked poset of rank $2n-1$ whose i^{th} row for $1 \leq i \leq n$ consists of the left G-cosets of H_{n+1-i}. The partial order between the first n rows is $gH \leq hK$ if and only if $gH \supseteq hK$ (*not* $gH \subseteq hK$). For $1 \leq i \leq n$, the elements in row $n+i-1$ are the left G-cosets of H_i. For gH and hK in rows greater than or equal to n, we have $gH \leq hK$ if and only if $gh \subset hK$ (*not* $gH \supseteq hK$). This makes $\mathcal{P}(H_1, \ldots, H_n)$ isomorphic to its opposite poset. The action of G on these cosets is given by left multiplication.

The poset $\mathcal{P}(H_1, \ldots, H_n)$ is related to the coset poset of K.S. Brown [3]. The *coset poset* of a finite group G is the set of all cosets in G ordered by inclusion. Brown found that the topological properties of the geometric realization of the coset poset of G was related to the probabilistic zeta function of G.

Example 8.0.2. Let S_n act on $[1, \ldots, n]$ in the usual way. For $n \geq 1$, we will identify $S_n \subset S_{n+1}$ with the subgroup of S_{n+1} which fixes $n+1$. Using this convention, we can describe $\mathcal{P}(S_1, S_2, \ldots, S_n)$ iteratively: the poset $\mathcal{P}(S_1)$ consists of a single point. If $\mathcal{P}(S_1, \ldots, S_{n-1})$ is defined, we can construct $\mathcal{P}(S_1, \ldots, S_n)$ by taking the disjoint union of n copies of $\mathcal{P}(S_1, \ldots, S_{n-1})$ and adding a minimal and maximal element. Note that this makes $\mathcal{P}(S_1, \ldots, S_{n-1})$ a clamped interval in $\mathcal{P}(S_1, \ldots, S_n)$. Below are the posets $\mathcal{P}(S_1, S_2)$ and $\mathcal{P}(S_1, S_2, S_3)$.



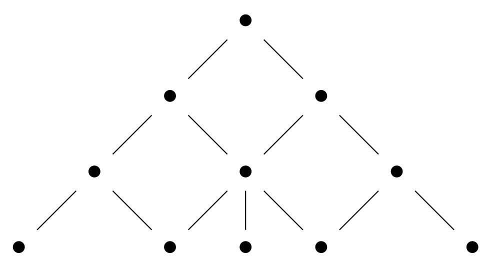


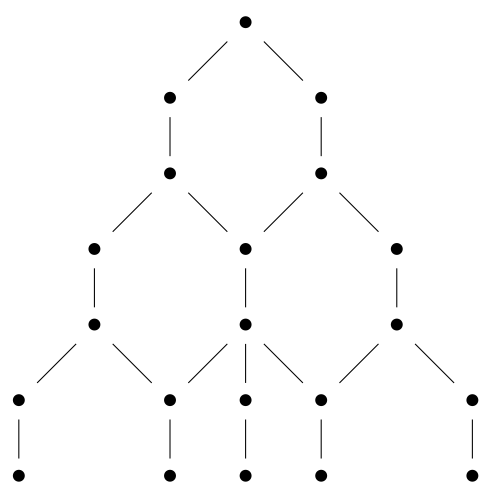
Before presenting an example, we recall the definition of Young's lattice of partitions. This is the ranked poset whose n row consists of the partitions of n. The partial order is generated by the covering relation $\lambda \leq \mu$ if and only if the Ferrers diagram of λ can be obtained by removing a leftmost node from the Ferrers diagram of μ. As in the rest of this document, we write the poset with the least element on top with greater elements below the lesser ones, and row n is below row $n-1$ for all n. For our purposes, we omit the empty partition from Young's lattice.

The Auslander-Reiten quiver of $D^b(\mathbb{C}\mathcal{P}(S_1, \ldots, S_n) \rtimes S_n)$ has a component that can be described in the following way.

Proposition 8.0.3. *The underlying graph of a slice of the component of the Auslander-Reiten quiver of* $D^b(\mathbb{C}\mathcal{P}(S_1, \ldots, S_n) \rtimes S_n)$ *containing the projective-injective modules* *is a modification of Young's lattice of partitions. Starting with the Hasse diagram for Young's lattice, we eliminate rows in positions greater than n, and for rows greater than the first, we replace each vertex by two vertices joined by a new edge. The bottom row of this modified lattice (i.e. row* $2n-1$*) corresponds to the projective-injective modules.*

An example of Young's lattice and the modification corresponding to $D^b(\mathbb{C}\mathcal{P}(S_1, S_2, S_3, S_4) \rtimes S_4)$ is shown below.





