Parametrizations of Irreducible Rational Representations of Coxeter Groups

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Dedication

To my family, for their unwavering support and encouragement throughout this journey.

Abstract

We develop two closely related methods for parametrizing the rational irreducible characters of an arbitrary finite Coxeter group W. The goal is to provide a uniform approach to such a parametrization, independent of Coxeter type. The two methods generalize two approaches to describing the irreducible representations of the symmetric groups, which coincide in type A but do not coincide generally. Our methods associate characters to pairs of reflection subgroups, in one case by considering common constituents of permutation and signed permutation modules, and in the other case by a generalization of the Specht modules. We ask whether, using either method, we can identify a set of subgroup pairs for which the matrix of multiplicities of rational irreducibles in the characters associated to these subgroup pairs is unitriangular. Such a unitriangular matrix provides a parametrization of the irreducible rational characters. For the noncrystallographic types H and I, we give a positive answer to this question. In type H, we show computationally that we can parametrize the irreducible rational characters of H_4 using generalized Specht modules, and we can parametrize the irreducible characters of H_3 using both methods. Moreover, we give an explicit decomposition of the generalized common constituents for the dihedral groups $I_2(n)$ for all n, and we prove that we can always exhibit a unitriangular multiplicity matrix using generalized common constituents. In type A our theory coincides with the classical theory of Specht modules and does not give any new information. In type B the approach we take is closely related to an existing parametrization of the irreducible characters, but it appears to have some novel elements.

Contents

Chapter 1

Introduction

The irreducible complex characters of finite Coxeter groups have been known for a long time. For the infinite families of Coxeter groups of types A, B and D, these characters are constructed using combinatorial data based on integer partitions. In exceptional Coxeter types that are not part of an infinite family, the character tables are known but they are constructed on a type-by-type basis. Our goal is not just to identify all irreducible characters but to parametrize them, by way of a bijection with some indexing set that is defined combinatorially or algebraically. We want to generalize the well-known parametrization by integer partitions in type A to an arbitrary finite Coxeter group W, with a construction defined only in terms of general properties of Coxeter groups and not on a type-by-type basis.

To do this, we consider two closely related but inequivalent approaches, which both generalize the classical representation theory of the symmetric groups. One such approach is the construction of the Specht modules S^λ as cyclic $\mathbb{Q}S_n$-modules generated by polytableaux. This approach is described in [20], for instance. In our construction, we define generalized Specht characters $\phi_{\text{Specht}}(P, Q)$, which are characters of cyclic submodules of $\mathbb{Q}W$ generated by certain group algebra elements analogous to polytableaux. The other, more implicit approach can be found in [11] where it is shown, for each partition λ of n, that there is a unique irreducible character which is a constituent of both $1 \uparrow_{S_\lambda}^{S_n}$ and $\epsilon \uparrow_{S_{\lambda'}}^{S_n}$, appearing in each with multiplicity 1. In our construction, we define a generalized character of common constituents $\phi_{\text{Const}}(P, Q)$, and this has ϕ_{Specht} as a summand. For an arbitrary Coxeter group, the generalized Specht module characters and the generalized common constituents are not equal in general, and they are not irreducible in general. We find, among other subtleties, that while the values of $\phi_{\text{Const}}(P, Q)$ are always determined by the Coxeter types of P and Q alone, the Specht characters $\phi_{\text{Specht}}(P, Q)$ are not. In most Coxeter types one or both generalizations allows us to establish a bijection between the irreducible rational characters of W and a certain collection of reflection subgroup pairs.

In may ways our approach differs from the classical theory, in great part out of the necessity to satisfy various properties our generalization ought to satisfy. First, while we want to associate characters to subgroups of a given Coxeter group such as the Young subgroups in type A, there are generally more irreducible characters than conjugacy classes of parabolic subgroups. It turns out that we need to consider all reflection subgroups, but this still does not resolve the cardinality issue. Examples of Coxeter groups which have more irreducible rational characters than reflection subgroup conjugacy classes include D_4, H_3, and H_4.

Moreover, we want to generalize a natural dual operation on the set of Young subgroups of S_n, arising from the dual operation on integer partitions. To our knowledge, for an arbitrary Coxeter group there is no meaningful way to associate to each reflection subgroup a dual subgroup. However, if we index not by subgroups but by subgroup pairs (P, Q), then we may ask whether the dual of the character constructed from (P, Q) is the tensor product of the character constructed from (Q, P) with the sign character ϵ. This is true for generalized common constituent characters, and Theorem 2.38 shows that this also holds for characters of generalized Specht modules. This generalizes the identity $S^{\lambda'} \simeq S^{\lambda} \otimes \epsilon$ in type A without reference to a combinatorial object that has P and Q as row and column stabilizers. In many cases, our parametrization will be compatible with tensor-epsilon duality of irreducible rational characters, in that whenever ϕ is mapped to (P, Q) in the parametrization, then $\phi \otimes \epsilon$ is mapped to (Q, P).

Finally, we choose to parametrize irreducible rational characters instead of irreducible complex characters, due to certain desired properties of parametrizations and due to the rationality of the generalized characters described above. In type A, we may think of Specht modules S^{λ} as not only parametrized by integer partitions, but parametrized using the unitriangular matrix

$$\left(\frac{\langle \chi(\mathbb{Q}\uparrow_{S_\mu}^{S_n}), \chi(S^{\lambda})\rangle}{\langle \chi(S^{\lambda}), \chi(S^{\lambda})\rangle}\right)_{\lambda,\,\mu\,\vdash\,n}$$

whose rows and columns are ordered consistently with the dominance order on partitions. We may then think of each module of the form $\mathbb{Q}\uparrow_{S_\lambda}^{S_n}$ as being the sum of a 'new' module S^{λ} appearing with multiplicity one, plus a sum of 'previously-encountered' modules S^{μ} with $\mu \trianglerighteq \lambda$. In our two approaches we attempt to construct a unitriangular matrix as above, but with $\mathbb{Q}\uparrow_{S_\lambda}^{S_n}$ replaced by ϕ_{Const} and ϕ_{Specht}. Since these characters are rational, we can only parametrize rational irreducible characters in this way. In particular, the reflection representation of a non-crystallographic Coxeter group cannot be written over $\mathbb{Q}$ and cannot be included in a parametrization as described above. However, a parametrization of rational irreducibles still gives a lot of representation-theoretic information, even in non-crystallographic types where the reflection representation is not itself a rational representation.

We point out that our approach to the parametrization of representations of Coxeter groups has been studied before, and there are many other approaches as well. In the work of Halicioglu [6, 7, 8] and Halicioglu-Morris [9] there are constructed generalized Specht modules with generalized polytableaux defined combinatorially using root systems. A parametrization of rational representations is not attempted by them. In [15] Macdonald gives a construction of certain irreducible representations that works for all finite Coxeter groups, and in types A and B his method constructs all irreducible representations. This is not so in type D, and for non-crystallographic groups his representations need not be rational. Lusztig's approach in [14] includes representations from left cells and special representations, but only applies to Weyl groups. In [4], Al-Aamily, Morris, and Peel exhibit the irreducible representations of B_n in arbitrary characteristic as quotients of generalized Specht modules. James and Peel [12] construct generalized Specht modules for symmetric groups by associating to each skew partition a pair of parabolic subgroups (the row and column stabilizers), and their construction coincides with ours in type A.

The structure of this document is as follows. First, in Chapter 2 we develop our general framework and define generalized common constituents and generalized Specht modules. We prove several useful lemmas at this level of generality and pose several properties which we are interested in investigating on a type-by-type basis.

Then, Chapter 3 reviews the classical constructions of irreducible characters in Types A and B, both as Specht characters and as common constituent characters. With either method, the characters constructed are absolutely irreducible rational characters, and all such irreducibles can be constructed in this way.

Chapter 4 details our original computational results for parametrizing irreducible rational characters in types H_3 and H_4. These are examples where certain properties of $\phi_{\text{Const}}(P, Q)$ and $\phi_{\text{Specht}}(P, Q)$ such as reducibility differ from type A. We prove that the irreducible rational characters of H_3 can be parametrized using generalized common constituents or using generalized Specht modules. We show that parametrizing the irreducible rational characters of H_4 using generalized common constituents is impossible, however there exists a parametrization using generalized Specht characters.

Chapter 5 describes our general computational process in constructing common constituents and Specht modules. We describe both the construction of generalized Specht modules over a finite field in GAP and the inference of the structure of the corresponding rational Specht characters.

Finally, in Chapter 6 we explicitly construct and decompose the common constituent characters for the non-crystallographic Coxeter groups of type $I_2(m)$, which are the dihedral groups D_{2m}. This allows us to give a parametrization of the irreducible rational characters using common constituents.

We establish notation that we use throughout this document. Here, K denotes a field of arbitrary characteristic, not necessarily algebraically closed. We use $\chi(-)$ to denote the mapping that sends a module (defined over some subfield of $\mathbb{C}$) to its character. However, when referring to characters we use ϕ to denote a rational character and χ to denote a complex character which may or may not be rational. We also sometimes refer to a $\mathbb{C}G$-module, its corresponding linear representation, and its character interchangeably, especially in the cases of the trivial representation 1, the sign representation ϵ of a finite Coxeter group W, and induced representations of the form $1 \uparrow_{W'}^{W}$ (permutation modules) and $\epsilon \uparrow_{W'}^{W}$ (signed permutation modules). We may distinguish between the trivial and sign representation of W and that of a subgroup W', for instance writing 1_W and $1_{W'}$, but we will omit suffixes in certain settings where the context is clear or the lack of distinction does not affect the discussion.

We also sometimes refer to a Coxeter group, its Coxeter type, and alternate descriptions interchangeably (e.g. $W(A_{n-1})$ vs. A_{n-1} vs. S_n). In data and discussions that reference a Coxeter type of a subgroup of a Coxeter group, either there is a canonical choice of subgroup which has that type, or the discussion does not depend on the choice of subgroup, or a choice of subgroup will be explicitly identified.

If v is an element of a group W or the group algebra KW and $w \in W$, we denote ${}^{w}v := wvw^{-1}$ and $v^{w} := w^{-1}vw$. If P is a subgroup of W and $w \in W$, then we denote by ${}^{w}P := wPw^{-1}$ and $P^{w} := w^{-1}Pw$ the conjugate subgroups of W. If $w \in W$ and V is a representation of a subgroup P of W, with character χ, then we denote by ${}^{w}V$ the conjugate representation of ${}^{w}P$.

If A is a finite-dimensional algebra over some field $\mathbb{F}$, V is an A-module equipped with a bilinear form, and U is an A-submodule of V, we denote by $U^{\perp}$ the orthogonal complement of U in V. We denote by $U^{\top}$ the annihilator of U in A. If U is generated by a vector u, then we write $u^{\top}$ instead of $U^{\top}$. We denote by U^{*} the linear dual $\mathrm{Hom}_{\mathbb{F}}(U, \mathbb{F})$.

Chapter 2

Parametrizations of Rational Characters by Subgroup Pairs

In this chapter we describe two closely related approaches to parametrizing the rational characters of an arbitrary finite Coxeter group: the generalized common constituents character $\phi_{\mathrm{Const}}(P, Q)$ and the generalized Specht character $\phi_{\mathrm{Specht}}(P, Q)$ where P and Q are reflection subgroups of W. For each approach, we define properties that can be thought of as criteria for a suitable parametrization of the rational irreducible characters.

Throughout, let W be a fixed finite Coxeter group. We recall (for example, from [5] or [10]) that any Coxeter group W has a presentation with a set of distinguished generators, called **Coxeter generators**. We write $S = \{s_i\}$ for the set of Coxeter generators. The Coxeter generators satisfy relations $s_i^2 = 1$ and $(s_i s_j)^{m(i,j)} = 1$, where $m(i, j) \geq 2$ for all $i \neq j$. We say that (W, S) is a **Coxeter system** with Coxeter generators S. We denote by $\ell(w)$ the **length** of w, which is a well-defined positive integer equal to the number of factors occurring in any reduced expression for w as a product of Coxeter generators.

The **Coxeter diagram** for W is the undirected graph with the Coxeter generators of W as vertices, and an edge between each pair $\{s_i, s_j\}$ with $m(i, j) > 2$. If $m(i, j) > 3$, then the edge is labeled with the value of $m(i, j)$. The classification of finite Coxeter groups of rank $n > 1$ is presented in Table 2.1, identifying Coxeter types in most cases when the underlying abstract groups are isomorphic.

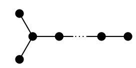
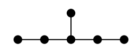
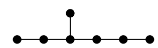
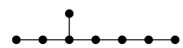
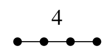
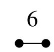
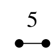
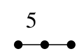
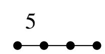
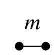
Coxeter type	Coxeter diagram
A_n	
$B_n \cong C_n$	
D_n	
E_6	
E_7	
E_8	
F_4	
$G_2 \cong I_2(6)$	
$H_2 \cong I_2(5)$	
H_3	
H_4	
$I_2(m)$	

Table 2.1: Classification of finite Coxeter groups of rank $n \geq 1$.

The **sign** of an element $w \in W$ is defined by $\epsilon(w) = (-1)^{\ell(w)}$. This defines a group homomorphism $\epsilon : W \to \{\pm 1\}$. The **alternating subgroup** of W, denoted $A(W)$, is the subgroup $\ker \epsilon$, or equivalently, the set of all elements of W equal to the product of an even number of Coxeter generators. The sign homomorphism extends to a nontrivial 1-dimensional representation of W, also denoted ϵ. The trivial representation 1 and the sign representation ϵ are two one-dimensional representations defined on every finite Coxeter group, over any field, and they are distinct whenever the characteristic of the field is not equal to 2.

Without explicitly defining the term 'parametrization,' we will say that we want to exhibit a bijection between the irreducible rational representations of W (or their characters) and a certain finite set. The finite set should have some algebraic or combinatorial structure in its own right, or at least it should be relevant to the algebraic or combinatorial structure of W. This set should generalize the set of partitions of a nonnegative integer n, which parametrizes the Specht modules of the symmetric group $S_n \cong W(A_{n-1})$. In practice, the index set will be a certain set of pairs of subgroups of W, which generalize the row and column stabilizers of a Young tableau. The bijection is often defined using a matrix whose rows are labeled by the rational irreducible characters, and whose columns are indexed by subgroup pairs. In this setup, the parametrization maps the i-th subgroup pair to the i-th irreducible character.

2.1 Reflection Subgroups

Every finite Coxeter group of rank n is isomorphic to a finite **reflection group**, which is a group of isometries of an n-dimensional real vector space generated by its reflections (see [5] or [10]). In our approach, we do not use the geometric structure of reflection groups, but we may still define reflections algebraically.

Definition 2.1. *Let W be a Coxeter group. A* ***reflection*** *is a conjugate of a Coxeter generator in W. A* ***reflection subgroup*** *is a subgroup of W generated by reflections. A* ***parabolic subgroup*** *is a conjugate of a subgroup of W generated by Coxeter generators.*

If T is a set of Coxeter generators (resp. reflections) in W, then we denote by W_T the parabolic subgroup (resp. reflection subgroup) of W generated by T. For a reflection subgroup P of W, denote by $T(P)$ the set of reflections in P. Denote by $\mathcal{R}$ (resp. $\mathcal{P}$) the set of reflection subgroups (resp. parabolic subgroups) of W, partially ordered by inclusion.

The structure of parabolic subgroups is well-known, see for instance [1] and [5]. Every parabolic subgroup is a Coxeter group: in particular, for each $J \subseteq S$, J is a set of Coxeter generators for W_J. The length function ℓ_J of W_J satisfies $\ell_J(w) = \ell(w)$ for all $w \in W_J$, and any reduced expression for w consists only of factors in J. Moreover, $W_J \cap W_K = W_{J\cap K}$ and $\langle W_J \cup W_K \rangle = W_{J\cup K}$. More generally, it holds that the intersection of any two parabolic subgroups is a parabolic subgroup (see Theorem 2.1.12 of [5] and the discussion following).

Reflection subgroups share many but not all of the properties afforded by parabolic subgroups. In particular, they are Coxeter groups. Letting T denote the set of reflections in W, define $N(w) = \{t \in T : \ell(tw) < \ell(w)\}$. Dyer proved in [3] that a reflection subgroup P of W is a Coxeter group with simple generators given by $\{t \in T : N(t) \cap P = \{t\}\}$. We have the following property of alternating subgroups of reflection subgroups:

Proposition 2.2. *Let $W' \leq W$ be a reflection subgroup. Then $A(W') = A(W) \cap W'$.*

Proof. Let $w \in A(W')$. Then w is a product of an even number of reflections in W'. Writing each reflection as a conjugate of a Coxeter generator, we have that we have that the reflections have odd length in W. Thus, w has even length in W, so $w \in A(W) \cap W'$. Analogously, if $w \in W'$ is a product of an odd number of reflections in W', then it has odd length in W. Therefore, an element of W' is contained in $A(W')$ if and only if it is contained in $A(W)$. □

We observe that the identity $\langle W_J \cup W_K \rangle = W_{J \cup K}$ still holds when J and K are sets of reflections. However, the intersection of two reflection subgroups is not a reflection subgroup in general. As a counterexample, let $W = W(B_2) = \langle a, b : a^2 = b^2 = (ab)^4 = 1 \rangle$. We take $P = \langle a, bab \rangle$ and $Q = \langle b, aba \rangle$ which are both Abelian subgroups isomorphic to $C_2 \times C_2$. Then $P \cap Q = \langle abab \rangle$ which is contained in the alternating subgroup $A(W)$. Since reflections all have sign -1, and $A(W)$ contains no elements of sign -1, we conclude that $P \cap Q$ is not generated by reflections.

Proposition 2.3. *The poset $\mathcal{R}$ is a lattice with join operation $W_J \vee W_K = W_{J \cup K}$ and meet operation $W_J \wedge W_K = W_{T(W_J) \cap T(W_K)}$.*

Proof. For any fixed sets of reflections J and K, we immediately observe $\langle W_J \cup W_K \rangle = W_{J \cup K}$. Now, if $W_J = W_{J'}$ and $W_K = W_{K'}$, then

$$W_{J \cup K} = \langle W_J \cup W_K \rangle = \langle W_{T(W_J)} \cup W_{T(W_K)} \rangle = \langle W_{T(W_{J'})} \cup W_{T(W_{K'})} \rangle = \langle W_{J'} \cup W_{K'} \rangle = W_{J' \cup K'}.$$

Thus the operation $W_J \vee W_K = W_{J \cup K}$ is a well-defined operation on the set of reflection subgroups of W. Now $W_J, W_K \leq W_J \vee W_K$, and any reflection subgroup containing W_J and W_K must contain $J \cup K$, so it contains $W_{J \cup K}$. Therefore $\mathcal{R}$ is a join-semilattice. Since $\mathcal{R}$ is finite and has the identity subgroup as a zero element, then $\mathcal{R}$ is a lattice with meet operation given by

$$W_J \wedge W_K = \bigvee_{P \in \mathcal{R}, P \leq W_J \text{ and } P \leq W_K} P.$$

We show that this equals $W_{T(W_J) \cap T(W_K)}$. Indeed, P is one of the terms in the join expression above if and only if $T(P)$ is a subset of W_J and of W_K. This is equivalent to $T(P) \subseteq T(W_J) \cap T(W_K)$, which is equivalent to $P \leq W_{T(W_J)\cap T(W_K)}$. □

We sometimes want to consider reflection subgroups of W taken up to conjugacy. Let Ω be a partially ordered set with an order-preserving W-action. We denote by $W\backslash\Omega$ the set of W-orbits of Ω.

Proposition 2.4. *Suppose W is finite. Then the set $W\backslash\Omega$ is partially ordered with $\overline{\omega} \leq_{W\backslash\Omega} \overline{\omega'}$ if there exists $\omega \in \overline{\omega}$, $\omega' \in \overline{\omega'}$ such that $\omega \leq \omega'$ in Ω.*

Proof. We have that $\leq_{W\backslash\Omega}$ is reflexive since $\leq_\Omega$ is reflexive. To show transitivity, suppose $\omega_0 \in \overline{\omega_0}$, $\omega_1, \omega_1' \in \overline{\omega_1}$, and $\omega_2 \in \overline{\omega_2}$ with $\omega \leq \omega_1$ and $\omega_1' \leq \omega_2$. Choose w such that $w \cdot \omega_1 = \omega_1'$. Then $\omega_0 \leq w^{-1} \cdot \omega_1' \leq w^{-1} \cdot \omega_2$ and therefore $\overline{\omega_0} \leq \overline{\omega_2}$.

Finally, to show antisymmetry, suppose $\overline{\omega_0} \leq \overline{\omega_1}$. and $\overline{\omega_1} \leq \overline{\omega_0}$. Take $\omega_0, \omega_0' \in \overline{\omega_0}$ and $\omega_1, \omega_1' \in \overline{\omega_1}$ with $\omega_0 \leq \omega_1$ and $\omega_1' \leq \omega_0'$. Choose $w_0, w_1 \in W$ such that $w_0 \cdot \omega_0 = \omega_0'$ and $w_1 \cdot \omega_1 = \omega_1'$. Then

$$\omega_0 \leq \omega_1 = w_1^{-1}\omega_1' \leq w_1^{-1}\omega_0' = w_1^{-1}w_0\omega_0.$$

Then $\omega_0 \leq (w_1^{-1}w_0)^n\omega_0$ for all $n \geq 0$. Since $w_1^{-1}w_0$ has finite order, it follows that $\omega_0 = (w_1^{-1}w_0)^n\omega_0$ for all n. In particular, $\omega_0 = w_1^{-1}w_0\omega_0$, and in view of the indented inequality above, we have that $\omega_1 = \omega_0 \in \overline{\omega_0}$, so $\overline{\omega_0} = \overline{\omega_1}$ and therefore $\leq_{W\backslash\Omega}$ is a partial order. □

When the action of W is conjugation (or pairwise conjugation) we use the notation Ω^{conj} instead of $W\backslash\Omega$. In particular, we are interested in the poset $\mathcal{R}^{\text{conj}}$ of conjugacy classes of reflection subgroups ordered by conjugate-inclusion. Informally, we will refer to elements of $\mathcal{R}^{\text{conj}}$ as reflection subgroups with the understanding that $P \leq Q$ if and only if P is conjugate to a subgroup of Q.

There are two posets of interest to us when considering pairs of reflection subgroups: $\mathcal{R}^{\text{conj}} \times \mathcal{R}^{\text{conj}}$ and $(\mathcal{R} \times \mathcal{R})^{\text{conj}}$. The product $\mathcal{R}^{\text{conj}} \times \mathcal{R}^{\text{conj}}$ consists of pairs of subgroups (P, Q) of W where $(P, Q) = (P', Q')$ if and only if P is conjugate to P' and Q is conjugate to Q'. In $(\mathcal{R} \times \mathcal{R})^{\text{conj}}$, however, $(P, Q) = (P', Q')$ if and only if there exists $w \in W$ such that $P^w = P'$ and $Q^w = Q'$. It will turn out that the Specht character $\chi(S(-,-))$ is not invariant under separate conjugacy of P and Q, so it is not a function of $\mathcal{R}^{\text{conj}} \times \mathcal{R}^{\text{conj}}$, but it is a function of $(\mathcal{R} \times \mathcal{R})^{\text{conj}}$.

We note that $\mathcal{R}^{\text{conj}}$ is not a lattice even though $\mathcal{R}$ is. In Chapter 3, we will see in type A that the conjugacy classes of reflection subgroups are the Young subgroups up to conjugacy, so they are in bijection with the set of partitions of n. The classical parametrization of irreducible characters of S_n uses the poset of partitions of n, which is a lattice that refines the conjugate-inclusion order on $\mathcal{R}^{\text{conj}}$.

2.2 Generalized Common Constituent Characters

We describe a first approach to parametrizing simple rational representations of arbitrary Coxeter groups. We will find, ultimately, that it does not give a full parametrization for all Coxeter groups (the group H_4 is an example), but that it does work for many. Because it is a natural generalization of the representation theory of the symmetric groups, and because it does give a full parametrization in many cases, we develop this approach here.

Definition 2.5. *Let* $(P, Q) \in \mathcal{R} \times \mathcal{R}$. *The* ***common constituent character of*** P ***and*** Q *(or the common constituents of* P *and* Q*) is defined as*

$$\phi_{\text{Const}}(P, Q) = \sum_{\chi \text{ irr. comp. char}} \min\left(\langle 1 \uparrow_P^W, \chi\rangle, \langle \epsilon \uparrow_Q^W, \chi\rangle\right)\chi.$$

Namely, the multiplicity of a complex character in $\phi_{\text{Const}}(P, Q)$ is the minimum of its multiplicities in the permutation module on P and the signed permutation module on Q. Observe that if P and P' are conjugate subgroups of W, then the representations $1 \uparrow_P^W$ and $1 \uparrow_{P'}^W$ are isomorphic, as are $\epsilon \uparrow_P^W$ and $\epsilon \uparrow_{P'}^W$. Therefore ϕ_{Const} is a well-defined function on $\mathcal{R}^{\text{conj}} \times \mathcal{R}^{\text{conj}}$.

We list some initial properties of ϕ_{Const}.

Proposition 2.6. $\phi_{\text{Const}}(P, Q)$ *is the character of a rational representation of* W.

Proof. Let ρ be an irreducible rational representation of W, and let ρ' be any rational representation of W. We may write $\rho = \rho_1^{a_1} \oplus \cdots \oplus \rho_k^{a_k}$ where ρ_i are the distinct irreducible complex constituents of ρ. Now, observe that we may write ρ' as a nonnegative integer direct sum of irreducible rational representations, and then write each irreducible rational constituent as a nonnegative integer direct sum of irreducible complex representations. Observe that distinct irreducible rational representations have distinct complex summands, since the space of homomorphisms is zero. It follows that the multiplicity of ρ_i in ρ' is equal to a_i times some integer $c_{\rho'}^{\rho}$ which is the multiplicity of ρ in ρ' as rational characters.

We apply this simultaneously to $\rho' = 1 \uparrow_P^W$ and $\rho' = \epsilon \uparrow_Q^W$ and now consider the characters of these representations. Let χ_i be the complex character of the irreducible representation ρ_i defined above, and denote by ϕ_ρ the rational character of ρ. We have that the multiplicity of χ_i in $\phi_{\mathrm{Const}}(P, Q)$ is $\min\{a_i c^{\rho}_{1\uparrow_P^W}, a_i c^{\rho}_{\epsilon\uparrow_Q^W}\}$. Therefore

$$\phi_{\mathrm{Const}}(P, Q) = \bigoplus_{\rho} \min\{c^{\rho}_{1\uparrow_P^W}, c^{\rho}_{\epsilon\uparrow_Q^W}\}\phi_\rho,$$

with this sum taken over a collection of rational irreducible representations ρ. Thus, $\phi_{\mathrm{Const}}(P, Q)$ is the character of a rational representation. □

The multiplicities defined in the above proof allow us to construct a rational representation whose character is $\phi_{\mathrm{Const}}(P, Q)$. However, there is no evident way to canonically identify this representation with a particular subrepresentation of $1 \uparrow_P^W$ or $\epsilon \uparrow_Q^W$.

Lemma 2.7. *If $(P_1, Q_1) \le (P_2, Q_2)$ in $(\mathcal{R} \times \mathcal{R})^{\mathrm{conj}}$ then $\phi_{\mathrm{Const}}(P_1, Q_1) - \phi_{\mathrm{Const}}(P_2, Q_2)$ is a character of W.*

The expression in the lemma is always a virtual character, and the point is that it is a character.

Proof. If $(P_1, Q_1) \le (P_2, Q_2)$ in $(\mathcal{R} \times \mathcal{R})^{\mathrm{conj}}$ then as subgroups of W and to within conjugacy, $P_1 \le P_2$ and $Q_1 \le Q_2$. Then we have $1 \uparrow_{P_2}^W$ is a summand of $1 \uparrow_{P_1}^W$ and $\epsilon \uparrow_{Q_2}^W$ is a summand of $\epsilon \uparrow_{Q_1}^W$. Thus, the common constituents of $1 \uparrow_{P_2}^W$ and $\epsilon \uparrow_{Q_2}^W$ appear in both $1 \uparrow_{P_1}^W$ and $\epsilon \uparrow_{Q_1}^W$, and the result follows. □

Lemma 2.8. *For all $P, Q \in (\mathcal{R} \times \mathcal{R})^{\mathrm{conj}}$, $\phi_{\mathrm{Const}}(Q, P) = \phi_{\mathrm{Const}}(P, Q) \otimes \epsilon$.*

Proof. We observe that the multiplicity of an irreducible complex character χ in a character χ' equals the multiplicity of $\chi \otimes \epsilon$ in $\chi' \otimes \epsilon$. Thus, for each irreducible χ, the minimum of the multiplicities of χ in $1 \uparrow_Q^W$ and $\epsilon \uparrow_P^W$ is equal to the minimum of the multiplicities of $\chi \otimes \epsilon$ in $(1 \uparrow_Q^W) \otimes \epsilon \cong \epsilon \uparrow_Q^W$ and $(\epsilon \uparrow_P^W) \otimes \epsilon \cong 1 \uparrow_P^W$. The claim follows. □

Common constituents of representations are closely related to homomorphisms between representations. The complex inner product $\langle \chi, \chi' \rangle$ is equal to the dimension of the space of W-equivariant linear maps between complex representations $\rho : V \to V$ and $\rho' : V' \to V'$ whose characters are χ and χ'. This value is often denoted $i(\rho, \rho')$ and referred to as the **intertwining number** of ρ and ρ'.

Over any field K containing $\mathbb{Q}$, the group algebra $\mathbb{C}W$ is semisimple and every finite-dimensional KW-module is a direct sum of irreducible representations. A nonzero homomorphism between U and V maps some nonzero submodule $U' \subseteq U$ isomorphically onto a submodule $V' \subseteq V$. Then $\chi(\rho|_{V'}) = \chi(\rho'|_{U'})$ is a summand of both χ and χ'. Thus, the modules U and V have a common constituent if and only if $i(\rho, \rho') = \langle \chi, \chi' \rangle \neq 0$.

In the case that $\chi_1 = 1 \uparrow_P^W$ and $\chi_2 = \chi \uparrow_Q^W$ where P and Q are reflection subgroups of W, and χ is a character of a one-dimensional representation of Q, we can give a more explicit characterization of $\langle \chi_1, \chi_2 \rangle$.

Proposition 2.9. *Let W be any finite group, and let P and Q be subgroups of W. Let χ be a character of Q of degree 1. Then*

$$\langle 1 \uparrow_P^W, \chi \uparrow_Q^W \rangle = |\{PwQ \in P\backslash W/Q : P \cap {}^wQ \leq \ker \chi\}|.$$

Proof. Let ρ be a representation of W whose character is χ. By Mackey's Decomposition Formula,

$$\begin{aligned}
\langle 1 \uparrow_P^W, \rho \uparrow_Q^W \rangle &= \langle 1_P, \rho \uparrow_Q^W \downarrow_P^W \rangle \\
&= \sum_{PwQ \in P\backslash W/Q} \left\langle 1_P, \left({}^w(\rho \downarrow^Q_{w^{-1}Pw \cap Q}) \right) \uparrow^P_{P \cap wQw^{-1}} \right\rangle \\
&= \sum_{PwQ \in P\backslash W/Q} \left\langle 1_P, ({}^w\rho) \downarrow^{wQw^{-1}}_{P \cap wQw^{-1}} \uparrow^P_{P \cap wQw^{-1}} \right\rangle \\
&= \sum_{PwQ \in P\backslash W/Q} \langle 1 \downarrow_{P \cap wQw^{-1}}, ({}^w\rho) \downarrow_{P \cap wQw^{-1}} \rangle.
\end{aligned}$$

Now, the characters $1 \downarrow_{P \cap wQw^{-1}}$ and ${}^w\rho \downarrow_{P \cap wQw^{-1}}$ are absolutely irreducible, as they have degree 1. Then the intertwining number of these characters is 1 if and only if the characters are equal, otherwise it is 0. Moreover, $({}^w\rho) \downarrow_{P \cap wQw^{-1}} \equiv 1$ if and only if $P \cap wQw^{-1} \in \ker({}^w\rho) = \ker(\rho) = \ker(\chi)$. The result follows. $\square$

We apply this to the restrictions of the degree-1 characters of W to reflection subgroups P and Q. When $\chi = 1$, as a character of Q, we have $\ker\chi = Q$ and we recover the well-known identity $\langle 1\uparrow_P^W, 1\uparrow_Q^W\rangle = |P\backslash W/Q|$ which holds for all finite groups and subgroup pairs. When $\chi = \epsilon_Q$, we obtain the following:

Corollary 2.10. *Let P and Q be reflection subgroups of W. Then we have the following:*

1. $\langle 1\uparrow_P^W, \epsilon\uparrow_Q^W\rangle = |\{PwQ \in P\backslash W/Q : P \cap {}^wQ \leq A(W)\}|$.
2. $\phi_{\mathrm{Const}}(P, Q) \neq 0$ *if and only if there exists* $w \in W$ *such that* $P \cap {}^wQ \leq A(W)$.

Proof. Observe that $\ker\epsilon_Q = A(Q)$ is the alternating subgroup of Q. By Proposition 2.2, $A(Q) = A(W) \cap Q$. Then Part 1 follows from Proposition 2.9, and Part 2 follows immediately from Part 1. □

Definition 2.11. *We say a pair of reflection subgroups (P, Q) has the* ***alternating intersection property*** *if there exists $w \in W$ such that $wPw^{-1} \cap Q \leq A(W)$.*

We will see later that in type A, the alternating intersection property becomes the **trivial intersection property** of pairs of Young subgroups $S_\lambda, S_\mu \leq S_n$, which states that there exists $\sigma \in S_n$ such that $\sigma S_\lambda \sigma^{-1} \cap S_\mu = 1$. This is because the intersection of two Young subgroups is also a Young subgroup.

2.2.1 Parametrizations via Generalized Common Constituents

We now describe how a parametrization of the irreducible rational characters can arise from common constituent characters. We take the viewpoint of ϕ_{Const} being a function whose domain is $\mathcal{R}^{\mathrm{conj}} \times \mathcal{R}^{\mathrm{conj}}$ and whose codomain is the vector space of complex-valued class functions on W. For any character-valued function ϕ whose domain is a partially-ordered set Ω, we define the **support** of ϕ as $\{\omega \in \Omega : \phi(\omega) \neq 0\}$.

Suppose that the rational irreducible characters of W are linearly ordered $\chi_1, \ldots, \chi_m$, and let $(P_j, Q_j)_{j=1}^m$ be a list of pairs of reflection subgroups of W (inequivalent up to either separate or joint conjugacy), and ϕ is a function that maps each subgroup pair listed above to a class function of W. Then we define the **multiplicity matrix** to be $M =$

$M^\phi = \left(\frac{\langle \phi(P_j,Q_j),\chi_i\rangle}{\langle \chi_i,\chi_i\rangle}\right)_{i,j=1}^m$. As discussed above, whenever we describe a parametrization in terms of a matrix whose rows and columns are labelled respectively by irreducible rational characters and by pairs of subgroups of W, the parametrization maps the i-th subgroup pair to the i-th character.

For the character-valued function $\phi = \phi_{\text{Const}}$ we consider the following properties:

Property 2.12. The values of ϕ_{Const} on the maximal elements of its support form a complete list, without repetition, of the irreducible rational characters of W.

Property 2.13. All values of ϕ_{Const} on maximal elements of its support are irreducible rational characters of W.

Property 2.14. All irreducible rational characters arise as values of ϕ_{Const} on maximal elements of its support.

Property 2.15. There is a linear ordering $\phi_1, \dots, \phi_N$ on the set of rational characters of W and a list of subgroup pairs $(P_j, Q_j)_{j=1}^N$ such that $M^{\phi_{\text{Const}}}$ is unitriangular.

Property 2.16. There is a linear ordering $\phi_1, \dots, \phi_N$ on the set of rational characters of W and a list of subgroup pairs $(P_j, Q_j)_{j=1}^N$ such that $M^{\phi_{\text{Const}}}$ is unitriangular. Whenever the resulting parametrization maps (P, Q) to ϕ, it maps (Q, P) to $\phi \otimes \epsilon$.

Property 2.17. For each rational character ϕ of G there is a pair of reflection subgroups (P, Q) for which ϕ occurs with multiplicity 1 in $\phi_{\text{Const}}(P, Q)$.

Remark 2.18. Property 2.16 implies that the multiplicity matrix is invariant under simultaneously interchanging the column of (P, Q) with (Q, P) and the row of ϕ with $\phi \otimes \epsilon$. This follows from Lemma 2.8, since $\phi_{\text{Const}}(Q, P) = \phi_{\text{Const}}(P, Q) \otimes \epsilon$.

Each of the above properties has the potential to provide a parametrization of the irreducible characters in terms of pairs of reflection subgroups. However, for some properties, a parametrization is not necessarily unique or canonical.

Proposition 2.19. *We have the following implications between the above properties:*

$$2.13 \Leftarrow 2.12 \Rightarrow 2.14 \Rightarrow 2.15 \Rightarrow 2.17, \qquad 2.16 \Rightarrow 2.15.$$

Proof. By definition, Property 2.12 implies both Properties 2.13 and 2.14. Moreover, Property 2.16 implies Property 2.15 by definition. Given Property 2.14, we may fix any ordering $\phi_1, \ldots, \phi_m$ on the irreducible rational characters, and then choose subgroup pairs (P_i, Q_i) such that $\phi_{\text{Const}}(P_i, Q_i)$ is irreducible and equal to ϕ_i. Then the multiplicity matrix is the identity matrix, so Property 2.15 holds. Finally, Property 2.15 directly implies Property 2.17 by definition. □

Problem 2.20. *Determine which properties hold for which finite Coxeter groups.*

Example 2.21. Let $W = A_2$ (isomorphic to the symmetric group S_3). This group has three conjugacy classes of reflection subgroups, of types A_2, A_1, and 1. These classes index the columns of Table 2.2, which gives character decompositions for all induced characters of the form $1 \uparrow_P^W$ and $\epsilon \uparrow_Q^W$. The character in the (P, Q)-entry is the common constituent character of $1 \uparrow_P^W$ and $\epsilon \uparrow_Q^W$.

		A_2	A_1	1
		ϵ	$\phi_2 + \epsilon$	$1 + 2\phi_2 + \epsilon$
A_2	1	0	0	1
A_1	$1 + \phi_2$	0	ϕ_2	$1 + \phi_2$
1	$1 + 2\phi_2 + \epsilon$	ϵ	$\phi_2 + \epsilon$	$1 + 2\phi_2 + \epsilon$

Table 2.2: Table of common constituents of permutation and signed permutation characters for $W = A_2$.

	$(A_2, 1)$	(A_1, A_1)	$(1, A_2)$
1	1	0	0
ϕ_2	0	1	0
ϵ	0	0	1

Table 2.3: Multiplicity matrix for selected common constituents for $W = A_2$.

Here, the maximal elements of the support of ϕ_{Const} are exactly the pairs chosen for the multiplicity matrix, and these give a complete list of the irreducible rational characters of A_2 without repetition. Thus Property 2.12 holds for A_2, which implies all other properties.

Example 2.22. We give an example in which the multiplicity matrix is not the identity matrix. All relevant reflection subgroup conjugacy classes and character direct sum decompositions were determined computationally in GAP.

Let $W = D_4$. This group has twelve conjugacy classes of reflection subgroups. There are three conjugacy classes of type D_3, denoted D_{3a}, D_{3b}, and D_{3c}, and three conjugacy classes of type D_2, denoted D_{2a}, D_{2b}, and D_{2c}. In $\mathcal{R}^{\text{conj}}$, the only conjugate-inclusion relations among these six classes are $D_{2a} \leq D_{3a}$, $D_{2b} \leq D_{3b}$, and $D_{2c} \leq D_{3c}$. The remaining subgroup classes are of type D_4, A_1^4, A_1^3, A_2, A_1, and 1. There are 13 irreducible complex characters, which are all rational. We denote these by $1, \epsilon, \phi_2, \phi_{3a}, \phi_{3b}, \phi_{3c}, \phi_{3d}, \phi_{3e}, \phi_{3f}, \phi_{4a}, \phi_{4b}, \phi_6$, and ϕ_8. Under this labeling, $\phi_{3a} \otimes \epsilon = \phi_{3d}$, $\phi_{3b} \otimes \epsilon = \phi_{3e}$, $\phi_{3c} \otimes \epsilon = \phi_{3f}$, and $\phi_{4a} \otimes \epsilon = \phi_{4b}$.

Table 2.4 gives the table of common constituents where the (P, Q) entry is the direct sum decomposition of $\phi_{\text{Const}}(P, Q)$ into irreducible characters. Ellipses denote a nonzero character which is omitted for conciseness, and we have also omitted the character decompositions of the permutation and signed permutation modules. Table 2.5 gives a unitriangular multiplicity matrix for one possible collection of subpairs, which has the property that (P, Q) is mapped to ϕ if and only if (Q, P) is mapped to $\phi \otimes \epsilon$. Thus, Property 2.16 holds. However, Property 2.13 does not hold, as for instance (A_2, A_2) is a maximal element of the support of ϕ_{Const}, but $\phi_{\text{Const}}(A_2, A_2) = \phi_6 + \phi_8$ which is reducible. Conversely, Property 2.14 does not hold, since the irreducible ϕ_6 is not equal to $\phi_{\text{Const}}(P, Q)$ for any (P, Q).

Proposition 2.23. *A parametrization of simple characters in the manner of Property 2.15 allows us to construct the irreducible rational characters of W recursively, using only representations induced from one-dimensional characters of reflection subgroups of W.*

	D_4	D_{3a}	D_{3b}	D_{3c}	A_1^4	A_1^3	A_2	D_{2a}	D_{2b}	D_{2c}	A_1	1
D_4	0	0	0	0	0	0	0	0	0	0	0	1
D_{3a}	0	0	0	0	0	0	0	ϕ_{3d}	0	0	$\phi_{3d} + \phi_{4b}$	$\cdots$
D_{3b}	0	0	0	0	0	0	0	0	ϕ_{3e}	0	$\phi_{3e} + \phi_{4b}$	$\cdots$
D_{3c}	0	0	0	0	0	0	0	0	0	ϕ_{3f}	$\phi_{3f} + \phi_{4b}$	$\cdots$
A_1^4	0	0	0	0	ϕ_2	ϕ_2	0	$\phi_2 + \phi_{3d}$	$\phi_2 + \phi_{3e}$	$\phi_2 + \phi_{3f}$	$\cdots$	$\cdots$
A_1^3	0	0	0	0	ϕ_2	$\phi_2 + \phi_8$	ϕ_8	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
A_2	0	0	0	0	0	ϕ_8	$\phi_6 + \phi_8$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
D_{2a}	0	ϕ_{3a}	0	0	$\phi_2 + \phi_{3a}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
D_{2b}	0	0	ϕ_{3b}	0	$\phi_2 + \phi_{3b}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
D_{2c}	0	0	0	ϕ_{3c}	$\phi_2 + \phi_{3c}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
A_1	0	$\phi_{4a} + \phi_{3a}$	$\phi_{4a} + \phi_{3b}$	$\phi_{4a} + \phi_{3c}$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
1	ϵ	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Table 2.4: Table of common constituents of permutation and signed permutation characters for $W = D_4$.