Proof. We prove this by induction on n. When $n = 1$, then the transporter category algebra is $\mathbb{C}$ and the underlying graph of the slice $\mathcal{T}$ is a single vertex. Let α and ω denote the minimal and maximal elements of $\mathcal{P}(S_1, \ldots, S_n)$, and let a and z denote the minimal and maximal elements of $\mathcal{P}(S_1, \ldots, S_{n-1})$. Note that (α, ω) consists of n copies of $\mathcal{P}(S_1, \ldots, S_{n-1})$, and S_n acts on these intervals transitively with stabilizers isomorphic to S_{n-1}. It follows that, as a category, we have $(\alpha, \omega) \rtimes S_n \simeq \mathcal{P}(S_1, \ldots, S_{n-1}) \rtimes S_{n-1}$, and there is one maximal clamped interval in $(\alpha, \omega) \rtimes S_n$ up to isomorphism. Thus by

Theorem 6.2.8, the underlying graph of the slice $\mathcal{T}$ of the component of the Auslander-Reiten quiver of $D^b(\mathbb{C}\mathcal{P}(S_1, \ldots, S_n) \rtimes S_n))$ containing the projective-injective modules is obtained by taking the the underlying graph $\tilde{\mathcal{T}}$ of the slice of the Auslander-Reiten quiver of $D^b(\mathbb{C}\mathcal{P}(S_1, \ldots, S_{n-1}) \rtimes S_{n-1})$ and adding $2k$ vertices $(v_{1,1}, v_{2,1}, \ldots, v_{1,k}, v_{2,k}$ to it where k is the number of isomorphism classes of simple $\mathbb{C}S_n$-modules. By the induction hypothesis, the underlying graph $\tilde{\mathcal{T}}$ of the slice is the modification of Young's lattice described in the proposition where the top row corresponds to the projective-injective $\mathbb{C}\mathcal{P}(S_1, \ldots, S_{n-1}) \rtimes S_{n-1}$-modules $P_{a,V}$. There is an edge between $v_{1,j}$ and $v_{2,j}$ for each j, and if $v_{2,j}$ corresponds to the simple $\mathbb{C}S_n$-module W, and V is a simple $\mathbb{C}S_{n-1}$-module, then the number of edges between $v_{2,j}$ and the vertex in $\tilde{\mathcal{T}}$ corresponding to $P_{a,V}$ is the multiplicity of V in $W \downarrow^{S_n}_{S_{n-1}}$. By the branching rule for representations of the symmetric group, the edges added between $v_{2,j}$ and the vertices in $\tilde{\mathcal{T}}$ are the same as the edges appearing between rows n and $n-1$ of Young's lattice. The result is the modification of Young's lattice. □

References

[1] I. Assem, D. Simson, and A. Skowronski, *Elements of the representation theory of associative algebras*, Cambridge University Press, 2006.

[2] M. Auslander and I. Reiten, *Representation theory of Artin algebras. III. Almost split sequences*, Comm. Algebra 3, No. 3 (1975), 239–294.

[3] K. S. Brown, *The coset poset and probabilistic zeta function of a finite group*, J. Algebra, 225 (2000), 989–1012.

[4] T. tom Dieck, *Transformation groups,* de Gruyter, 1987.

[5] K. Diveris, M. Purin, and P. Webb, *The bounded derived category of a poset*, preprint, url https://arxiv.org/abs/1709.03227, 2017.

[6] G. Drozdowski and D. Simson, *Remarks on posets of finite representation type*, unpublished, accessed at https://www-users.mat.umk.pl/~simson/DrozdowskiSimson1978.pdf.

[7] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, Cambridge University Press, 1988.

[8] H. Krause, *Homological theory of representations*, preprint, accessed at https://www.math.uni-bielefeld.de/~ hkrause/HomTheRep.pdf, 2021.

[9] S. Liu, *Auslander-Reiten theory in a Krull-Schmidt category*, São Paulo J. Math. Sci., 4 (2010), 425–472.

[10] M. Loupias, *Indecomposable representations of finite ordered sets*, Representations of Algebras (Proc. Internat. Conf., Carleton Univ., Ottawa, Ont., 1974), pp. 201–209, Lecture Notes in Math., Vol. 488, Springer, Berlin, (1975).

[11] B. Mitchell, *Rings with several objects*, Advances in Math. 8 (1972), 1–161.

[12] I. Reiten and C. Reidtmann, *Skew group rings in the representation theory of Artin algebras*, J. Algebra, 92 (1983), 224–282.

[13] S. Scherotzke, *Finite and bounded Auslander-Reiten components in the derived category*, Journal of Pure and Applied Algebra, 215 (2010), 232–241.

[14] R.W. Thomason, *Homotopy colimits in the category of small categories*, Mathematical Proceedings of the Cambridge Philosophical Society, 85 (1979), 91–109.

[15] P. Webb, A course in finite group representation theory, Cambridge University Press, 2016.

[16] P. Webb, *An introduction to the representation theory and cohomology of categories*, in *Group representation theory*, EPFL Press (Lausanne), 149–173, 2007.

[17] F. Xu, *Homological properties of category algebras*, PhD. thesis, University of Minnesota 2006.