	$(D_4, 1)$	(D_{3a}, A_1)	(D_{3a}, D_{2a})	(D_{3b}, D_{2b})	(D_{3b}, D_{2b})	(A_2, A_2)	(A_1^3, A_1^3)	(A_1^4, A_1^4)	(A_1, D_{3a})	(D_{2a}, D_{3a})	(D_{2b}, D_{3b})	(D_{2c}, D_{3c})	$(1, D_4)$
1	1	0	0	0	0	0	0	0	0	0	0	0	0
ϕ_{4b}	0	1	0	0	0	0	0	0	0	0	0	0	0
ϕ_{3d}	0	1	1	0	0	0	0	0	0	0	0	0	0
ϕ_{3e}	0	0	0	1	0	0	0	0	0	0	0	0	0
ϕ_{3f}	0	0	0	0	1	0	0	0	0	0	0	0	0
ϕ_6	0	0	0	0	0	1	0	0	0	0	0	0	0
ϕ_8	0	0	0	0	0	1	1	0	0	0	0	0	0
ϕ_2	0	0	0	0	0	0	1	1	0	0	0	0	0
ϕ_{4a}	0	0	0	0	0	0	0	0	1	0	0	0	0
ϕ_{3a}	0	0	0	0	0	0	0	0	1	1	0	0	0
ϕ_{3b}	0	0	0	0	0	0	0	0	0	0	1	0	0
ϕ_{3c}	0	0	0	0	0	0	0	0	0	0	0	1	0
ϵ	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 2.5: Multiplicity matrix for selected common constituents for $W = D_4$.

Proof. Without loss of generality, we may assume that the multiplicity matrix $M_{i,j}$ is upper unitriangular. Then we have $\phi_1 = M_{1,1} = \phi_{\mathrm{Const}}(P_1, Q_1)$ which must be an irreducible character, and recursively for $j > 1$ we have

$$\phi_j = \phi_{\mathrm{Const}}(P_j, Q_j) - \sum_{i<j} M_{i,j}\phi_i.$$

□

Proposition 2.23 implies that, if we know the multiplicities of implicitly-defined irreducibles in direct sum decompositions of permutation modules and signed permutation modules (but not necessarily the values of the irreducible summands themselves), then we can determine the values of the irreducibles using a unitriangular multiplicity matrix. An example of this construction is given in Chapter 6, where we discuss the dihedral groups. Even without this prior knowledge, a parametrization can be thought of as a way of imposing a structure on the set of irreducible characters, or alternatively, on a certain set of subgroup pairs for which a parametrization exists.

At one time we conjectured that Property 2.15 might hold for all finite Coxeter groups, but we will see later that this is false: it does not hold for the group H_4. It does, however, hold for the group H_3, as well as many others. We will also see a refinement of the parametrization we consider here, using a generalization of the Specht modules for the symmetric groups, that does parametrize the rational characters of H_4. It remains interesting to determine the Coxeter groups for which the function ϕ_{Const} exhibits a 'nice' parametrization using the properties above (or similar).

At this point we comment that we can also consider the parametrization of characters using pairs of parabolic subgroups of W, instead of pairs of reflection subgroups. Let ϕ'_{Const} be the restriction of ϕ_{Const} to the set of pairs of parabolic subgroups of W. We may consider analogous properties, denoted for example as Property 2.12', with ϕ_{Const} replaced by ϕ'_{Const}. Notice that, in the case of symmetric groups, the reflection subgroups are all parabolic so $\phi_{\text{Const}} = \phi'_{\text{Const}}$ for these groups. The same is not true for other Coxeter types and, indeed, parabolic subgroups are inadequate for the kind of parametrizations we consider. The easiest example of this occurs with the dihedral groups $I_2(m)$. Here $I_2(m)$ has three or four conjugacy classes of parabolic subgroups when n is odd or even respectively, but the number of simple rational characters is $\tau(m) + 1$ or $\tau(m) + 2$ respectively, where $\tau(m)$ is the number of divisors of m.

Because of this, we do not consider using the poset $\mathcal{P}^{\text{conj}} \times \mathcal{P}^{\text{conj}}$ after this section. We observe that Property 2.14' implies Property 2.14, since whenever an irreducible character arises on a pair of parabolic subgroups (P', Q'), then we may choose a pair of reflection subgroups (P, Q) greater than or equal to (P', Q') in $\mathcal{R}^{\mathrm{conj}} \times \mathcal{R}^{\mathrm{conj}}$, which is maximal on the support of ϕ_{Const}. The value of ϕ_{Const} on (P, Q) is necessarily equal to $\phi_{\mathrm{Const}}(P', Q')$.

Moreover, Property 2.15' implies Property 2.15, Property 2.16' implies Property 2.16, and Property 2.17' implies Property 2.17, because because the choice of pairs of parabolic subgroups that establishes each of these properties is also a set of pairs of reflection subgroups. The remaining properties do not in general have an analogous relationship.

We emphasize that computations involving common constituent characters depend on a prior knowledge of the irreducible complex characters of W. The standard GAP package "CHEVIE" includes the complex character tables for all Coxeter groups, so common constituent characters are a viable computational tool. They cannot be used to 'generate' the irreducible rational characters or determine their values starting with no assumed knowledge of the characters of W.

2.3 Generalized Specht Modules

In this section, we describe another approach to parametrization generalizing the Specht modules for the symmetric groups. The characters of generalized Specht modules will be summands of the common constituent characters, and in Coxeter types other than A, they are often proper summands. Thus, this approach is a refinement of the approach via generalized common constituents. That said, there are limitations to what combinatorial properties of the classical Specht modules can be generalized in this way. In particular, the dominance order has no natural generalization to arbitrary Coxeter type.

The generalized Specht modules $S(P, Q) = S_{\mathbb{F}}(P, Q)$ will be indexed by pairs of reflection subgroups P and Q of a Coxeter group W, where the subgroup pair (P, Q) is taken up to simultaneous conjugacy. We will define $S_{\mathbb{F}}(P, Q)$ in two equivalent ways: as submodules of permutation modules; and as left ideals of the group algebra $\mathbb{F}W$. We note that there are related constructions in [4, 6, 7, 8, 9, 12]. Many of our results hold when $\mathbb{F}$ is an arbitrary field, or a field of characteristic not dividing $|W|$. Some results and proofs that are particularly relevant to constructing and decomposing the modules $S_{\mathbb{F}}(P, Q)$ are postponed until Chapter 5.

Definition 2.24. *For each reflection subgroup P, let*

$$P^+ := \sum_{g \in P} g, \qquad P^- := \sum_{g \in P} \epsilon(g) g,$$

which are elements of the group algebra $\mathbb{F}W$. Then, for a pair of reflection subgroups (P, Q), define

$$\kappa_{(P,Q)} := Q^- P^+ \quad \in \mathbb{F}W.$$

Definition 2.25. *Let W be a finite Coxeter group, and let P and Q be reflection subgroups of W. Let $\mathbb{F}$ be a field.*

1. *(First definition) The generalized Specht module $S(P, Q) = S_{\mathbb{F}}(P, Q)$ is the $\mathbb{F}W$-submodule of $\mathbb{F} \uparrow_P^W$ generated by $Q^- \bar{1}$.*

2. *(Second, equivalent definition) The generalized Specht module $S(P,Q)$ is the $\mathbb{F}W$-submodule of $\mathbb{F}W$ generated by $\kappa_{(P,Q)}$.*

Example 2.26. Let $W = S_n$ be the symmetric group, and let P and Q be the parabolic subgroups $P = R_t$ and $Q = C_t$, which are the row- and column-stabilizers of a Young tableau t (all to be defined in Chapter 3. Then it is well known that $S(P,Q)$ is the ordinary Specht module S^λ corresponding to the partition λ.

In view of this, we will frequently omit the word 'generalized' and refer simply to 'Specht modules' for a Coxeter group, meaning generalized Specht modules. This is an extension of the standard terminology, even in the case of symmetric groups, in that there is a Specht module defined for each pair of reflection subgroups.

Lemma 2.27. *1. The two definitions of the Specht module $S(P,Q)$ in Definition 2.25 are equivalent: the left ideal $\mathbb{F}W\kappa_{(P,Q)}$ of the group algebra $\mathbb{F}W$ is isomorphic to the $\mathbb{F}W$-submodule of $\mathbb{F}\uparrow_P^W$ generated by $Q^-\overline{1}$.*

2. *$S(P,Q)$ is isomorphic to a quotient module of the signed permutation module $\epsilon\uparrow_Q^W$.*

3. *The Specht modules $S_{\mathbb{Q}}(P,Q)$ are rational representations of W.*

Proof. 1. We first define a $\mathbb{F}W$-module homomorphism $f : \mathbb{F}WP^+ \to \mathbb{F}\uparrow_P^W$ by $f(P^+) = \overline{1}$. Observe that the stabilizers of P^+ and $\overline{1}$ in W are both equal to P. The set $O := \{wP^+ | w \in W\}$ is then equivalent as a W-set to W/P. The distinct elements of O have disjoint supports in the standard basis for $\mathbb{F}W$, so the set O is linearly independent. Since O spans $\mathbb{F}WP^+$ by definition, it is a basis. This establishes that f is a well-defined linear map as well as a $\mathbb{F}W$-module homomorphism. The map given by $\overline{w} \mapsto wP^+$ is similarly shown to be a well-defined $\mathbb{F}W$-module homomorpism, and it is the inverse to f.

Now let g be the restriction of f to $(\mathbb{F}W)\kappa_{(P,Q)}$ which must be injective. Then for all $w \in W$, $g(w\kappa_{(P,Q)}) = wQ^-\overline{1}$, so the image of g is $\mathbb{F}WQ^-\overline{1} = S(P,Q)$ which completes the proof.

2. We define a homomorphism $\epsilon \uparrow_Q^W \to \mathbb{F} \uparrow_P^W$ of $\mathbb{F}W$-modules whose image is $\mathbb{F}WQ^{-}\overline{1}$, as follows. The element $Q^{-}\overline{1} \in \mathbb{F} \uparrow_P^W$ spans a $\mathbb{F}Q$-submodule that is an image of ϵ, because each $q \in Q$ acts on it as multiplication by $\epsilon(q)$. This homomorphism extends to the desired homomorphism by the universal property of induction, and its image is evidently the Specht module $\mathbb{F}WQ^{-}\overline{1}$.

3. This is immediate because each Specht module is defined as a $\mathbb{Q}W$-submodule of a permutation module over $\mathbb{Q}$. □

Although we do not exploit this here, the Specht modules $S_{\mathbb{Q}}(P, Q)$ are evidently defined over $\mathbb{Z}$. We denote $\phi_{\mathrm{Specht}}(P, Q) := \chi(S_{\mathbb{Q}}(P, Q))$.

Corollary 2.28. *The character $\phi_{\mathrm{Specht}}(P, Q)$ is a summand of $\phi_{\mathrm{Const}}(P, Q)$.*

Proof. By Part 1 of Lemma 2.27, $S(P, Q)$ is a quotient of $\epsilon \uparrow_Q^W$, so its character is a summand of the character of $\epsilon \uparrow_Q^W$. Likewise, by the first characterization in Definition 2.25, $\phi_{\mathrm{Specht}}(P, Q)$ is a summand of $\chi(\mathbb{Q} \uparrow_P^Q)$. Taking the minimum of the induced character multiplicities gives the result. □

This means that Specht modules have the potential to discriminate more finely between different irreducible characters of W than the common constituents of permutation and signed permutation modules. We must be careful, though, when discussing the invariance of the Specht module and its character.

Proposition 2.29. *Let $\mathbb{F}$ be any field. Let P and Q be reflection subgroups of W, and let $w \in W$. Then $S_{\mathbb{F}}(P, Q) \cong S_{\mathbb{F}}({}^wP, {}^wQ)$. When $\mathbb{F} = \mathbb{Q}$, we have that ϕ_{Specht} is a well-defined function on $(\mathcal{R} \times \mathcal{R})^{\mathrm{conj}}$.*

Proof. Observe that

$$({}^wP)^+ = \sum_{\sigma \in {}^wP} \sigma = \sum_{p \in P} wpw^{-1} = w \left[\sum_{p \in P} p \right] w^{-1} = {}^w(P^+).$$

Moreover, since conjugation preserves the sign character, we also have

$$\begin{aligned}({}^{w}Q)^{-} &= \sum_{\sigma\in {}^{w}Q} \epsilon(\sigma)\sigma \\ &= \sum_{q\in Q} \epsilon(wqw^{-1})wqw^{-1} \\ &= \sum_{q\in Q} \epsilon(q)wqw^{-1} \\ &= w\left[\sum_{q\in Q} \epsilon(q)q\right]w^{-1} \\ &= {}^{w}(Q^{-}).\end{aligned}$$

Combining these, we have $\kappa_{({}^{w}P,\, {}^{w}Q)} = ({}^{w}Q)^{-}({}^{w}P)^{+} = {}^{w}(Q^{-}P^{+}) = {}^{w}\kappa_{(P,Q)}$. We claim there is a well-defined map $f : \mathbb{F}W\kappa_{(P,Q)} \to \mathbb{F}W\kappa_{({}^{w}P,{}^{w}Q)}$ given by $v \mapsto vw^{-1}$. Indeed, if $v = \sum_{\sigma\in W} a_\sigma \sigma Q^{-}P^{+}$, then

$$\begin{aligned}vw^{-1} &= \sum_{\sigma\in W} a_\sigma \sigma Q^{-}P^{+}w^{-1} \\ &= \sum_{\sigma\in W} a_\sigma \sigma w^{-1}\left[wQ^{-}P^{+}w^{-1}\right] \\ &= \sum_{\sigma\in W} a_\sigma \sigma w^{-1}\,{}^{w}(\kappa_{(P,Q)}) && \in \mathbb{F}W\kappa_{({}^{w}P,{}^{w}Q)}.\end{aligned}$$

This map is invertible with inverse given by right multiplication by w. Moreover, f commutes with left multiplication by elements of W, so it is a $\mathbb{F}W$−module isomorphism as required.

□

While $S(P, Q)$ is invariant up to joint conjugacy of P and Q, the following example shows it is not invariant up to separate conjugacy:

Example 2.30. We mention an example whose Specht character calculations we postpone until Chapter 4. When $W = H_4$, consider subgroups P of Coxeter type A_2^2 and

Q of type A_1^4. We have calculated all possible characters of Specht modules across all possible pairs of conjugates of P and Q. Our computations show that there are two possible Specht modules that may arise, depending on the particular realization of P and Q as subgroups of H_4. Their characters, call them ϕ and ϕ', have the following rational decomposition:

$$\phi = \phi_{10} + \phi_{18} + \phi_{25b} + \phi_{40} + \phi_{48a},$$

$$\phi' = \phi_{8a} + \phi_{10} + \phi_{18} + \phi_{40} + \phi_{48a}.$$

Here the subscripts give the degrees of the irreducible rational constituents, consistent with Table 4.6. We see that neither character is a summand of the other, so there is no canonical choice of subgroup conjugates that gives the 'largest' Specht module. In our calculation that establishes this Specht module decomposition, this pair of reflection subgroups corresponds to ϕ_{40} in the parametrization given there. Either of the two possible Specht modules could be used in our parametrization.

Thus, in the context of Specht modules, while we may speak of reflection subgroups P and Q having the alternating intersection property $P \cap Q^w \leq A(W)$, the choice of w matters.

We have a result for the existence of nonzero Specht modules analogous to Corollary 2.10. This result is particularly relevant for computations, and we postpone the proof to Chapter 5.

Proposition 2.31. *Let $\mathbb{F}$ be a field. Then $S(P, Q) \neq 0$ if and only if $P \cap Q \leq \mathrm{Ker}(\epsilon)$ and $|P \cap Q| \neq 0$ in $\mathbb{F}$.*

From this, we have the following observation that can be quite useful for certain Coxeter types.

Proposition 2.32. *Let P and Q be reflection subgroups of W. If $\phi_{\mathrm{Const}}(P, Q)$ is nonzero, then there exists $w \in W$ such that $\phi_{\mathrm{Specht}}(P, {}^wQ)$ is nonzero. If additionally $\phi_{\mathrm{Const}}(P, Q)$ is irreducible, then $\phi_{\mathrm{Specht}}(P, {}^wQ)$ is irreducible and w arises from a unique (P, Q)-double coset.*

Proof. If $\phi_{\text{Const}}(P, Q)$ is nonzero, then by Corollary 2.10, there exists $w \in W$ such that $P \cap {}^wQ \leq A(W)$. Since $\text{Ker}(\epsilon) = A(W)$ over the rational numbers, $\phi_{\text{Specht}}(P, {}^wQ)$ is nonzero by 2.31. By Corollary 2.28, $\phi_{\text{Specht}}(P, {}^wQ)$ is a summand of $\phi_{\text{Const}}(P, {}^wQ) = \phi_{\text{Const}}(P, Q)$, so it must be irreducible and equal to $\phi_{\text{Const}}(P, Q)$. The uniqueness of w also follows from Corollart 2.10. □

Thus, if we can show that $i(\mathbb{Q} \uparrow_P^W, \epsilon \uparrow_Q^W) = 1$ using Proposition 2.9, then there exists an irreducible Specht module of the form $S(P, {}^wQ)$, $w \in W$, such that $P \cap {}^wQ \leq A(W)$, and any two such w, w' lie in the same (P, Q)-double coset.

We summarize our criteria determined from Propositions 2.31 and 2.32, along with Corollary 2.10 for common constituent characters:

Theorem 2.33. *The following are equivalent:*

1. $\langle 1 \uparrow_P^W, \epsilon \uparrow_Q^W \rangle \neq 0$,
2. $\phi_{\text{Const}}(P, Q) \neq 0$,
3. *There exists* $w \in W$ *such that* $\phi_{\text{Specht}}(P, {}^wQ) \neq 0$,
4. *There exists* $w \in W$ *such that* $P \cap {}^wQ \leq A(W)$.

Moreover, $\phi_{\text{Const}}(P, Q)$ *is irreducible if and only if there exists a unique* (P, Q)*-double coset* PwQ *such that* $P \cap {}^wQ \leq \ker(\epsilon)$. *In this case,* $\phi_{\text{Specht}}(P, {}^wQ)$ *is also irreducible.*

Example 2.34. Let $W = W(B_2) = \langle a, b : a^2 = b^2 = (ab)^4 = 1 \rangle$ be the rank 2 Coxeter group of type B. Let $P = \langle a, bab \rangle$ and $Q = \langle b, aba \rangle$ which are both Abelian subgroups isomorphic to $C_2 \times C_2$. Since P and Q have index 2 in W, they are normal subgroups. There is a unique double coset in $P \backslash W / Q$ which we may write as $P1Q$. This double coset has the alternating intersection property since $P \cap Q = \langle abab \rangle \leq A(W)$. Therefore $i(\mathbb{Q} \uparrow_P^W, \epsilon \uparrow_Q^W) = 1$, and $S(P, Q)$ is irreducible with character $\phi_{\text{Const}}(P, Q)$.

The following may be a helpful tool for determining direct sum decompositions of generalized Specht modules.

Proposition 2.35. *Let $\mathbb{F}$ be a field of characteristic not dividing $|W|$. Let P, Q, P', Q' be reflection subgroups of W with $P \leq P'$ and $Q \leq Q'$. Then $S(P, Q') \mid S(P, Q)$ and $S(P', Q) \mid S(P, Q)$.*

Proof. First, let X be a transversal for the set of left cosets of Q in Q'. Then

$$\begin{aligned} S(P, Q') &= \mathbb{F}W\kappa_{(P,Q')} \\ &= \mathbb{F}W \sum_{q' \in Q'} \sum_{p \in P} \epsilon(q')q'p \\ &= \mathbb{F}W \sum_{x \in X} \sum_{q \in Q} \sum_{p \in P} \epsilon(x)x\epsilon(q)qp \\ &= \mathbb{F}W \sum_{x \in X} \epsilon(x)x \sum_{q \in Q} \sum_{p \in P} \epsilon(q)qp \\ &= \mathbb{F}W \sum_{x \in X} \epsilon(x)x\kappa_{(P,Q)}. \end{aligned}$$

Since for each $x \in X$, $\epsilon(x)x\kappa_{(P,Q)} \in S(P, Q)$, the cyclic generator $\sum_{x \in X} \epsilon(x)x\kappa_{(P,Q')}$ of $S(P, Q')$ is contained in $S(P, Q)$ as well, and the first claim follows from semisimplicity of $\mathbb{F}W$.

Next, let X be a transversal for the set of right cosets of P in P'. Then

$$\begin{aligned} S(P', Q) &= \mathbb{F}W\kappa_{(P',Q)} \\ &= \mathbb{F}W \sum_{q \in Q} \sum_{p' \in P'} \epsilon(q)qp' \\ &= \mathbb{F}W \sum_{q \in Q} \sum_{p \in P} \sum_{x \in X} \epsilon(q)qpx \\ &= \mathbb{F}W \sum_{q \in Q} \sum_{p \in P} \epsilon(q)qp \sum_{x \in X} x \\ &= \mathbb{F}W\kappa_{(P,Q)} \sum_{x \in X} x. \end{aligned}$$

Then $S(P', Q)$ is the image of $S(P, Q)$ under the mapping $\mathbb{F}W \to \mathbb{F}W$ given as right multiplication by $\sum_{x \in X} x$. Then $S(P', Q)$ is a homomorphic image of $S(P, Q)$, and the second claim again follows from semisimplicity. □

We emphasize that this result depends on the particular choices of P, Q, P', and Q'. If Q is merely conjugate to a subgroup of Q', then $S(P, Q)$ may not be isomorphic to a summand of $S(P, Q')$. The Coxeter group $W = H_3$ is a counterexample, see Table 4.5 in Section 4.1.

We will now show that $S_{\mathbb{F}}(Q, P) \cong S_{\mathbb{F}}(P, Q) \otimes \epsilon$ whenever $\mathbb{F}W$ is semisimple. This generalizes the identity $S^{\lambda'} \cong S^{\lambda} \otimes \epsilon$ which holds for Specht modules in type A. In the following discussion, let $\mathbb{F}$ be a field whose characteristic does not divide $|W|$, and fix reflection subgroups P and Q of W.

We have an isomorphism $\alpha : \epsilon \otimes \mathbb{F}W \to \mathbb{F}W$ specified as follows. We take the basis elements of $\mathbb{F}W$ to be the group elements $w \in W$. Then the elements $e \otimes w$ form a basis of $\epsilon \otimes \mathbb{F}W$. We check that the specification $\alpha(e \otimes w) = \epsilon(w)w$ defines an $\mathbb{F}W$-module isomorphism:

$$\begin{aligned}
\alpha(g(e \otimes w)) &= \alpha(\epsilon(g)e \otimes gw) \\
&= \epsilon(g)\,\epsilon(gw)gw \\
&= g(\epsilon(w)w) \\
&= g\alpha(e \otimes w).
\end{aligned}$$

When P and Q are subgroups of W, observe that $\alpha(e \otimes P^+) = P^-$ and $\alpha(e \otimes Q^-) = Q^+$.

Proposition 2.36. $\epsilon \otimes S(P, Q) \cong \mathbb{F}WQ^+P^-$.

Proof. Consider the diagram

$$\begin{array}{ccc} \mathbb{F}WQ^- & \xrightarrow{1\mapsto P^+} & \mathbb{F}WP^+ \\ \big\downarrow & & \big\downarrow \\ \mathbb{F}W & \xrightarrow{1\mapsto P^+} & \mathbb{F}W \end{array}$$

Right multiplication by P^+ sends wQ^- to wQ^-P^+, so the image of the top map is $S(P, Q)$. Now tensor with ϵ (an exact functor)

$$\begin{array}{ccccc} \epsilon\otimes(\mathbb{F}WQ^-) & & \xrightarrow{\epsilon\otimes 1\mapsto\epsilon\otimes P^+} & & \epsilon\otimes(\mathbb{F}WP^+) \\ & \searrow\!\!\!\!\searrow & & \nearrow & \\ \big\downarrow & & \epsilon\otimes S(P,Q) & & \big\downarrow \\ \epsilon\otimes\mathbb{F}W & & \xrightarrow{\epsilon\otimes 1\mapsto\epsilon\otimes P^+} & & \epsilon\otimes\mathbb{F}W \end{array}$$

and apply α to get

$$\begin{array}{ccccc} \mathbb{F}WQ^+ & & \xrightarrow{1\mapsto P^-} & & \mathbb{F}WP^- \\ & \searrow\!\!\!\!\searrow & & \nearrow & \\ \big\downarrow & & \mathbb{F}WQ^+P^- & & \big\downarrow \\ \mathbb{F}W & & \xrightarrow{1\mapsto P^-} & & \mathbb{F}W \end{array}$$

We deduce $\epsilon\otimes S(P, Q) \cong \mathbb{F}WQ^+P^-$. □

Proposition 2.37. $S(Q, P)^* \cong \mathbb{F}WQ^+P^-$.

Proof. Let $P^{-\top}$ be the left annihilator of P^- in $\mathbb{F}W$ and let $Q^{+\top}$ be the left annihilator of Q^+ in $\mathbb{F}W$. Observe that $(P^{-\top})^2 = |P|P^{-\top}$, with the right hand side being nonzero by our assumption on $\mathbb{F}$. Then the image of the map $\mathbb{F}W \to \mathbb{F}W$ given by $x \mapsto xP^-$ is $\mathbb{F}WP^-$, the kernel is $P^{-\top}$ by definition, and these ideals have trivial intersection in $\mathbb{F}W$. It follows that $\mathbb{F}W \cong (\mathbb{F}WP^-) \oplus P^{-\top}$. Similarly, $(Q^{+\top})^2 = Q^{+\top} \neq 0$ and $\mathbb{F}W \cong (\mathbb{F}WQ^+) \oplus Q^{+\top}$. Thus,

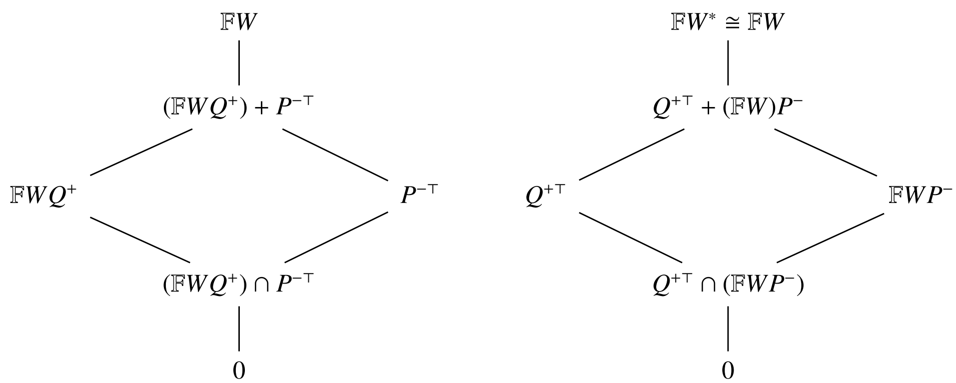
$$\mathbb{F}W/P^{-\top} \cong \mathbb{F}WP^- \subseteq \mathbb{F}W$$

and

$$\mathbb{F}W/Q^{+\top} \cong \mathbb{F}WQ^{+} \subseteq \mathbb{F}W.$$

Also, as described in Exercises 13 and 14 from Chapter 4 in [22], under the isomorphism $\mathbb{F}W \cong (\mathbb{F}W)^{*}$ of Exercise 14, we have $P^{-\top} \cong (\mathbb{F}WP^{-})^{\perp}$ and $Q^{+\top} \cong (\mathbb{F}WQ^{+})^{\perp}$.

Consider the following submodule diagrams:



The diagram on the right shows subspaces of the dual space $\mathbb{F}W^{*} = \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}W, \mathbb{F})$ arranged so that, on inverting the diagram, each submodule is in the same position as its perpendicular space under the canonical pairing of $\mathbb{F}W$ and $\mathbb{F}W^{*}$. By duality, sections of the diagram on the right are isomorphic to the dual of corresponding sections of the diagram on the left, after inversion.

In the left diagram

$$((\mathbb{F}WQ^{+}) + P^{-\top})/P^{-\top} \cong \mathbb{F}WQ^{+}P^{-}$$

with isomorphism given as the restriction of the map from $\mathbb{F}W/P^{-\top}$ to $\mathbb{F}WP^{-}$ sending $1 + P^{-\top}$ to P^{-}. The image of $Q^{+} + P^{-\top}$ in this module is $Q^{+}P^{-}$, so the restricted map is surjective and thus an isomorphism. By duality, the quotient module $((\mathbb{F}WQ^{+}) + P^{-\top})/P^{-\top}$ is isomorphic to the dual of the quotient

$$\mathbb{F}WP^{-}/(Q^{+\top} \cap (\mathbb{F}WP^{-}))$$

in the diagram on the right, and by the second isomorphism theorem this is isomorphic to

$$(Q^{+\top} + (\mathbb{F}WP^{-}))/Q^{+\top}.$$

This, in turn, is the image of $\mathbb{F}WP^{-}$ in $\mathbb{F}W/Q^{+\top} \cong \mathbb{F}WQ^{+}$, namely $S(Q, P)$. □

Putting this together we get

Theorem 2.38. *Let $\mathbb{F}$ be a field whose characteristic does not divide $|W|$. Let P and Q be reflection subgroups of W. Then $\epsilon \otimes S_{\mathbb{F}}(P, Q) \cong S_{\mathbb{F}}(Q, P)^{*} \cong S_{\mathbb{F}}(Q, P)$.*

Proof. Propositions 2.36 and 2.37 give the first of the isomorphisms. The second follows from semisimplicity and self-injectivity of $\mathbb{F}W$. □

2.3.1 Parametrizations via Generalized Specht Modules

We may consider Properties 2.39 through 2.44 analogous to those defined in the previous chapter but defined for the generalized Specht characters ϕ_{Specht} instead of ϕ_{Const}.

Property 2.39. The maximal elements of the support of ϕ_{Specht} form a complete list, without repetition, of the irreducible rational characters of W.

Property 2.40. All values of ϕ_{Specht} on maximal elements of its support are irreducible rational characters of W.

Property 2.41. All irreducible rational characters arise as values of ϕ_{Specht} on maximal elements of its support.

Property 2.42. There is a linear ordering $\phi_1, \ldots, \phi_N$ on the set of rational characters of W and a list of pairs of reflection subgroups $(P_j, Q_j)_{j=1}^{N}$ such that the multiplicity matrix $M^{\phi_{\text{Specht}}}$ is unitriangular.

Property 2.43. There is a linear ordering $\phi_1, \ldots, \phi_N$ on the set of rational characters of W and a list of subgroup pairs $(P_j, Q_j)_{j=1}^{N}$ such that $M^{\phi_{\text{Specht}}}$ is unitriangular. Whenever the resulting parametrization maps (P, Q) to ϕ, it maps (Q, P) to $\phi \otimes \epsilon$.

Property 2.44. For each rational character ϕ of G there is a pair of reflection subgroups (P, Q) for which ϕ occurs with multiplicity 1 in $\phi_{\text{Specht}}(P, Q)$.

Proposition 2.45. *The following implications hold:*

$$2.40 \Leftarrow 2.39 \Rightarrow 2.41 \Rightarrow 2.42 \Rightarrow 2.44 \quad 2.43 \Rightarrow 2.42.$$

Proof. Analogous to the proof of Proposition 2.19. ◻

The logical connections between these properties and Properties 2.39 through 2.44 are more subtle. We do have the following:

Proposition 2.46. *Property 2.14 implies Property 2.41: if the irreducible rational characters of W arise as values of ϕ_{Const} on maximal elements of its support, then they arise as values of ϕ_{Specht} on maximal elements of its support.*

Proof. Suppose that the irreducible complex characters arise as common constituent characters $\phi_{\text{Const}}(P_i, Q_i)$. By Proposition 2.32, for each i, there exists $w_i \in W$ so that $P_i \cap {}^{w_i}Q_i \subset A(W)$, and the value of ϕ_{Specht} is irreducible on this pair, equal to $\phi_{\text{Const}}(P_i, {}^{w_i}Q_i) = \phi_{\text{Const}}(P_i, Q_i)$. Now, we may choose subgroups P'_i, Q'_i such that $P_i \leq P'_i$, ${}^{w_i}Q_i \leq Q'_i$, and (P'_i, Q'_i) is maximal on the support of ϕ_{Specht}. Again, we must have that $\phi_{\text{Specht}}(P'_i, Q'_i) = \phi_{\text{Specht}}(P_i, Q_i)$ is irreducible. Thus, the irreducible complex characters are equal to the Specht characters $\phi_{\text{Specht}}(P'_i, Q'_i)$, which proves the result. ◻

However, there are no other analogous logical implications among the other properties.

We pose the following conjecture:

Conjecture 2.47. *Property 2.42 holds for all finite Coxeter groups W: there is a linear ordering $\phi_1, \ldots, \phi_m$ on the set of rational characters of W and a list of pairs of reflection subgroups $(P_j, Q_j)_{j=1}^m$ such that the multiplicity matrix $M^{\phi_{\text{Specht}}}$ is unitriangular.*

We will see that for all the Coxeter groups considered here, this conjecture does indeed hold.

Chapter 3

Parametrizations in classical types *A* and *B*

The indecomposable finite crystallographic Coxeter groups are parametrized by Coxeter diagrams of types A, B, D, E, F and G, and they are the indecomposable Coxeter groups that appear as Weyl groups of finite dimensional complex semi-simple Lie algebras. In this chapter we reformulate the known parametrizations of characters of Coxeter groups of types A and B in the terms we have been describing. In both of these cases, the complex irreducible representations can be realized over $\mathbb{Q}$, and we will see that it is possible to parametrize them using common constituents of induced representations in the manner of Section 2.2. The more refined approach of generalized Specht modules is not needed for these groups. In type A the parametrization by common constituents is really the same as certain standard results, but in type B the deduction appears to have some novel elements. Our main result in type B is Theorem 3.33. Aside from types A and B, we note that the group of Coxeter type G_2 is dihedral of order 12, and the parametrization of its characters is done in Chapter 6 on dihedral groups. We leave the parametrization of characters of the remaining crystallographic groups to further investigation.

3.1 Coxeter type A

The Coxeter group of type A_{n-1} has Coxeter diagram given by

and is the group of isometries of the regular simplex of dimension $n-1$. It is abstractly isomorphic to the symmetric groups S_n, and under this isomorphism the Coxeter generators are identified with the adjacent transpositions $(i, i+1)$, $1 \leq i < n$. We discuss algebraic and combinatorial properties of Young subgroups. Then, we summarize two classical approaches to identifying the irreducible characters of S_n: as common constituent characters, and as characters of Specht modules. In type A both of these characters coincide, though the two approaches can be seen as somewhat independent. This is because the former is an implicit approach and the latter is an explicit construction of representations.

The purpose of this summary is mostly to describe the existing theory in terms of our generalized framework, as well as to establish notation for the following section. This chapter has no new results about the symmetric groups, but proofs are sometimes given for the sake of exposition.

3.1.1 Young subgroups

If $\Delta = \{\Delta_i, i \in I\}$ is a collection of disjoint subsets of $[n]$, then for each i, S_{Δ_i} denotes the set of permutations of Δ_i. Observe that the S_{Δ_i} have pairwise trivial intersection and are contained in each others' centralizers in S_n. Thus the product $\prod_i S_{\Delta_i}$ is isomorphic to a direct product $S_{\Delta_1} \times S_{\Delta_2} \times \cdots$.

Definition 3.1. *A* ***Young subgroup*** *is a subgroup of* S_n *of the form* $S_\Delta = \prod_i S_{\Delta_i}$, *where* Δ *is a partition of* $\{1, \ldots, n\}$.

The reflections in S_n are the transpositions (i, j), and the Coxeter generators in S_n are the adjacent transpositions $(i, i+1)$. From this it follows that every Young subgroup is a reflection subgroup. The following is well-known, but we reproduce the argument here:

Proposition 3.2. *For the Coxeter groups $W = A_{n-1} = S_n$, $n \geq 1$, all reflection subgroups are parabolic subgroups, and these are exactly the Young subgroups of S_n.*

Proof. Let W' be a reflection subgroup of S_n. Whenever W' contains the reflections $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$, it contains the symmetric group $S_{\{i_1,\ldots,i_k\}}$. Define a relation on $\{1, \ldots, n\}$ as follows: $i \sim j$ if, for some $k \geq 0$, there are reflections $(i_1, i_2), \ldots, (i_{k-1}, i_k)$ in W' such that $i_1 = i$ and $i_k = j$. This is an equivalence relation that induces a set partition of $\{1, \ldots, n\}$ into blocks $\Delta_1, \ldots, \Delta_m$. Then W' contains each subgroup S_{Δ_i} and thus it contains $\prod_{j=1}^m S_{\Delta_j}$. Moreover, each transposition in W is contained in $\prod_{j=1}^m S_{\Delta_j}$ since it transposes two indices in Δ_j for some j. Thus, $\prod_{j=1}^m S_{\Delta_j}$ contains all reflections and it is equal to W'.

Finally, we must show W' is conjugate to a subgroup generated by adjacent transpositions. We may order $\{1, \ldots, n\}$ as follows: If $j \in \Delta_k$ and $j' \in \Delta_{k'}$, then $j \prec j'$ if and only if $k < k'$, or $k = k'$ and $j < j'$. This gives a total ordering of the form $\{\sigma(1) \prec \sigma(2) \prec \ldots \prec \sigma(n)\}$ for some permutation $\sigma \in S_n$. Then the subsets $\sigma(\Delta_i)$ are intervals in $\{1, \ldots, n\}$. If for a fixed i, $\sigma(\Delta_i) = \{a, a+1, \ldots, b\}$, then $S_{\sigma(\Delta_i)}$ is generated by the adjacent transpositions $(a, a+1), \ldots, (b-1, b)$. Therefore $\sigma W' \sigma^{-1}$ is generated by adjacent transpositions and W' is a parabolic subgroup. □

Definition 3.3. *A* ***composition*** *λ of a positive integer n is a sequence of nonnegative integers $\lambda_1, \lambda_2, \ldots$ such that $\sum_{i=1}^\infty \lambda_i = n$ and $\lambda_{k+1} = 0$ whenever $\lambda_k = 0$. A* ***partition*** *of n is a composition λ such that $\lambda_i \geq \lambda_{i+1}$ for all $i \geq 1$.*

We write $\lambda \vDash n$ when λ is an arbitrary composition of n and $\lambda \vdash n$ when λ is a partition of n. Denote by $\Lambda(n)$ the set of compositions of n and $\Lambda^+(n)$ the set of partitions of n. We will often write a partition λ in shorthand, condensing multiple parts of size i. For example, we write $\lambda = (3, 3, 2, 1, 1, 1) \vdash 11$ as $(3^2, 2, 1^3)$.

We associate with each partition λ its **Ferrers diagram**, which is a top- and left-justified orthogonal array of squares with λ_i squares in the i-th row from the top.

In the proof of Proposition 3.2, we may compose the permutation σ with a permutation that rearranges the blocks of the partition Δ in weakly decreasing order of cardinality. This implies the following:

Proposition 3.4. *Every Young subgroup is conjugate to a Young subgroup of the form* $S_{\{1,\dots,\lambda_1\}} \times S_{\{\lambda_1+1,\dots,\lambda_1+\lambda_2\}} \cdots \times S_{\{\lambda_1+\cdots+\lambda_{k-1}+1,\dots,\lambda_1+\cdots+\lambda_k\}}$*, where* $\lambda = (\lambda_1, \dots, \lambda_k)$ *is a partition of* n*.*

We say a Young subgroup of S_n is of **type** λ if it is isomorphic to $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$. From here on, we will use S_λ to denote "the" Young subgroup of type λ, with the understanding that we may need to refer to a particular conjugacy class representative depending on the context.

Before describing common constituents, we address how our alternating intersection property for an arbitrary Coxeter group coincides with the trivial intersection property in type A:

Proposition 3.5. *The intersection of two Young Subgroups is a Young subgroup. This intersection is contained in the alternating subgroup if and only if it is equal to the identity subgroup.*

Proof. Let $P = \prod_i S_{\Delta_i}$ and $Q = \prod_i S_{\Delta'_i}$ be Young subgroups corresponding to set partitions Δ and Δ'. Then $P \cap Q$ permutes the elements of each nonempty intersection $\Delta_i \cap \Delta'_j$ transitively, and it can only consist of permutations that fix each subset $\Delta_i \cap \Delta'_j$. Thus $P \cap Q$ is the Young subgroup corresponding to the common refinement of Δ and Δ'. The second statement follows immediately from Proposition 3.2. □

3.1.2 Common constituent characters

We summarize exposition from Chapters 1-2 of [11] which proves the existence of irreducible representations S^λ occurring as the unique common constituents of $1 \uparrow_{S_\lambda}^{S_n}$ and $\epsilon \uparrow_{S_{\lambda'}}^{S_n}$. This construction depends closely on the combinatorics of integer partitions, especially the dominance order. However, it does not explicitly construct a submodule of $1 \uparrow_{S_\lambda}^{S_n}$ or even use Young tableaux at all.

Definition 3.6. *Let $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ be partitions of n. Then $\lambda \trianglelefteq \mu$ if for all $k \geq 0$, $\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i$. This ordering on $\Lambda^+(n)$ is called the* ***dominance order****.*

We write $\lambda \lessdot \mu$ if $\lambda \triangleleft \mu$ and there does not exist a partition $\nu \neq \lambda, \mu$ such that $\lambda \triangleleft \nu \triangleleft \mu$.

Proposition 3.7. *The following are equivalent:*

1. $\lambda \lessdot \mu$,
2. *There exist i and j such that*
   (a) $i < j$,
   (b) $\mu_j = \lambda_j - 1$ *and* $\mu_i = \lambda_i + 1$, *and for* $k \neq i, j$, *we have* $\lambda_k = \mu_k$,
   (c) $i = j - 1$ *or* $\lambda_i = \lambda_j$.
3. *The Ferrers diagram of μ can be obtained by moving the rightmost box of some row of the Ferrers diagram of λ upwards to the end of another row, that row being the lowest possible row where the resulting shape is the Ferrers diagram of a partition.*

The dual operation on Coxeter group representations given by $V \mapsto V \otimes \epsilon$ will have a particular significance for the symmetric groups, in terms of a combinatorial dual operation on integer partitions which we define below.

Definition 3.8. *Let $\lambda \dashv n$ be a partition. Then the* ***conjugate*** *of λ, denoted λ', has λ'_j equal to the number of parts of λ greater than or equal to j.*

Equivalently, λ' is the partition whose Ferrers diagram is obtained from the Ferrers diagram of λ by transposing about the northwest / southeast diagonal.

Proposition 3.7 implies the following:

Proposition 3.9. *$\lambda \trianglelefteq \mu$ if and only if $\mu' \trianglelefteq \lambda'$.*

The dominance order and the trivial intersection property also have a close connection:

Proposition 3.10. *Let λ and μ be partitions of n. Then the number of 0-1 matrices with row sums $\lambda_1, \lambda_2, \cdots$ and column sums $\mu'_1, \mu'_2, \ldots$ is equal to the number of double cosets $S_\lambda \sigma S_{\mu'}$ such that $S_\lambda \cap \sigma S_{\mu'} \sigma^{-1} = \{1\}$. This number is nonzero if and only if $\lambda \trianglelefteq \mu$.*

Proof. The first statement is Corollary 1.3.13 of [11]. The second statement was originally proven by Gale and Ryser using 0-1 matrices, and the proof appears in [19]. □

Applying Equation 1.3.7 in [11] (which is generalized by our Proposition 2.9), we have the following result:

Proposition 3.11. *$\lambda \trianglelefteq \mu$ if and only if $1 \uparrow_{S_\lambda}^{S_n}$ and $\epsilon \uparrow_{S_{\mu'}}^{S_n}$ have nonzero common constituents.*

If $\lambda = \mu$, then we can recover a unique (irreducible) common constituent.

Proposition 3.12. *There exists a unique 0-1 matrix with with row sums $\lambda_1, \lambda_2, \cdots$ and column sums $\lambda'_1, \lambda'_2, \ldots$*

Proof. Suppose M is such a matrix. If λ has k parts, then for each i, $1 \leq i \leq k$, the i-th row of M has λ_i 1's. The j-th column sum of M is λ'_j which is the number of parts of λ which are greater than or equal to j. Then whenever $\lambda_i \geq j$, row i must have one of its ones at the (i, j)-th position by the pigeonhole principle. Therefore M is uniquely determined with $M_{ij} = 1$ if and only if $j \leq \lambda_i$. This matrix indeed has row sums λ_i and λ'_j, completing the proof. □

The following is Theorem 2.1.3, Equation 2.1.8, Lemma 2.1.10, and Theorem 2.1.11 from [11].

Proposition 3.13.

1. *For all $\lambda \dashv n$, the representations $1 \uparrow_{S_\lambda}^{S_n}$ and $\epsilon \uparrow_{S_{\lambda'}}^{S_n}$ have a unique shared irreducible summand S^λ, necessarily a rational representation, which occurs in these representations with multiplicity one.*

2. *$S^\lambda \otimes \epsilon \cong S^{\lambda'}$ for all $\lambda \dashv n$.*

3. *The multiplicity $\langle 1 \uparrow_{S_\lambda}^{S_n}, S^\mu \rangle$ is nonzero if and only if $\lambda \trianglelefteq \mu$.*

4. *The set* $\{S^\lambda \mid \lambda \dashv n\}$ *is a complete set of irreducible representations for* S_n.

We mention that our Lemma 2.8 generalizes the deduction that $S^\lambda \otimes \epsilon \cong S^{\lambda'}$ that appears in Equation 2.1.8 of [11].

We now connect the above theorem to our list of parametrization properties for the function ϕ_{Const}.

Lemma 3.14. *Let* λ *and* μ *be partitions of n. Then the multiplicity* $\langle \epsilon \uparrow_{S_\lambda}^{S_n}, S^\mu \rangle$ *is nonzero if and only if* $\mu \trianglelefteq \lambda'$.

Proof. Observe that $\langle \epsilon \uparrow_{S_\lambda}^{S_n}, S^\mu \rangle \neq 0$ if and only if $\langle \epsilon \otimes \epsilon \uparrow_{S_\lambda}^{S_n}, \epsilon \otimes S^\mu \rangle \neq 0$. But $\epsilon \otimes S^\mu \cong S^{\mu'}$ and $\epsilon \otimes \epsilon \uparrow_{S_\lambda}^{S_n} \cong 1 \uparrow_{S_\lambda}^{S_n}$. By Part 3 of Proposition 3.13, this multiplicity is nonzero exactly when $\lambda \trianglelefteq \mu'$, which occurs if and only if $\mu \trianglelefteq \lambda'$. □

Proposition 3.15. *For all partitions* λ *and* μ, $\phi_{\text{Const}}(1 \uparrow_{S_\lambda}^{S_n}, \epsilon \uparrow_{S_{\mu'}}^{S_n})$ *is irreducible if and only if* $\lambda = \mu'$.

Proof. Let λ, μ and ν be partitions. By Lemma 3.14, S^ν is a constituent of both $1 \uparrow_{S^\lambda}^{S_n}$ and $\epsilon \uparrow_{S^\mu}^{S_n}$ if and only if $\lambda \trianglelefteq \nu \trianglelefteq \mu'$. In particular, if $\phi_{\text{Const}}(S_\lambda, S_\mu)$ is nonzero then $\lambda \trianglelefteq \mu'$. Moreover, if $\lambda \neq \mu'$ then both S^λ and $S^{\mu'}$ are distinct constituents of $\phi_{\text{Const}}(S_\lambda, S_\mu)$ which proves the result. □

Before establishing our parametrization, we define another partial order on the set of partitions of n:

Definition 3.16. *The* ***lexicographic ordering*** *on the set of partitions of n has, for* $\lambda = (\lambda_i)_{i\geq 1}$ *and* $\mu = (\mu_i)_{i \geq 1}$, $\lambda \leq_{\text{lex}} \mu$ *if and only if* $\lambda = \mu$ *or there is some* $m \geq 1$ *such that* $\lambda_i = \mu_i$ *for all* $i < m$, *and* $\lambda_m < \mu_m$.

Observe that the lexicographic ordering is a linear ordering that refines the dominance order.

Theorem 3.17. *For* $W = S_n$, *Properties 2.12 and 2.16 hold for all n. Thus, all listed properties hold for the function* ϕ_{Const}.

Proof. We prove both Properties 2.12 and 2.16 by exhibiting a parametrization of the rational characters $\phi_{\text{Const}}(S_\lambda, S_{\lambda'})$, on a collection of subgroup pairs which are maximal on the support of ϕ_{Const}, such that the multiplicity matrix is the identity matrix. For our choice of reflection subgroup pairs, we may choose subgroup pairs of the form $(S_\lambda, S_{\lambda'})$, $\lambda \vdash n$ such that $(S_{\lambda'}, S_\lambda)$ is chosen whenever $(S_\lambda, S_{\lambda'})$ is. We order these by the lexicographic ordering on the partitions λ appearing in the first coordinate.

Propositions 3.11 and 3.15 imply that the chosen subgroup pairs are maximal on the support of ϕ_{Const}, and the values of ϕ_{Const} on these pairs, $\chi(S^\lambda)$, are irreducible. By Part 4 of Proposition 3.13, the S^λ are a complete list of irreducibles, so Property 2.12 holds.

Property 2.16 follows from our choice of subgroup pairs and by Part 2 of Proposition 3.13, since $\phi_{\text{Const}}(S_\lambda, S_{\lambda'}) \otimes \phi_\epsilon = \chi(S^\lambda) \otimes \phi_\epsilon = \chi(S^{\lambda'}) = \phi_{\text{Const}}(S_{\lambda'}, S_\lambda)$. □

3.1.3 Specht modules

In type A, the Specht modules are defined in terms of Young tableaux, combinatorial objects that model many algebraic properties of the symmetric groups. Our summary most closely follows [20].

Definition 3.18. *A **Young tableau of shape** λ (or λ-tableau) of a partition λ is a diagram $t = t^\lambda$ obtained from the Ferrers diagram of λ by labeling the boxes in bijection with the indices $1, \ldots, n$.*

Let $t_{i,j}$ denote the label of t in row i and column j. Denote by t' the λ'-tableau obtained by reflecting t along the main diagonal. Explicitly, $t'_{i,j} = t_{j,i}$ whenever $t_{j,i}$ is defined.

The symmetric group acts transitively on the set of all λ-tableaux, with $(\sigma t)_{i,j} = \sigma(t_{i,j})$.

Definition 3.19. *Let $t = t^\lambda$ be a λ-tableau. Then the **row stabilizer** R_t of t is the Young subgroup S_Δ where the blocks of Δ are the rows of t. The **column stabilizer** C_t of t is the Young subgroup $S_{\Delta'}$ where the blocks of Δ' are the columns of t.*

Immediately we have that R_t is of type λ and C_t is of type λ'. Observe that $\sigma R_t \sigma^{-1} = R_{\sigma t}$ and $\sigma C_t \sigma^{-1} = C_{\sigma t}$, so the pairs $(R_{\sigma t}, C_{\sigma t})$, $\sigma \in S_n$, all correspond to the same element of $(\mathcal{R} \times \mathcal{R})^{\text{conj}}$.

We note as well that $R_{t'} = C_t$ and $C_{t'} = R_t$.

Definition 3.20. *A* ***tabloid*** *(of shape λ) is an orbit of the action of R_t on the set of λ-tableaux. We denote by M^λ the free $\mathbb{C}$-vector space with basis the set of tabloids of shape λ.*

The action of S_n on the set of λ-tableaux is free and transitive and descends to a well-defined action on tabloids. The set of tabloids, as an S_n-set, is isomorphic to S_n/R_t. Thus, M^λ is isomorphic to $1 \uparrow_{S_\lambda}^{S_n}$.

Definition 3.21. *A* ***polytabloid*** *is an element of M^λ of the form $e_t = (C_t)^-\{t\}$ where $\{t\}$ is a λ-tabloid.*

Definition 3.22. *The (classical)* ***Specht module*** *S^λ is the cyclic module $\mathbb{C}S_n(C_t)^-\{t\}$ where $(C_t)^-\{t\}$ is any polytabloid.*

We omit the justification that S^λ does not depend on the choice of t. We also reuse the notation S^λ knowing this will turn out to coincide with the common constituent representation S^λ defined earlier. In our notation of generalized Specht modules, Lemma 2.27 implies that the classical Specht module S^λ is isomorphic to

$$S(R_t, C_t) = \mathbb{C}S_n(C_t)^-(R_t)^+ = \mathbb{C}S_n(R_{t'})^-(R_t)^+ = S(R_t, R_{t'})$$

.

Irreducibility of the Specht modules in characteristic zero follows from James' Submodule Theorem, which holds in arbitrary characteristic:

Proposition 3.23. *Any submodule of M^λ either contains S^λ or is contained in $(S^\lambda)^\perp$.*

Moreover, both the dimension of the space of homomorphisms of S^λ into M^μ, and the dimension of S^λ itself, may be enumerated by combinatorial objects associated to λ and μ.

Definition 3.24. *A **generalized Young tableau** (of shape λ) is a diagram t obtained from the Ferrers diagram of λ by labeling the boxes with any positive integers. The **content** of t is the composition $\alpha = (\alpha_i)_{i\geq 1}$ where α_i is the number of i's occurring in t.*

Definition 3.25. *A **semistandard Young tableau** is a generalized Young tableau t whose labels are weakly increasing along rows and strictly increasing along columns. If both rows and columns are strictly increasing and the label set is $\{1, \ldots, n\}$, then t is a **standard Young tableau**.*

Definition 3.26. *The **Kostka numbers** are the multiplicities $K_{\lambda,\mu}$ of S^{λ} in M^{μ}.*

The following is Corollary 2.4.7 and Theorem 2.11.2 of [20].

Proposition 3.27. *For all $\lambda, \mu \vdash n$, $K_{\lambda,\mu}$ is equal to the number of semistandard Young tableaux with shape λ and content μ. If $K_{\lambda,\mu} > 0$, then $\lambda \trianglerighteq \mu$. Moreover $K_{\lambda,\lambda} = 1$ for all partitions λ.*

In particular, the Specht modules are pairwise distinct, so they are in bijection with the set of conjugacy classes of S_n, namely, the set of partitions. Thus, the Specht modules give a full list of the irreducible complex S_n-representations (though this may be deduced without a combinatorial description of $K_{\lambda,\mu}$). As in our proof of Lemma 2.27, we note that the Specht modules are rational by definition, as they are generated by polytableaux, which are rational vectors in M^{μ}.

The following is Parts 1 and 2 of Theorem 2.6.5 of [20], noting as in this text that Part 2 also follows from Proposition 3.27 as the special case $\mu = [1^n]$.

Proposition 3.28. *The set $\{e_t \mid t$ is a standard Young tableaux of shape $\lambda\}$ is a basis for S^{λ}. The dimension f^{λ} of S^{λ} equals the number of standard Young tableaux of shape λ.*

Finally, we observe that the identity $S^{\lambda'} \cong S^{\lambda} \otimes \epsilon$ can be proven analogously to our proofs of Lemma 2.27 and Theorem 2.38, by way of an isomorphic identification of S^{λ} with a certain cyclic left ideal in the group algebra.

The preceding exposition establishes that the characters of the Specht modules of symmetric groups are irreducible and equal to the common constituent characters $\phi_{\text{Const}}(S_{\lambda}, S_{\lambda'})$. Thus, we have the following:

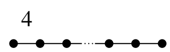
Theorem 3.29. *Properties 2.39 and 2.43 (and thus all other listed properties for the function $\phi_{\text{Specht}}(-,-)$) hold for $W = S_n$.*

Proof. For each λ, choose a λ-tableau t such that the tableau corresponding to λ' is t'. We take our collection of subgroup pairs to be $\{(R_{t_\lambda}, C_{t_\lambda})\}$, $\lambda \vdash n$, ordered lexicographically. By Propositions 3.23 and 3.27 and the discussions following, the representations $S(R_{t_\lambda}, C_{t_\lambda}) \cong S^\lambda$ are pairwise distinct irreducible representations, and the subgroup pairs $(R_{t_\lambda}, C_{t_\lambda})$ are maximal elements of the support of $\phi_{\text{Specht}}(-,-)$. This establishes Property 2.39. The multiplicity matrix is thus the identity matrix. Since $(C_t, R_t) = (R_{t'}, C_{t'})$, then the collection of subgroup pairs includes (Q, P) whenever it includes (P, Q). The corresponding parametrized rational characters are $\phi(S^\lambda)$ and $\phi(S^{\lambda'}) = \phi(S^\lambda) \otimes \epsilon$, which establishes Property 2.43. □

We remark that there many choices of subgroup pairs that give a unitriangular parametrization of the irreducibles S^λ. In the proofs of Theorems 3.17 and 3.29, the multiplicity matrix was the identity matrix. We may also take our collection of subgroup pairs to be $\{(S_\lambda, 1), \lambda \vdash n\}$ where S_λ is any Young subgroup of type λ. With this choice, $\phi_{\text{Const}}(S_\lambda, 1) = \phi_{\text{Specht}}(S_\lambda, 1) = \chi(M^\lambda)$, and the multiplicity matrix is equal to the unitriangular matrix $(K_{\lambda,\mu})_{\lambda,\mu}$. This parametrization gives a recursive construction of the irreducible characters, since each of the characters $\phi_{\text{Const}}(P, Q)$ (or $\phi_{\text{Specht}}(P, Q)$) consists of a 'new' character ϕ associated with (P, Q), together with 'previously calculated' characters associated with subgroup pairs preceding (P, Q) in the ordering. However, this parametrization is not invariant under tensor product with the sign representation, as $M^{\lambda'} \ncong M^\lambda \otimes \epsilon$. We believe that this additional invariance may be of interest to arbitrary finite Coxeter groups, but it may give no additional information in type A.

3.2 Coxeter type B

The Coxeter group of type B_n has Coxeter diagram given by



and is the group of isometries of the hypercube of dimension n. Our main result is Theorem 3.33, which states that the already-known complex irreducible characters $\chi_{\lambda,\mu}$ of B_n arise as common constituent characters. From this, we may obtain a parametrization satisfying Property 2.14 of ϕ_{Const}, and also a parametrization satisfying Property 2.41 of ϕ_{Specht}.

3.2.1 Common constituent characters

The irreducible complex representations of Coxeter groups of type B are described in the book of Geck and Pfeiffer [5] and they are all realizable over $\mathbb{Q}$. We summarize their construction here. The Coxeter group of type B_n is a semidirect product $W_n = C_2^n \rtimes S_n$. The symmetric group S_n permutes the n factors C_2 regularly. Letting t be a generator for the first C_2 factor and letting $s_1, s_2, \ldots s_{n-1}$ be Coxeter generators for S_n, the conjugates $t, s_1 t s_1, \ldots, s_{n-1} \cdots s_1 t s_1 \cdots s_{n-1}$ form a basis for the subgroup C_2^n as an $\mathbb{F}_2$-vector space. Furthermore, $\{t, s_1, s_2, \ldots s_{n-1}\}$ is a set of Coxeter generators of W_n, showing that it is a Coxeter group of type B_n (see [5, Sec. 1.4.1]).

The sign representation ϵ of W_n has $\epsilon(t) = \epsilon(s_i) = -1$ for all i, and there is another homomorphism $\epsilon^\dagger : W_n \to \{\pm 1\}$ specified by $\epsilon^\dagger(t) = -1$, $\epsilon^\dagger(s_i) = 1$ for all i. Now $W_n^\dagger := \operatorname{Ker} \epsilon^\dagger$ is a subgroup of the form $V \rtimes S_n$ that is a Coxeter group of type D_n, and V identifies as the 'coordinate-sum-zero' subspace of C_2^n.

The irreducible representations are conventionally parametrized by pairs of partitions (λ, μ) where $|\lambda| + |\mu| = n$. We write $|\lambda| = a$ and $|\mu| = b$, and these numbers determine a reflection subgroup $W_a \times W_b \leq W_n$ which in turn contains a reflection subgroup $W_a \times W_b^\dagger$. The irreducible character $\chi_{(\lambda,\mu)}$ of W_n has the form

$$\chi_{(\lambda,\mu)} = (\tilde{\chi}_\lambda \boxtimes \epsilon^\dagger \tilde{\chi}_\mu) \uparrow_{W_a \times W_b}^{W_n}$$

where $\tilde{\chi}_\lambda$ is the represention of W_a that is the irreducible representation of S_a corresponding to λ, made into a representation of W_a via the homomorphism $W_a \to S_a$, and similarly with $\tilde{\chi}_\mu$. We write $\epsilon^\dagger$ to denote the homomorphim $W_a \to \{\pm 1\}$, not the homomorphism $W_n \to \{\pm 1\}$. The outer tensor product $\boxtimes$ of irreducible representations of W_a and W_b is an irreducible representation of $W_a \times W_b$. The proof that this describes the irreducible characters of W_n depends on Clifford theory with respect to the normal subgroup C_2^n of W_n in a way that is well described in section 8.2 of [21].

For each partition λ of a (and similarly with partitions μ of b) we define the subgroup W_λ of W_a as $W_\lambda = C_2^a \rtimes S_\lambda$, where S_λ is a Young subgroup of S_a of type λ. If $\lambda = (\lambda_1, \lambda_2, \ldots)$, then W_λ is a product $W_\lambda \cong W_{\lambda_1} \times W_{\lambda_2} \times \cdots$ of Coxeter groups of type B. We also define $W_\lambda^\dagger = W_{\lambda_1}^\dagger \times W_{\lambda_2}^\dagger \times \cdots$, and this is a product of Coxeter groups of type D. By convention we set $W_{(0)} = W_{(0)}^\dagger = 1$. These are all reflection subgroups of W_n. We denote the conjugate partition of λ by λ'.

Our main result is Theorem 3.33 and there will be several identifications in the calculations required for the proof. We present these identifications as the following lemmas.

Lemma 3.30. *Let J be a subgroup of G and let K be a subgroup of H. If U is a representation of J and V is a representation of K then*

$$(U \boxtimes V) \uparrow_{J\times K}^{G\times H} \cong U \uparrow_J^G \boxtimes V \uparrow_K^H .$$

Proof. We may break up the induction into two steps: $\uparrow_{J\times K}^{J\times H}$ followed by $\uparrow_{J\times H}^{G\times H}$, and because these are handled similarly it suffices to consider only the first step $\uparrow_{J\times K}^{J\times H}$. Taking a set of left coset representatives $\{h_1, \ldots, h_t\}$ of K in H, the elements $\{(1, h_1), \ldots, (1, h_t)\}$ form a set of coset representatives of $J \times K$ in $J \times H$ and so $(U \boxtimes V) \uparrow_{J\times K}^{J\times H}$ is a direct sum of spaces $(1, h_i)(U \otimes V)$, which identifies with the sum of spaces $U \otimes h_i V$ that is

$U \boxtimes V \uparrow_K^H$. ☐

Lemma 3.31. *Let λ be a partition of a. Then $1 \uparrow_{W_\lambda^\dagger}^{W_\lambda} = 1 + \epsilon^\dagger$ as representations of W_λ.*

Proof. We have that $W_\lambda^\dagger$ is a subgroup of index 2 in W_λ with coset representatives $\{1, t\}$ so the induced representation is the sum of the trivial representation and a 1-dimensional representation on which t acts as -1 and all the s_i act as 1, because they lie in $W_\lambda^\dagger$. This 1-dimensional representation must then be $\epsilon^\dagger$ with $W_\lambda^\dagger$ as its kernel. ☐

Lemma 3.32. *Let k be a positive integer, and let $\lambda \vdash k$. Let $\tilde{\chi}_\lambda^{W_k}$ be the irreducible S_k-character corresponding to λ, inflated to W_k. Then $(\epsilon \cdot \epsilon^\dagger) \cdot \tilde{\chi}_\lambda^{W_k} = \tilde{\chi}_{\lambda'}^{W_k}$ as W_k-characters.*

Proof. The 1-dimensional W_k-character $\epsilon \cdot \epsilon^\dagger$ is equal to ϵ when restricted to S_k and equal to 1 on C_2^k. It follows that both sides of the identity are trivial on C_2^k, so they descend to characters on S_n and it suffices to show that they are equal on S_n. The left-hand character becomes $(\epsilon^{S_k}) \cdot \chi_\lambda^{S_k}$ and the right-hand character becomes $\chi_{\lambda'}^{S_k}$, and these are equal due to Theorem 3.13. ☐

Theorem 3.33. *Let λ be a partition of a and μ a partition of b, where $a + b = n$. Then $\phi_{\text{Const}}(W_\lambda \times W_\mu^\dagger, W_{\lambda'}^\dagger \times W_{\mu'}) = \chi_{(\lambda,\mu)}$.*

Proof. For reasons of explanation we find it easier to exclude the cases $a = n, b = 0$ and $a = 0, b = n$ at first, although the argument in these cases is the same. We consider $1 \uparrow_{W_\lambda \times W_\mu^\dagger}^{W_n}$ and $\epsilon \uparrow_{W_{\lambda'}^\dagger \times W_{\mu'}}^{W_n} = \epsilon \cdot (1 \uparrow_{W_{\lambda'}^\dagger \times W_{\mu'}}^{W_n})$ and factor both of these inductions through the intermediate group $W_a \times W_b$. We may write

$$1 \uparrow_{W_\lambda}^{W_a} = \tilde{\chi}_\lambda^{W_a} + \sum_{\theta > \lambda} K_{\theta,\lambda} \tilde{\chi}_\theta^{W_a}$$

where the natural numbers $K_{\theta,\lambda}$ are the Kostka numbers, and where we use a superscript in $\tilde{\chi}_\lambda^{W_a}$ to indicate the group for which this is a character. Similarly

$$1 \uparrow_{W_\mu}^{W_b} = \tilde{\chi}_\mu^{W_b} + \sum_{\eta > \mu} K_{\eta,\mu} \tilde{\chi}_\eta^{W_b}$$

so that

$$1\uparrow_{W_\mu^\dagger}^{W_b} = (1+\epsilon^\dagger)(\tilde{\chi}_\mu^{W_b} + \sum_{\eta>\mu} K_{\eta,\mu}\tilde{\chi}_\eta^{W_b})$$

by Lemma 3.31. It follows that

$$\begin{aligned}
1\uparrow_{W_\lambda\times W_\mu^\dagger}^{W_n} &= (1\boxtimes 1)\uparrow_{W_\lambda\times W_\mu^\dagger}^{W_a\times W_b}\uparrow_{W_a\times W_b}^{W_n} \\
&= (1\uparrow_{W_\lambda}^{W_a}\boxtimes 1\uparrow_{W_\mu^\dagger}^{W_b})\uparrow_{W_a\times W_b}^{W_n} \\
&= ((\tilde{\chi}_\lambda^{W_a} + \sum_{\theta>\lambda} K_{\theta,\lambda}\tilde{\chi}_\theta^{W_a})\boxtimes(1+\epsilon^\dagger)(\tilde{\chi}_\mu^{W_b} + \sum_{\eta>\mu} K_{\eta,\mu}\tilde{\chi}_\eta^{W_b}))\uparrow_{W_a\times W_b}^{W_n} \\
&= ((\tilde{\chi}_\lambda^{W_a} + \sum_{\theta>\lambda} K_{\theta,\lambda}\tilde{\chi}_\theta^{W_a})\boxtimes(\tilde{\chi}_\mu^{W_b} + \sum_{\eta>\mu} K_{\eta,\mu}\tilde{\chi}_\eta^{W_b}))\uparrow_{W_a\times W_b}^{W_n} \\
&\quad + ((\tilde{\chi}_\lambda^{W_a} + \sum_{\theta>\lambda} K_{\theta,\lambda}\tilde{\chi}_\theta^{W_a})\boxtimes\epsilon^\dagger(\tilde{\chi}_\mu^{W_b} + \sum_{\eta>\mu} K_{\eta,\mu}\tilde{\chi}_\eta^{W_b}))\uparrow_{W_a\times W_b}^{W_n}.
\end{aligned}$$

The first term in the last expression (before the main + sign) consists of characters that are 1 on the subgroup C_2^n, and the second term is a sum of characters that are neither trivial, nor a multiple of the sign ϵ on C_2^n, provided $a \neq 0 \neq b$. Similarly (and suppressing some of the analogous computation) we have

$$\begin{aligned}
\epsilon\uparrow_{W_{\lambda'}^\dagger\times W_{\mu'}}^{W_n} &= \epsilon\cdot(1\uparrow_{W_{\lambda'}^\dagger\times W_{\mu'}}^{W_n}) \\
&= \epsilon\cdot((\tilde{\chi}_{\lambda'}^{W_a} + \sum_{\theta>\lambda'} K_{\theta,\lambda'}\tilde{\chi}_\theta^{W_a})\boxtimes(\tilde{\chi}_{\mu'}^{W_b} + \sum_{\eta>\mu'} K_{\eta,\mu'}\tilde{\chi}_\eta^{W_b}))\uparrow_{W_a\times W_b}^{W_n} \\
&\quad + \epsilon\cdot(\epsilon^\dagger(\tilde{\chi}_{\lambda'}^{W_a} + \sum_{\theta>\lambda'} K_{\theta,\lambda'}\tilde{\chi}_\theta^{W_a})\boxtimes(\tilde{\chi}_{\mu'}^{W_b} + \sum_{\eta>\mu'} K_{\eta,\mu'}\tilde{\chi}_\eta^{W_b}))\uparrow_{W_a\times W_b}^{W_n} \\
&= \epsilon\cdot((\tilde{\chi}_{\lambda'}^{W_a} + \sum_{\theta>\lambda'} K_{\theta,\lambda'}\tilde{\chi}_\theta^{W_a})\boxtimes(\tilde{\chi}_{\mu'}^{W_b} + \sum_{\eta>\mu'} K_{\eta,\mu'}\tilde{\chi}_\eta^{W_b}))\uparrow_{W_a\times W_b}^{W_n} \\
&\quad + ((\tilde{\chi}_\lambda^{W_a} + \sum_{\theta>\lambda'} K_{\theta,\lambda'}\tilde{\chi}_{\theta'}^{W_a})\boxtimes\epsilon^\dagger(\tilde{\chi}_\mu^{W_b} + \sum_{\eta>\mu'} K_{\eta,\mu'}\tilde{\chi}_{\eta'}^{W_b}))\uparrow_{W_a\times W_b}^{W_n}.
\end{aligned}$$

Here the first term in the last expression (before the main + sign) consists of characters that are ϵ on C_2^n and the second term is a sum of characters that are neither trivial, nor a multiple of the sign ϵ on C_2^n, provided $a \neq 0 \neq b$.

The characters from $1 \uparrow_{W_\lambda \times W_\mu^\dagger}^{W_n}$ that are trivial on C_2^n do not appear in the expression for $\epsilon \uparrow_{W_{\lambda'}^\dagger \times W_{\mu'}}^{W_n}$ and the characters in $\epsilon \uparrow_{W_{\lambda'}^\dagger \times W_{\mu'}}^{W_n}$ that are ϵ on C_2^n do not appear in the expression for $1 \uparrow_{W_\lambda \times W_\mu^\dagger}^{W_n}$. Thus the common constituents are the common constituents of

$$((\tilde{\chi}_\lambda^{W_a} + \sum_{\theta > \lambda} K_{\theta,\lambda} \tilde{\chi}_\theta^{W_a}) \boxtimes \epsilon^\dagger (\tilde{\chi}_\mu^{W_b} + \sum_{\eta > \mu} K_{\eta,\mu} \tilde{\chi}_\eta^{W_b})) \uparrow_{W_a \times W_b}^{W_n}$$

and

$$((\tilde{\chi}_\lambda^{W_a} + \sum_{\theta > \lambda'} K_{\theta,\lambda'} \tilde{\chi}_{\theta'}^{W_a}) \boxtimes \epsilon^\dagger (\tilde{\chi}_\mu^{W_b} + \sum_{\eta > \mu'} K_{\eta,\mu'} \tilde{\chi}_{\eta'}^{W_b})) \uparrow_{W_a \times W_b}^{W_n} .$$

The characters that appear here have the form $(\tilde{\chi}_\theta^{W_a} \boxtimes \epsilon^\dagger \tilde{\chi}_\eta^{W_b}) \uparrow_{W_a \times W_b}^{W_n}$ where $\theta \geq \lambda$ and $\eta \geq \mu$ in the first case and $\theta \leq \lambda$ and $\eta \leq \mu$ in the second. We see that there is a unique common constituent, and it is $(\tilde{\chi}_\theta^{W_a} \boxtimes \epsilon^\dagger \tilde{\chi}_\eta^{W_b}) \uparrow_{W_a \times W_b}^{W_n}$ with multiplicity 1.

Returning to the cases $a = n, b = 0$ and $a = 0, b = n$, when $a = n, b = 0$ we have $W_b = W_\mu = W_\mu^\dagger = 1$. The terms on the right of the $\boxtimes$ are all the trivial character. The expression for $1 \uparrow_{W_\lambda}^{W_n}$ consists entirely of characters that are the identity on C_2^n. In the expression for $\epsilon \uparrow_{W_{\lambda'}^\dagger} W_n$ the terms with a factor ϵ are not common constituents, and the analysis of the remaining terms is the same as before. When $a = 0, b = n$ the argument is similar. □

While the subgroup pairs $(W_\lambda \times W_\mu^\dagger, W_{\lambda'}^\dagger \times W_{\mu'})$ may or may not be maximal elements of the support of ϕ_{Const}, each such pair is less than or equal to such a maximal pair (P, Q). Necessarily $\phi_{\text{Const}}(W_\lambda \times W_\mu^\dagger, W_{\lambda'}^\dagger \times W_{\mu'}) = \phi_{\text{Const}}(P, Q)$ by Lemma 2.7, because the value on the left is irreducible. This establishes our main result for common constituent characters:

Corollary 3.34. *Property 2.14 of the function ϕ_{Const} holds for Coxeter groups of type B.*

We see from this that the common constituents function ϕ_{Const} provides a parametrization of the simple representation in type B that is essentially the same as the usual parametrization of these representations.

We note that the analogous result to Corollary 3.34 holds for generalized Specht modules as a consequence of our general theory, so that these modules also provide a construction of the irreducible modules in type B.

Corollary 3.35. *Property 2.41 of the function ϕ_{Specht} holds for Coxeter groups of type B.*

Proof. This follows immediately from Proposition 2.46. □

Chapter 4

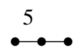
Parametrizations in Coxeter type H

In this chapter we parametrize the characters of rational representations of the non-crystallographic Coxeter groups H_3 of order 120, abstractly isomorphic to $A_5 \times C_2$, and H_4 of order 14400. These are the groups of isometries of a regular icosahedron and of the 600-cell.

The approach of this section is to investigate Properties 2.15 and 2.42 for the groups H_3 and H_4. We show that Property 2.15 holds for H_3. Moreover, Property 2.42 holds for both H_3 and H_4 which establishes part of Conjecture 2.42. We will see also that Property 2.15 does not hold for H_4. From this it follows by Proposition 2.19 that Properties 2.12 and 2.14 do not hold. The main conclusion is, in Theorems 4.2 and 4.6, that the irreducible rational characters of H_3 and H_4 can be parametrized by pairs of reflection subgroups. In the case of H_3 it can be done using common constituents, but for H_4 this approach is insufficient and generalized Specht modules are needed. The situation with H_4 provides a justification for the introduction of the generalized Specht modules. The proofs are highly computational and, in the case of H_4, the computational feasibility approaches the limits of what is currently available. We present our results in this chapter, and the computations that underlie them are described in Chapter 5.

4.1 The group H_3

The Coxeter group of type H_3 has Coxeter diagram given by



and is the group of isometries of the dodecahedron and its dual polytope, the icosahedron.

Table 4.1 gives the complex character table of H_3, which is well-known. In this table, we denote $a := -\zeta_5 - \zeta_5^4$ where ζ_5 is a primitive complex 5th root of unity.

Order of element	1	2	2	3	5	10	5	6	10	2
χ_1	1	1	1	1	1	1	1	1	1	1
χ_ϵ	1	-1	1	1	1	-1	1	-1	1	-1
χ_{3a}	3	-1	1	-1	a	$\overline{a}$	$\overline{a}$	0	a	3
χ_{3b}	3	-1	1	-1	$\overline{a}$	a	a	0	$\overline{a}$	3
χ_{3c}	3	1	0	-1	a	$-\overline{a}$	$\overline{a}$	0	$-a$	-3
χ_{3d}	3	1	0	-1	$\overline{a}$	$-a$	a	0	$-\overline{a}$	-3
χ_{4a}	4	0	1	0	-1	-1	-1	1	-1	4
χ_{4b}	4	0	1	0	-1	1	-1	-1	1	-4
χ_{5a}	5	1	-1	1	0	0	0	-1	0	5
χ_{5b}	5	-1	-1	1	0	0	0	1	0	-5

Table 4.1: Complex character table for $W = H_3$.

As for the rational characters, we define several characters as follows. Let $\phi_1 := \chi_1$, $\phi_\epsilon := \chi_\epsilon$, $\phi_{4a} := \chi_{4a}$, $\phi_{4b} := \chi_{4b}$, $\phi_{5a} := \chi_{5a}$, and $\phi_{5b} := \chi_{5b}$. Then, define $\phi_{6a} := \chi_{3a}+\chi_{3b}$ and $\phi_{6b} := \chi_{3c}+\chi_{3d}$. We observe that these characters are rational, and as complex characters they are either irreducible or the sum of two distinct complex irreducibles which are complex conjugates. Thus these characters are irreducible rational characters, and since every complex irreducible is a summand of one such character, this gives a complete list of the rational irreducible characters.

The list of reflection subgroups can be found in [13]. We list them here by Coxeter

	1	A_1	A_1^2	A_2	A_1^3	$I_2(5)$	H_3
H_3	ϕ_1	0	0	0	0	0	0
$I_2(5)$	$\phi_1 + \phi_{6b} + \phi_{5a}$	$\phi_{6b} + \phi_{5a}$	ϕ_{5a}	0	0	0	0
A_1^3	$\phi_1 + \phi_{4a} + 2\phi_{5a}$	$\phi_{4a} + 2\phi_{5a}$	$\phi_{4a} + \phi_{5a}$	ϕ_{4a}	0	0	0
A_2	$\phi_1 + \phi_{6b} +$ $+\phi_{4a} + \phi_{4b} + \phi_{5a}$	$\phi_{6b} + \phi_{4a} +$ $+\phi_{4b} + \phi_{5a}$	$\phi_{4a} + \phi_{4b} + \phi_{5a}$	$\phi_{4a} + \phi_{4b}$	ϕ_{4b}	0	0
A_1^2	$\phi_1 + \phi_{6b} + \phi_{4a} +$ $\phi_{4b} + 2\phi_{5a} + \phi_{5b}$	$\phi_{6b} + \phi_{4a} +$ $+\phi_{4b} + 2\phi_{5a} + \phi_{5b}$	$\phi_{4a} + \phi_{4b} +$ $+\phi_{5a} + \phi_{5b}$	$\phi_{4a} + \phi_{4b} + \phi_{5b}$	$\phi_{4b} + \phi_{5b}$	ϕ_{5b}	0
A_1	$\cdots$	$\cdots$	$\phi_{6a} + \phi_{4a} +$ $+\phi_{4b} + \phi_{5a} + 2\phi_{5b}$	$\phi_{6a} + \phi_{4a} +$ $+\phi_{4b} + \phi_{5b}$	$\phi_{4b} + 2\phi_{5b}$	$\phi_{6a} + \phi_{5b}$	0
1	$\cdots$	$\cdots$	$\phi_\epsilon + \phi_{6a} + \phi_{4a} +$ $+\phi_{4b} + \phi_{5a} + 2\phi_{5b}$	$\phi_\epsilon + \phi_{6a} +$ $+\phi_{4a} + \phi_{4b} + \phi_{5b}$	$\phi_\epsilon + \phi_{4b} + 2\phi_{5b}$	$\phi_\epsilon + \phi_{6a} + \phi_{5b}$	ϕ_ϵ

Table 4.2: Table of common constituents of reflection subgroup pairs, for $W = H_3$.

type, noting that all reflection subgroups of a given Coxeter type are conjugate in H_3:

$$1,\ A_1,\ A_1^2,\ A_2,\ A_1^3,\ I_2(5),\ H_3.$$

4.1.1 Common constituent characters for H_3

We now set about presenting a parametrization of the characters by pairs of reflection subgroups, verifying that Property 2.15 holds for H_3. Table 4.2 has rows and columns indexed by reflection subgroups P, Q, and the (P, Q)-entry is the decomposition $\phi_{\text{Const}}(P, Q)$ into irreducible rational characters. The reflection subgroups are denoted by their Coxeter type. The entries for $(1, 1)$, $(1, A_1)$, $(A_1, 1)$ and (A_1, A_1) are omitted because they are lengthy and do not contribute.

We immediately have the following observation:

Proposition 4.1. *Property 2.13 does not hold for H_3: There exist maximal pairs on the support of ϕ_{Const} with reducible common constituents.*

Proof. From Table 4.2 we see that the maximal elements of the support of ϕ_{Const} for H_3 are as follows:

$$(H_3, 1), (I_2(5), A_1^2), (A_1^3, A_1^2), (A_1^3, A_2), (A_2, A_2), (A_2, A_1^3), (A_1^2, A_1^3), (A_1^2, I_2(5)), (1, H_3).$$

Moreover, $\phi_{\text{Const}}(A_1^3, A_1^2) = \phi_{4a} + \phi_{5a}$, $\phi_{\text{Const}}(A_1^2, A_1^3) = \phi_{4b} + \phi_{5b}$, and $\phi_{\text{Const}}(A_2, A_2) = \phi_{4a} + \phi_{4b}$, which establishes the claim. □

To establish a parametrization of characters of rational representations of H_3 using Property 2.15, we exhibit a multiplicity matrix showing the values of ϕ_{Const} on a collection of reflection subgroup pairs. We take our collection to be

$$(H_3, 1), (I_2(5), A_1), (I_2(5), A_1^2), (A_1^3, A_2), (A_2, A_1^3), (A_1^2, I_2(5)), (A_1, I_2(5)), (1, H_3).$$

Table 4.3 gives the multiplicity matrix for this collection and for a suitable ordering of the rational characters. Table 4.4 gives a summary of the parametrization, indicating which rational character is associated to each subgroup pair.

	$(H_3, 1)$	$(I_2(5), A_1)$	$(I_2(5), A_1^2)$	(A_1^3, A_2)	(A_2, A_1^3)	$(A_1, I_2(5))$	$(A_1^2, I_2(5))$	$(1, H_3)$
ϕ_1	1	0	0	0	0	0	0	0
ϕ_{5a}	0	1	0	0	0	0	0	0
ϕ_{6b}	0	1	1	0	0	0	0	0
ϕ_{4a}	0	0	0	1	0	0	0	0
ϕ_{4b}	0	0	0	0	1	0	0	0
ϕ_{6a}	0	0	0	0	0	1	0	0
ϕ_{5b}	0	0	0	0	0	1	1	0
ϕ_ϵ	0	0	0	0	0	0	0	1

Table 4.3: Multiplicity matrix for selected subgroup pairs for $W = H_3$.

We observe that the chosen subgroup pairs have the property that (P, Q) labels the i-th column from the left if and only if (Q, P) labels the i-th column from the right. Moreover, the character labelling the j-th lowest row is equal to the tensor product of the character labelling the j-th highest row with the sign character. This establishes the following result:

Theorem 4.2. *The irreducible rational representations of H_3 admit a parametrization using pairs of reflection subgroups, using Property 2.16 of the function ϕ_{Const}.*

Rational irr. char.	Complex char. sum	Associated subgroup pair
ϕ_1	χ_1	$(H_3, 1)$
ϕ_{5a}	χ_{5a}	$(I_2(5), A_1)$
ϕ_{6b}	$\chi_{3c} + \chi_{3d}$	$(I_2(5), A_1^2)$
ϕ_{4a}	χ_{4a}	(A_1^3, A_2)
ϕ_{4b}	χ_{4b}	(A_2, A_1^3)
ϕ_{6a}	$\chi_{3a} + \chi_{3b}$	$(A_1^2, I_2(5))$
ϕ_{5b}	χ_{5b}	$(A_1, I_2(5))$
ϕ_ϵ	χ_ϵ	$(1, H_3)$

Table 4.4: Summary of chosen subgroup pairs for unitriangular parametrization, for $W = H_3$.

4.1.2 Specht modules for H_3

For most reflection subgroup pairs (P, Q), the Specht modules $S(P, Q^w)$ are all isomorphic, with character equal to the common constituents. The only exceptions are (A_2, A_1^2), (A_1^2, A_2), (A_1^2, A_1), (A_1, A_1^2), and (A_1, A_1).

The pairs (A_2, A_1^2) and (A_1^2, A_2) have two equivalence classes of pairs of subgroups when taken up to simultaneous conjugacy, which admit a nonzero Specht module. For one of these pairs, the Specht module character is equal to the common constituents, and for the other the Specht module is a proper submodule of the common constituents. For the pair (A_2, A_1^2), the common constituents character is $\phi_{4a} + \phi_{4b} + \phi_{5a}$. The proper submodule has character $\phi_{5a} + \phi_{4b}$. We may obtain the corresponding characters for (A_1^2, A_2) by tensoring with ϕ_ϵ, by Theorem 2.38.

The pairs (A_1^2, A_1) and (A_1, A_1^2) have four equivalence classes of pairs of subgroups when taken up to simultaneous conjugacy, which admit a nonzero Specht module. For three of these pairs, the Specht module character is equal to the common constituents, and for the fourth subgroup pair the Specht module is a proper submodule of the common constituents. For the pair (A_1^2, A_1) the common constituents character is $\phi_{6b} + \phi_{4a} + \phi_{4b} + 2\phi_{5a} + \phi_{5b}$ The proper submodule has character $\phi_{6b} + \phi_{4b} + \phi_{5b}$. Again, we may obtain the corresponding characters for (A_1, A_1^2) by tensoring with ϕ_ϵ.

The pair (A_1, A_1) has five equivalence classes of pairs of subgroups when taken up to simultaneous conjugacy, which admit a nonzero Specht module. For two of these pairs, the Specht modules have dimension 48 and their characters are equal to $\phi_{\text{Const}}(A_1, A_1) = \phi_{6b} + 2\phi_{5a} + 2\phi_{4a} + 2\phi_{4b} + 2\phi_{5b} + \phi_{6a}$. One subgroup pair has a 40-dimensional Specht module with character equal to $\phi_{6b}+2\phi_{5a}+\phi_{4a}+\phi_{4b}+2\phi_{5b}+\phi_{6a}$. The other two subgroup pairs have 30-dimensional Specht modules with characters equal to $\phi_{6b} + \phi_{5a} + \phi_{4a} + \phi_{4b} + \phi_{5b} + \phi_{6a}$.

	1	A_1	A_1^2	A_2	A_1^3	$I_2(5)$	H_3
H_3	ϕ_1	0	0	0	0	0	0
$I_2(5)$	$\phi_1 + \phi_{6b} + \phi_{5a}$	$\phi_{6b} + \phi_{5a}$	ϕ_{5a}	0	0	0	0
A_1^3	$\phi_1 + \phi_{4a} + 2\phi_{5a}$	$\phi_{4a} + 2\phi_{5a}$	$\phi_{4a} + \phi_{5a}$	ϕ_{4a}	0	0	0
A_2	$\phi_1 + \phi_{6b}+$ $+\phi_{4a} + \phi_{4b} + \phi_{5a}$	$\phi_{6b} + \phi_{4a}+$ $+\phi_{4b} + \phi_{5a}$	$\phi_{4b} + \phi_{5a}$	$\phi_{4a} + \phi_{4b}$	ϕ_{4b}	0	0
A_1^2	$\phi_1 + \phi_{6b} + \phi_{4a}+$ $\phi_{4b} + 2\phi_{5a} + \phi_{5b}$	$\phi_{6b} + \phi_{4a} + \phi_{5a}$	$\phi_{4a} + \phi_{4b}+$ $+\phi_{5a} + \phi_{5b}$	$\phi_{4a} + \phi_{5b}$	$\phi_{4b} + \phi_{5b}$	ϕ_{5b}	0
A_1	$\cdots$	$\phi_{6b} + \phi_{4a} + \phi_{5a}$ $+\phi_{5b} + \phi_{4b} + \phi_{6a}$	$\phi_{6a} + \phi_{4b} + \phi_{5b}$	$\phi_{6a} + \phi_{4a}+$ $+\phi_{4b} + \phi_{5b}$	$\phi_{4b} + 2\phi_{5b}$	$\phi_{6a} + \phi_{5b}$	0
1	$\cdots$	$\cdots$	$\phi_\epsilon + \phi_{6a} + \phi_{4a}+$ $+\phi_{4b} + \phi_{5a} + 2\phi_{5b}$	$\phi_\epsilon + \phi_{6a}+$ $+\phi_{4a} + \phi_{4b} + \phi_{5b}$	$\phi_\epsilon + \phi_{4b} + 2\phi_{5b}$	$\phi_\epsilon + \phi_{6a} + \phi_{5b}$	ϕ_ϵ

Table 4.5: Table of Specht module characters of least dimension, for $W = H_3$.

For all subgroup pairs we found it sufficient to consider the Specht module of least dimension, which for $W = H_3$ we found to be unique up to isomorphism. In Table 4.5, the (P, Q) entry gives the direct sum decomposition of the character of the form $\phi_{\text{Specht}}(P, Q^w)$ which has least dimension, omitting a few values for conciseness. In view of the comment following Corollary 2.35, we observe that the (P, Q)-entry is not always a summand of the (P, Q')-entry or the (P', Q)-entry when $P' \leq P$ and $Q' \leq Q$ in $\mathcal{R}^{\text{conj}}$.

We record some results established by these computations:

Proposition 4.3. *1. For each pair of Coxeter types of reflection subgroups of $W = H_3$, there is a subgroup pair (P, Q) having those Coxeter types such that $\phi_{\text{Specht}}(P, Q) = \phi_{\text{Const}}(P, Q)$.*

2. Suppose P and P' have the same Coxeter type and Q and Q' have the same

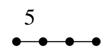
Coxeter type, and suppose $\dim(S(P, Q)) \leq \dim(S(P', Q'))$. *Then* $S(P, Q)$ *is isomorphic to a summand of* $S(P', Q')$.

3. *Each pair in the collection chosen to verify Property 2.16 in Theorem 4.2 has all possible Specht modules isomorphic to one another, with character equal to the common constituents. That is, the collection of pairs of Coxeter types chosen to verify Property 2.16 of common constituents also verifies Property 2.43 of Specht modules, for any choice of subgroup pairs having those Coxeter types.*

Corollary 4.4. *The irreducible rational representations of* H_3 *admit a parametrization using pairs of reflection subgroups, using Property 2.43 of the function* ϕ_{Specht}.

4.2 The group H_4

The Coxeter group of type H_4 has Coxeter diagram given by



and is the group of isometries of the 120-cell and its dual polytope, the 600-cell.

The complex character table of H_4 was first obtained by Read in [17]. From that table we may deduce the characters of rational representations of H_4, although we do not immediately need the complete argument to proceed with our own parametrization. The list of characters of rational representations appears below in Table 4.6, and also in Table 4.8 that gives their parametrization in terms of reflection subgroup pairs. It turns out that in all cases where there is an irrational entry in the complex character table, the sum of that character and its algebraic conjugate is the character of a rational representation. Aside from this, there is a complex character with rational values and Schur index 2, as documented in [5, Theorem 6.3.8]. It is the unique irreducible complex character of degree 48. It means that representation cannot be realized over $\mathbb{Q}$, but that the direct sum of two copies of it can be so realized, so there is an irreducible rational representation of degree 96. This representation has to be treated carefully in our computations, which were done over the finite field $\mathbb{F}_{23}$. Because there all finite division algebras are commutative, the irreducible representation of degree 48 can in fact be realized over this field. Whenever this representation appears with multiplicity $2k$ in a generalized Specht module, the corresponding 96-degree rational representation must appear in the rational generalized Specht module with multiplicity k.

In practice, our computations with Specht modules constructed as many rational representations as there are characters of the form just described, and so we are able to deduce the form of the rational characters, but we should point out that they should be regarded as already known in the literature. Table 4.6 gives a listing of the rational characters ϕ_d of H_4, indexed by their degree d and with their decomposition as a sum of complex characters. With the exception of ϕ_{96}, every irreducible rational character is either absolutely irreducible or the sum an irreducible complex character with its

	Rational irreducible character of H_4	Complex character decomposition
1	ϕ_1	χ_1
2	ϕ_{8d}	$\chi_{4c} + \chi_{4d}$
3	ϕ_{16d}	χ_{16d}
4	ϕ_{18b}	$\chi_{9c} + \chi_{9d}$
5	ϕ_{16b}	χ_{16b}
6	ϕ_{12}	$\chi_{6a} + \chi_{6b}$
7	ϕ_{8b}	χ_{8b}
8	ϕ_{36b}	χ_{36b}
9	ϕ_{32}	$\chi_{16e} + \chi_{16f}$
10	ϕ_{48b}	$\chi_{24c} + \chi_{24d}$
11	ϕ_{60}	$\chi_{30a} + \chi_{30b}$
12	ϕ_{96}	$2\chi_{48}$
13	ϕ_{40}	χ_{40}
14	ϕ_{48a}	$\chi_{24a} + \chi_{24b}$
15	ϕ_{25b}	χ_{25b}
16	ϕ_{10}	χ_{10}
17	ϕ_{18c}	χ_{18}
18	ϕ_{8a}	χ_{8a}
19	ϕ_{8c}	$\chi_{4a} + \chi_{4b}$
20	ϕ_{16c}	χ_{16c}
21	ϕ_{36a}	χ_{36a}
22	ϕ_{18a}	$\chi_{9a} + \chi_{9b}$
23	ϕ_{16a}	χ_{16a}
24	ϕ_{25a}	χ_{25a}
25	ϕ_ϵ	χ_ϵ

Table 4.6: Decomposition of irreducible rational characters of $W = H_4$.

algebraic conjugate.

We also list the reflection subgroups up to conjugacy, which can also be found in [13]. As with H_3, there is a unique reflection subgroup of each listed Coxeter type:

$$1,\ A_1,\ A_1^2,\ A_1^3,\ A_1^4,\ A_2,\ A_2A_1,\ A_2^2,\ A_3,\ A_4,\ D_4,\ H_3,\ H_3A_1,\ I_2(5),\ I_2(5)A_1,\ I_2(5)^2,\ H_4.$$

4.2.1 Common constituent characters for H_4

We construct the common constituent characters by way of constructing the complex character table, which is computationally feasible in GAP. We may readily calculate multiplicities of these characters in permutation characters and in the tensor products of these characters with the sign character.

We show that Property 2.15 fails for H_4. This means that, unlike H_3, we are unable to parametrize the irreducible rational representations using the function ϕ_{Const}.

Proposition 4.5. *Property* 2.15 *does not hold for the Coxeter group* H_4.

Proof. We will show that there are two irreducible rational characters ϕ and ϕ' of H_4 such that $\langle \phi_{\text{Const}}(P,Q), \phi \rangle \neq 0$ if and only if $\langle \phi_{\text{Const}}(P,Q), \phi' \rangle \neq 0$ for all H, K. We take ϕ to be an irreducible rational 48-dimensional character (either ϕ_{48a} or ϕ_{48b}, which are related by $- \otimes \phi_\epsilon$), and we take ϕ' to be the unique 40-dimensional character ϕ_{40} of H_4. Table 4.7 gives character data for the multiplicities of ϕ and ϕ' in $\phi_{\text{Const}}(P,Q)$, among all maximal choices of (P,Q) for which $\langle \phi_{\text{Const}}(P,Q), \phi \rangle$ or $\langle \phi_{\text{Const}}(P,Q), \phi' \rangle$ is nonzero.

(P,Q)	$\langle \phi_{\text{Const}}(P,Q), \chi \rangle$	$\langle \phi_{\text{Const}}(P,Q), \chi' \rangle$
$(I_2(5) \times A_1, I_2(5) \times A_1)$	1	2
$(I_2(5) \times A_1, A_3)$	1	1
$(I_2(5) \times A_1, A_2^2)$	1	1
$(I_2(5) \times A_1, A_1^4)$	1	2
$(A_3, I_2(5) \times A_1)$	1	1
(A_3, A_3)	1	1
(A_3, A_2^2)	1	1
(A_3, A_1^4)	1	1
$(A_2^2, I_2(5) \times A_1)$	1	1
(A_2^2, A_3)	1	1
(A_2^2, A_2^2)	1	1
(A_2^2, A_1^4)	1	1
$(A_1^4, I_2(5) \times A_1)$	1	2
(A_1^4, A_3)	1	1
(A_1^4, A_2^2)	1	1
(A_1^4, A_1^4)	3	5

Table 4.7: Multiplicities of $\phi = \phi_{48a}$ and $\phi' = \phi_{40}$ in $\phi_{\text{Const}}(P,Q)$.

For all other subpairs, either $\langle \phi_{\text{Const}}(P,Q), \chi \rangle = \langle \phi_{\text{Const}}(P,Q), \chi' \rangle = 0$, or P and Q are both subgroups of one of $I_2(5) \times A_1$, A_3, A_2^2, or A_1^4. In the latter case $\phi_{\text{Const}}(P,Q)$ has as a summand $\phi_{\text{Const}}(P',Q')$ for some (P',Q'). $\square$

4.2.2 Specht modules for H_4

We now show that if we use generalized Specht modules associated to pairs of reflection subgroups, encoded in the function ϕ_{Specht}, rather than ϕ_{Const}, then we can indeed parametrize the irreducible rational characters of H_4 using Property 2.42. The result depends upon extensive calculation, and we describe these calculations in Chapter 5.

Theorem 4.6. *The irreducible rational representations of H_4 may be parametrized by pairs of reflection subgroups, using generalized Specht modules, to exhibit Property 2.42.*

	Reflection subgroup pair (P_i, Q_i)	Specht module dimensions	Chosen dimension	Parametrized rat. character	Complex char. Decomposition
1	$(H_4, 1)$	1(1)	1	ϕ_1	χ_1
2	(H_3, A_1)	60(1), 119(44)	60	ϕ_{8d}	$\chi_{4c} + \chi_{4d}$
3	(H_3, A_1^2)	111(16)	111	ϕ_{16d}	χ_{16d}
4	$(H_3 \times A_1, A_1^2)$	59(8)	59	ϕ_{18b}	$\chi_{9c} + \chi_{9d}$
5	$(H_3 \times A_1, A_2)$	41(3)	41	ϕ_{16b}	χ_{16b}
6	$(I_2(5), I_2(5))$	144(2), 508(50), 900(40)	144	ϕ_{12}	$\chi_{6a} + \chi_{6b}$
7	$(A_2 \times A_1, A_2 \times A_1)$	440(6), 504(4), 572(2), 620(8), 664(8), 716(2), 742(2), 744(4), 748(4), 750(2), 796(2), 814(20), 822(4), 840(8)	440	ϕ_{8b}	χ_{8b}
8	$(H_3, I_2(5))$	36(1)	36	ϕ_{36b}	χ_{36b}
9	$(I_2(5) \times A_1, I_2(5) \times A_1)$	216(2), 364(8), 392(8)	364	ϕ_{32}	$\chi_{16e} + \chi_{16f}$
10	$(I_2(5) \times A_1, A_3)$	166(1), 292(8), 310(4)	310	ϕ_{48b}	$\chi_{24c} + \chi_{24d}$
11	$(I_2(5) \times A_1, A_1^4)$	241(30)	241	ϕ_{60}	$\chi_{30a} + \chi_{30b}$
12	(A_2^2, A_2^2)	220(4)	220	ϕ_{96}	$2\chi_{48}$
13	(A_2^2, A_1^4)	124(12), 141(6)	124	ϕ_{40}	χ_{40}
14	(A_2^2, A_3)	66(1), 192(4), 210(4)	66	ϕ_{48a}	$\chi_{24a} + \chi_{24b}$
15	$(H_3 \times A_1, A_2 \times A_1)$	25(1)	25	ϕ_{25b}	χ_{25b}
16	$(I_2(5)^2, A_2^2)$	28(2)	28	ϕ_{10}	χ_{10}
17	$(I_2(5)^2, A_3)$	18(1)	18	ϕ_{18c}	χ_{18}
18	(D_4, A_3)	8(1)	8	ϕ_{8a}	χ_{8a}
19	(A_1, H_3)	60(1), 119(44)	60	ϕ_{8c}	$\chi_{4a} + \chi_{4b}$
20	(A_1^3, H_3)	52(3), 77(2)	52	ϕ_{16c}	χ_{16c}
21	$(I_2(5), H_3)$	36(1)	36	ϕ_{36a}	χ_{36a}
22	$(A_1^2, H_3 \times A_1)$	59(8)	59	ϕ_{18a}	$\chi_{9a} + \chi_{9b}$
23	$(A_2, H_3 \times A_1)$	41(3)	41	ϕ_{16a}	χ_{16a}
24	$(A_2 \times A_1, H_3 \times A_1)$	25(1)	25	ϕ_{25a}	χ_{25a}
25	$(1, H_4)$	1(1)	1	ϕ_ϵ	χ_ϵ

Table 4.8: Reflection subgroup pairs used in unitriangular parametrization of Specht module characters for $W = H_4$.

ϕ	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
ϕ_1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ϕ_{8d}	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ϕ_{16d}	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ϕ_{18b}	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ϕ_{16b}	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ϕ_{12}	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ϕ_{8b}	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ϕ_{36b}	0	1	1	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ϕ_{32}	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ϕ_{48b}	0	0	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ϕ_{60}	0	0	0	0	0	1	1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
ϕ_{96}	0	0	0	0	0	0	1	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
ϕ_{40}	0	0	0	0	0	0	1	0	2	1	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0
ϕ_{48a}	0	0	0	0	0	0	1	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
ϕ_{25b}	0	0	1	1	1	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0
ϕ_{10}	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	1	0	0	0	0	0	0	0	0	0
ϕ_{18c}	0	0	0	0	0	0	1	0	0	1	1	1	1	1	0	1	1	0	0	0	0	0	0	0	0
ϕ_{8a}	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0
ϕ_{8c}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
ϕ_{16c}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0
ϕ_{36a}	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
ϕ_{18a}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
ϕ_{16a}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0
ϕ_{25a}	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0
ϕ_{ϵ}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 4.9: Multiplicity matrix of chosen generalized Specht modules, for $W = H_4$.

Proof. We have computed Specht modules for a collection of reflection subgroup pairs given in Table 4.8, then chosen a module for each pair and exhibited direct sum decompositions with multiplicities given in Table 4.9.

In Table 4.8, notation such as 60(1), 119(44) in the list of Specht module dimensions means there are 45 Specht modules computed for this pair of subgroups, of which 1 had dimension 60 and the other 44 had dimension 119. The Specht modules for subgroup pairs (P, Q^w) are all computed as submodules of $F \uparrow_P^W$ as indicated in Lemma 5.1. These correspond to regular orbits of Q on the cosets W/P which, in turn, biject with the double cosets QwP in W in which $P \cap Q^w = 1$ by Lemma 5.2. This does not give a complete list of all possible Specht modules, but enough to choose Specht modules of small enough dimension that we may compute a satisfactory multiplicity matrix.

In the multiplicity matrix given in Table 4.9, the i-th parametrized rational character from Table 4.8 appears as a consituent of $\phi_{\text{Specht}}(P_i, Q_i)$ with multiplicity 1 and corresponds to the (i, i)-entry equal to 1. Because this matrix is unitriangular, the theorem is proven.

□

In many instances, Specht modules of different dimension could have been chosen in the proof of Theorem 4.6 that would still produce a unitriangular multiplicity matrix. Most often, the Specht module of smallest dimension for that subgroup pair was chosen, but sometimes a larger Specht module had to be chosen in order to include all rational irreducible characters as summands in the decomposition.

Currently, we do not know whether a suitable choice of subgroup pairs exhibits the slightly stronger Property 2.43.

Chapter 5

Computational methods for determining Specht modules and their composition factor multiplicities

Our results for the types H_3 and H_4 were entirely based on computations in the software GAP. Our implementation generally used the expanded features of Peter Webb's GAP software 'reps' [23]. Sometimes, though, we directly used Meataxe routines on which 'reps' relies [16]. We give an overview of the computation.

5.1 Constructing generalized Specht modules in GAP

First, we constructed a list of representatives of conjugacy classes of reflection subgroups of W. For each pair of listed reflection subgroups (P, Q), we constructed for $\mathbb{F} = \mathbb{F}_{23}$ a complete list of all nonzero Specht modules of the form $S_{\mathbb{F}}(P, {}^{w}Q)$ for which wP is a regular orbit in the action of Q on W/P. Through trial and error we selected a list of Specht modules whose module decompositions contained a complete set of irreducible representations over $\mathbb{F}_{23}$, and whose multiplicity matrix was unitriangular. From this we inferred that the multiplicity matrix for the corresponding rational characters was also unitriangular.

Because the 'reps' package only allows for module constructions over a finite field, we constructed permutation modules and generalized Specht modules in characteristic p coprime to the order of W, where the representations are still semisimple. From these constructions in characteristic p, in most cases it is possible to completely deduce the structures of the analogous constructions in characteristic zero. We chose $\mathbb{F}_{23}$ as the field as it has the following properties:

1. The prime 23 is not a factor of the order of any Coxeter group of rank 4 or less.
2. The group of units $\mathbb{F}_{23}^{\times}$ is such that no roots of unity exist of order dividing $|W|$, other than the roots of unity that already exist in $\mathbb{Q}$ (namely $\{\pm 1\}$).
3. The prime 23 is the smallest prime with these properties.

Condition (1) has the effect that all $\mathbb{F}_{23}W$-modules are semisimple. Furthermore, over the algebraic closure, the irreducible $\overline{\mathbb{F}}_{23}W$-modules have the same dimension as the irreducible $\mathbb{C}$-modules, and the decomposition of modules such as permutation modules is the same (see [22]).

Condition (2) has the effect that irreducible representations over $\mathbb{Q}$ do not split further when reduced to representations over $\mathbb{F}_{23}$, with one exception. The exception is the complex representation of degree 48, which has Schur index 2 (see Theorem 6.3.8 of Geck and Pfeiffer [5], page 193). This means that the representation cannot be realized over $\mathbb{Q}$ although its character is rational-valued, but the direct sum of two copies of it can be so realized. Over $\mathbb{F}_{23}$ there are no non-commutative division algebras by Wedderburn's Theorem ([18], Theorem C-2.31) and a single copy of the degree 48 representation can be realized.

We constructed the Specht module for a given pair (P, Q) following the definition of $S(P, Q)$ as a submodule of the permutation module $\mathbb{F}\uparrow_P^W$ on the cosets of P in W. In particular, we constructed a signed sum of basis vectors for the action of Q on $\mathbb{F}\uparrow_P^W$. Initially, we generated a list of distinct vectors of the form σv, $\sigma \in Q$ for a fixed vector v. Recursively, we would append with new vectors of the form $-s\sigma v$ where s is one of the Coxeter generators of W and σv is already listed. Taking the sum of these gives the signed sum in the definition of $S(P, Q)$.

Our method for computing all Specht modules for pairs of reflection subgroups (P, Q^w) as w ranges over W depends on the following result. It means that they can all be computed as submodules of a single permutation module, namely the submodules $\mathbb{F}W \cdot Q^- \cdot wP$ of the permutation module on the cosets of P. This streamlines the computation.

Lemma 5.1. *Let $\mathbb{F}$ be a field. Let (P, Q) be a pair of reflection subgroups of W and let $w \in W$. Then the Specht module $S_{\mathbb{F}}({}^wP, Q) \cong S_{\mathbb{F}}(P, Q^w)$ is isomorphic to $\mathbb{F}W \cdot Q^- \cdot (wP)$, where wP is the coset of P determined by w in the permutation module on the cosets of P.*

Proof. The fact that $S({}^wP, Q) \cong S(P, Q^w)$ is the content of Proposition 2.29. For the remaining isomorphism, there is an isomorphism of left W-sets $W/P \to W/({}^wP)$ that sends $xP \mapsto xw^{-1}{}^wP$. Under this isomorphism the coset wP is sent to the coset ${}^wP$. The isomorphism extends to an isomorphism of permutation modules $\mathbb{F}[W/P] \to \mathbb{F}[W/({}^wP)]$, under which $Q^- \cdot (wP)$ is sent to $Q^-({}^wP)$, so that $\mathbb{F}W \cdot Q^- \cdot (wP)$ is mapped isomorphically to the submodule $\mathbb{F}W \cdot Q^-({}^wP) = S({}^wP, Q)$.

□

We now prove Proposition 2.31 which says, over an arbitrary field $\mathbb{F}$, that $S(P, Q) \neq 0$ if and only if $P \cap Q \leq \mathrm{Ker}(\epsilon)$ and $|P \cap Q| \neq 0$ in $\mathbb{F}$.

Proof. First, observe that the cyclic module $S(P, Q)$ is zero if and only if its generator Q^-P^+ is zero. We write

$$
\begin{aligned}
Q^-P^+ &= (\sum_{\sigma\in Q}\epsilon(\sigma)\sigma)(\sum_{\tau\in P}\tau)\\
&= \sum_{(\sigma,\tau)\in Q\times P}\epsilon(\sigma)\sigma\tau.
\end{aligned}
$$

Suppose $(P \cap Q) \nleq \ker(\epsilon)$. Then there exists $\alpha \in P \cap Q$ with $\epsilon(\alpha) = -1$ and every element of Q with $\epsilon(\sigma) = -1$ can be written $\sigma = \sigma'\alpha$ where $\sigma' = \sigma\alpha^{-1}$ and $\epsilon(\sigma') = 1$. Since left multiplication by α permutes the elements of P, we have the following:

$$
\begin{aligned}
\sum_{(\sigma,\tau)\in Q\times P}\epsilon(\sigma)\sigma\tau &= \sum_{(\sigma,\tau)\in Q\times P:\epsilon(\sigma)=1}\sigma\tau - \sum_{(\sigma,\tau)\in Q\times P:\epsilon(\sigma)=-1}\sigma\tau\\
&= \sum_{(\sigma,\tau)\in Q\times P:\epsilon(\sigma)=1}\sigma\tau - \sum_{(\sigma',\tau)\in Q\times P:\epsilon(\sigma')=1}\sigma'\alpha\tau\\
&= \sum_{(\sigma,\tau)\in Q\times P:\epsilon(\sigma)=1}\sigma\tau - \sum_{(\sigma,\tau)\in Q\times P:\epsilon(\sigma)=1}\sigma\tau\\
&= 0.
\end{aligned}
$$

So now suppose $(P \cap Q) \leq \ker(\epsilon)$. Then the number of unique group algebra basis elements appearing in Q^-P^+ is $[Q : P \cap Q]|P|$, and each basis element appears with multiplicity $|P \cap Q|$. Then $Q^-P^+ \neq 0$ if and only if $|P \cap Q| \neq 0$ in $\mathbb{F}$. □

Due to the computation time in calculating Specht modules for the group H_4, we had to develop more efficient techniques. To do this, we only considered subgroup pairs (P, Q) such that $P \cap Q = 1$.

Lemma 5.2. *If $w \in W$ and $P \cap Q^w = \{1\}$, then in the action of Q on the cosets of P in W, the orbit of the coset wP is a regular orbit. Moreover, $S(P, Q^w)$ is nonzero.*

Proof. Suppose $P \cap Q^w = \{1\}$, and suppose $\sigma \in Q$ fixes the coset wP. Then $\sigma^w \in Q^w$

fixes P, so σ^w is an element of $P \cap Q^w = \{1\}$. Then $\sigma^w = 1$ and so $\sigma = 1$. Therefore the elements $(\sigma wP)_{\sigma \in Q}$ are all distinct and the orbit of wP is a regular orbit. That $S(P, Q^w) \neq 0$ follows from the second part of Lemma 5.1. □

Our method for constructing a Specht module started with a list of all regular Q-orbits on the set W/P. We obtained the regular orbits by using the built-in GAP command that finds orbits and tests which had size $|Q|$. We constructed the signed sum of elements in this orbit by tensoring the permutation representation on W/P with the sign representation, and then taking the unsigned sum of the elements of Q on the orbit. Though this does not give all possible nonzero Specht modules of the form $S(P, {}^wQ)$, it still gives many inequivalent Specht modules of unequal dimensions (see Table 4.8).

When decomposing a Specht module, we used the Meataxe command that finds a complete list of composition factors of a module. Since semisimplicity holds over our chosen field, this gave a list of irreducible modules V, which are all direct summands of $S(P, Q)$. To determine the multiplicity of a fixed simple module U in $S(P, Q)$, we iterated over all summands V and checked how many times $\dim(\mathrm{Hom}(U, V)) \neq 0$ using the DimHom() command from Peter Webb's package 'reps'.

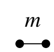
We had to identify which complex characters were summands of $S(P, Q)$ over $\mathbb{C}$ by way of our construction in positive characteristic. By our choice of finite field, the dimensions of the irreducible constituents of $S(P, Q)$ over $\mathbb{F}_{23}$ equaled the dimensions of the simple summands of $S(P, Q)$ over $\mathbb{Q}$ (with the 48-dimensional exception). Though the group H_4 has several sets of rational characters with the same dimension, these could be disambiguated by considering which are common constituents of $1 \uparrow_P^W$ and $\epsilon \uparrow_Q^W$, for which we had computational data.

In determining the rational ireducible representations of H_4 we took the complex character table and used the fact that all Schur indices are 1 except for the representation of degree 48, which has Schur index 2, as stated in Theorem 6.3.8 of Geck and Pfeiffer [5], page 193. This means that the irreducible complex representation of degree 48 is not realizable over the rationals, even though its character is rational valued. However, the direct sum of two copies of this representation can be written in the rationals. For the remaining rational representations their characters were taken by summing Galois conjugacy classes of characters.

Chapter 6

Parametrizations in Coxeter Type I

In this chapter we discuss the dihedral groups $I_2(m)$, $m > 2$, also commonly denoted D_{2m} as they are of order $2m$. These have Coxeter diagrams given by



and are the groups of isometries of regular polygons.

Our main result in this chapter is Theorem 6.10, which states that for all m, the rational irreducible characters of $I_2(m)$ admit a parametrization by generalized common constituents that satisfies Property 2.16.

Our main results are Theorems 6.10 and 6.12, which show that for all m, the rational irreducible characters of $I_2(m)$ admit a parametrization by generalized common constituents that satisfies Property 2.16, and also that $\phi_{\mathrm{Const}}(P, Q)$ is irreducible whenever (P, Q) is maximal, in the manner of Property 2.12. We discuss several examples and explicitly give the decomposition matrices in these cases. We do not discuss generalized Specht modules of dihedral groups.

Throughout this section, D_{2m} will always denote the Coxeter group of type $I_2(m)$, with presentation $\langle x, y \mid x^2 = y^2 = (xy)^m = 1\rangle$. Let C_m denote the cyclic subgroup $\langle xy\rangle \leq D_{2m}$, and let ζ_m denote the complex number $e^{\frac{2\pi i}{m}}$.

6.1 Parametrization of rational characters in type I with common constituents

The complex character tables for D_{2m} are well known; see for example [22]. We include these as Table 6.1:

$D_{2m},\ m$ odd						
g	1	x	xy	$(xy)^2$	$\dots$	$(xy)^{\frac{m-1}{2}}$
χ_1	1	1	1	1	$\dots$	1
χ_ϵ	1	-1	1	1	$\dots$	1
$\chi_{\zeta_m^s}\uparrow_{C_m}^{D_{2m}}$ $(1 \le s \le \frac{m-1}{2})$	2	0	$\zeta_m^s + \zeta_m^{-s}$	$\zeta_m^{2s} + \zeta_m^{-2s}$	$\dots$	$\zeta_m^{\frac{m-1}{2}s} + \zeta_m^{-\frac{m-1}{2}s}$

$D_{2m},\ m$ even								
g	1	x	y	xy	$(xy)^2$	$\dots$	$(xy)^{\frac{m-2}{2}}$	$(xy)^{\frac{m}{2}}$
χ_1	1	1	1	1	1	$\dots$	1	1
χ_ϵ	1	-1	-1	1	1	$\dots$	1	1
χ_{1a}	1	-1	1	-1	1	$\dots$	$(-1)^{\frac{m-2}{2}}$	$(-1)^{\frac{m}{2}}$
χ_{1b}	1	1	-1	-1	1	$\dots$	$(-1)^{\frac{m-2}{2}}$	$(-1)^{\frac{m}{2}}$
$\chi_{\zeta_m^s}\uparrow_{C_m}^{D_{2m}}$ $(1 \le s \le \frac{m-2}{2})$	2	0	0	$\zeta_m^s + \zeta_m^{-s}$	$\zeta_m^{2s} + \zeta_m^{-2s}$	$\dots$	$\zeta_m^{\frac{m-2}{2}s} + \zeta_m^{-\frac{m-2}{2}s}$	-2

Table 6.1: Complex character tables for $W = D_{2n}$.

From now on, we write ϕ_1, ϕ_{1a}, ϕ_{1b}, and ϕ_ϵ instead of $\chi_1, \chi_{1a}, \chi_{1b}$, and χ_ϵ to emphasize that they are rational characters. We note that $\phi_{1b} = \phi_{1a} \otimes \phi_\epsilon$.

We identify a family of rational characters of D_{2m} constructed using subgroups of C_m. We denote for every $s \in \mathbb{Z}$ the complex character $\chi_{2,s} := \chi_{\zeta_m^s}\uparrow_{C_m}^{D_{2m}}$. We immediately observe the following:

Lemma 6.1. *The following properties hold:*

1. *For all $a \in \mathbb{Z}$, $\chi_{2,am+s} = \chi_{2,s}$.*

2. *For all $s \in \mathbb{Z}$, $\chi_{2,m-s} = \chi_{2,s}$.*

3. *We have that $\chi_{2,0}$ is rational and decomposes as $\phi_1 + \phi_\epsilon$.*

4. *When m is even, we have that $\chi_{2,\frac{m}{2}}$ is rational and decomposes as $\phi_{1a} + \phi_{1b}$.*

5. *For all $s \in \mathbb{Z}$, $\chi_{2,s} \otimes \chi_\epsilon = \chi_{2,s}$.*

We establish some preliminary results and then give a description of all rational characters of D_{2m}. For a finite group G let CF(G) denote the complex space of class functions on G.

Proposition 6.2. *If k is odd and relatively prime to m, then the mapping $\Psi_k : \mathrm{CF}(D_{2m}) \to \mathrm{CF}(D_{2m})$ given by $\Psi_k f(g) = f(g^k)$, for all $g \in D_{2m}$, is a linear isomorphism which permutes the irreducible complex characters and is the identity on rational characters. Moreover, the character identity*

$$\langle \chi, \chi_{2,s} \rangle = \langle \Psi_k \chi, \chi_{2,ks} \rangle$$

holds.

Proof. If f is a class function and $g, h \in D_{2m}$ then $\Psi_k f(hgh^{-1}) = f((hgh^{-1})^k) = f(hg^k h^{-1}) = f(g^k) = \Psi_k f(g)$, so $\Psi_k f$ is also a class function. Then Ψ_k is well-defined, and it is clearly linear.

Now suppose that χ is any character of degree 1. Since k is odd, $g^k = g$ when g is one of 1, x, or y. Moreover, when m is even, the character table shows that $\chi((xy)^d) = -1$ if and only if $\chi((xy)^{kd}) = (-1)^k = -1$. Therefore Ψ_k fixes each one-dimensional character of D_{2m}.

For the remaining irreducible characters, we again have for all s that $\Psi_k \chi_{2,s}$ evaluates to 2, 0, and 0 on 1, x, and y respectively. Moreover,

$$\Psi_k \chi_{2,s}((xy)^d) = \chi_{2,s}((xy)^{kd}) = \zeta_m^{kds} + \zeta_m^{-kds} = \chi_{2,ks}((xy)^d).$$

Therefore $\Psi_k \chi_{2,s} = \chi_{2,ks}$ which establishes that Ψ_k takes irreducible characters to irreducible characters. Since k is relatively prime to n, then Ψ_k is invertible on the two-dimensional irreducibles. Thus Ψ_k is a permutation on irreducible characters, so it is a linear isomorphism on CF(D_{2m}). The character identity also immediately follows.

We will now show that Ψ_k is the identity on rational characters. Observe that the complex characters of D_{2m} all take on values in the cyclotomic field $\mathbb{Q}(\zeta_m)$. The Galois group of this field is naturally isomorphic to $(\mathbb{Z}/m\mathbb{Z})^\times$, where the residue $k \pmod m$ with $\gcd(k, m) = 1$ corresponds to the field automorphism σ_k which sends ζ_m^d to ζ_m^{dk}. Then for all integers s and d,

$$\begin{aligned}\Psi_k\chi_{2,s}((xy)^d) &= \chi_{2,s}((xy)^{kd}) \\ &= \zeta_m^{kds} + \zeta_m^{-kds} \\ &= \sigma_k(\zeta_m^{ds} + \zeta_m^{-ds}) \\ &= \sigma_k(\chi_{2,s}((xy)^d)).\end{aligned}$$

This gives the identity $\Psi_k\chi(g) = \sigma_k(\chi(g))$ for any two-dimensional irreducible character χ and for any g of the form $(xy)^d$. If χ is a one-dimensional character or g is not a power of xy, then the character table shows that $\chi(g)$ is rational, so it is fixed by σ_k: $\sigma_k(\chi(g)) = \chi(g)$. In the case that χ is a one-dimensional character, then we saw earlier that $\Psi_k(\chi) = \chi$ and so $\Psi_k\chi(g) = \chi(g) = \sigma_k\chi(g)$. In the case that g is not a power of xy, then $g^k = g$ since k is odd and $|xy|$ is 1 or 2. Therefore $\Psi_k\chi(g) = \chi(g^k) = \chi(g) = \sigma_k\chi(g)$.

Therefore $\Psi_k\chi(g) = \sigma_k(\chi(g))$ holds for all irreducible complex characters χ and all $g \in D_{2m}$. Taking linear combinations, we see that this identity holds for all class functions on D_{2m}.

Then suppose that ϕ is an arbitrary rational character (which may be irreducible as a rational character but reducible as a complex character). Then

$$\Psi_k\phi(g) = \sigma_k(\phi(g)) = \phi(g)$$

since σ_k fixes the rational subfield. Thus Ψ_k fixes all rational characters pointwise. □

Corollary 6.3. *1. Let ϕ be a rational character of D_{2m}. If $s \in \mathbb{Z}$ and k is relatively prime to m, then $\langle\phi, \chi_{2,s}\rangle = \langle\phi, \chi_{2,ks}\rangle$.*

2. *Let ϕ be as above. For all divisors d of m, $2 < d < \frac{m}{2}$, the multiplicities in ϕ of all characters $\chi_{2,s}$ such that* $\gcd(s,m) = \frac{m}{d}$*, are all equal.*

3. *Now let $\hat{\phi}$ be a rational character of the cyclic subgroup C_m. Then for all divisors d of m, $d > 2$, the multiplicities in $\hat{\phi}$ of all characters ζ_m^s such that* $\gcd(s,m) = \frac{m}{d}$*, are all equal.*

Proof. 1. Notice that if m is odd and k happens to be even, then $m + k$ is odd and will serve as well as k, so we may assume that k is relatively prime to $2m$. Proposition 6.2 implies that $\langle \phi, \chi_{2,s} \rangle = \langle \psi_k \phi, \chi_{2,ks} \rangle$, and since ϕ is rational the right-hand side is equal to $\langle \phi, \chi_{2,ks} \rangle$, again by Proposition 6.2.

2. Let d be a divisor of m with $d > 2$. Then the condition $\gcd(s,m) = m/d$ implies $\langle \phi, \chi_{2,s} \rangle = \langle \phi, \chi_{2,\gcd(s,m)\frac{s}{\gcd(s,m)}} \rangle = \langle \phi, \chi_{2,\gcd(s,m)} \rangle$ by Part 1 of this corollary, since $\gcd(\frac{s}{\gcd(s,m)}, m) = 1$. Thus the multiplicities of $\chi_{2,s}$ where $\gcd(s,m) = \frac{m}{d}$ are all equal to $\langle \phi, \chi_{2,\frac{m}{d}} \rangle$.

3. Let $\hat{\phi}$ be a rational character of C_m and suppose that $\gcd(s,m) = \gcd(s',m) = \frac{m}{d}$. If $\langle \hat{\phi}, \zeta_m^s \rangle = a$, then taking complex conjugates we have $\langle \hat{\phi}, \zeta_m^{-s} \rangle = a$ as well. Inducing to D_{2m}, we observe that a complex summand of $\chi_{2,s}$ in $\hat{\phi} \uparrow_{C_m}^{D_{2m}}$ arises from inducing a summand of $\hat{\phi}$ equal to either ζ_m^s or ζ_m^{-s}. Thus, $\langle \hat{\phi} \uparrow_{C_m}^{D_{2m}}, \chi_{2,s} \rangle = 2a$. Likewise if $\langle \hat{\phi}, \zeta_m^{s'} \rangle = a'$, then $\langle \hat{\phi} \uparrow_{C_m}^{D_{2m}}, \chi_{2,s'} \rangle = 2a'$. By Part 2 of this corollary, $a = a'$ which completes the proof. □

Corollary 6.3 implies that whenever an irreducible rational character has a constituent $\chi_{2,s}$ where $|\zeta_m^s| = d$, then it also has as a constituent any other character $\chi_{2,s'}$ with $|\zeta_m^{s'}| = d$.

From here on, we define reflection subgroups based on the presentation $D_{2m} = \langle x, y \mid x^2 = y^2 = (xy)^m = 1 \rangle$. For each divisor d of m, let $D_{2d,x} = \langle x, (yx)^{\frac{m}{d}-1} y \rangle$ and $D_{2d,y} = \langle y, (xy)^{\frac{m}{d}-1} x \rangle$. Note in the case of $d = 1$ that the subgroups $D_{2,x}$ and $D_{2,y}$ are rank one Coxeter groups with $D_{2,x} = \langle x \rangle$ and $D_{2,y} = \langle y \rangle$. When m is odd, $D_{2d,x}$ and $D_{2d,y}$ are conjugate, and we denote $D_{2d} = D_{2d,x}$. We record the following observation:

Proposition 6.4. *Every nonidentity reflection subgroup is conjugate to $D_{2d,x}$ or $D_{2d,y}$ for some $d \mid m$. If $d \neq m$, then each of the subgroups $D_{2d,x}$ and $D_{2d,y}$ contains a unique Coxeter generator.*

Proof. Let P be a nonidentity reflection subgroup. If P has rank one then it is generated either by a conjugate of x or a conjugate of y. Otherwise, P is dihedral, and we may simultaneously conjugate its generators so that one of them is equal to x or y. This establishes the first claim, and the second claim follows immediately. □

We will now show that there exist representations of D_{2m} with rational characters of the form

$$\sum_{1 \leq s < \frac{m}{2}, \gcd(s,m) = \frac{m}{d}} \chi_{2,s}$$

which are necessarily irreducible.

Proposition 6.5. *1. The cyclic group $C_m = \langle g \rangle$ has irreducible rational characters $\hat{\phi}_H$ in bijection with its subgroups H, such that*

$$\mathbb{Q} \uparrow_H^{C_m} = \sum_{H' \geq H} \hat{\phi}_{H'}$$

for each subgroup H. If $|C_m : H| = d$, the value $\hat{\phi}_H$ at g is equal to the sum of the primitive d-th roots of unity.

2. *Each of the representations $\hat{\phi}_H$ of C_m extends to an irreducible representation ϕ_J of D_{2m} whenever J is a dihedral subgroup whose alternating subgroup is equal to H.*

3. *The irreducible rational representations of D_{2m} are the above representations, together with the sign representation.*

4. *For all divisors d of m, the virtual complex character*

$$\sum_{1 \leq s < \frac{m}{2}, \gcd(s,m) = \frac{m}{d}} \chi_{2,s}$$

is the character of a rational representation of D_{2m}. When $d > 2$, this character is irreducible.

Proof. 1. Fix a divisor d of m. If $H = \langle g^{\frac{m}{d}} \rangle$ then $\mathbb{Q} \uparrow_H^{C_m}$ has a basis of d elements cyclically permuted by g. Thus the characteristic polynomial for the action of g is $x^d - 1$. Then we define the complex representation $\hat{\rho}_H$ to be the sum of the ζ_d^k-eigenspaces of $\mathbb{Q} \uparrow_H^{C_m}$, for all k with $\gcd(k, d) = 1$. This has dimension $\phi(d)$ (as in the Euler phi function). Denote by $\hat{\phi}_H$ the character of $\hat{\rho}_H$.

We will show by induction on $|C_m : H|$ that $\mathbb{Q} \uparrow_H^{C_m}$ is a sum of rational representations of the form $\hat{\rho}_{H'}$ with characters $\hat{\phi}_{H'}$ as specified. For the base case, when $|C_m : H| = 1$ (that is, $d = m$), then $\hat{\rho}_H$ is the trivial module, which is equal to $\mathbb{Q} \uparrow_H^{C_m}$.

Now suppose that $|C_m : H| > 1$. Let d' be a divisor of m with $d \mid d'$, so $H' := \langle g^{\frac{m}{d'}} \rangle \geq \langle g^{\frac{m}{d}} \rangle = H$. Then $\mathbb{Q} \uparrow_{H'}^{C_m}$ is a direct summand of $\mathbb{Q} \uparrow_H^{C_m}$. The former of these permutation representations is, by induction, the sum of $\hat{\rho}_{H''}$ where $H'' \geq H'$. In particular, $\hat{\rho}_{H'}$ is a summand of $\mathbb{Q} \uparrow_H^{C_m}$. It follows that $\mathbb{Q} \uparrow_H^{C_m}$ has each such $\hat{\rho}_{H'}$ as a constituent, and by definition $\hat{\rho}_H$ is also a constituent. All of these constituents have trivial intersections because the actions of g on these representations have distinct eigenvalues. Then by dimension counting, the multiplicities of the $\hat{\rho}_{H'}$, $H' \geq H$, must equal one since $d = \sum_{d'|d} \phi(d')$. Thus $\mathbb{Q} \uparrow_H^{C_m} = \sum_{H' \geq H} \hat{\rho}_{H'}$, and if all $\hat{\rho}_{H'}$, $H' > H$, are rational characters, then so is $\hat{\rho}_H$. To conclude Part 1, the irreducibility of the $\hat{\rho}_H$ follows from Part 3 of Corollary 6.3, except in the case that m is even and $|C_m : H| = 2$. In this case, $\hat{\rho}_H$ is equal to the nontrivial character of C_m of degree 1 in which g^k acts as multiplication by $(-1)^k$.

2. We may extend the representations $\hat{\rho}_H$ to D_{2m}-representations, as follows. The C_m-representation $\mathbb{Q} \uparrow_H^{C_m}$ has a basis of cosets which are permuted cyclically by the action of g. We also consider the linear transformation θ that sends each $g^j v$ to $g^{-j} v$, which has a rational matrix. Now if J is one of the dihedral subgroups of the form $D_{2d,x}$ or $D_{2d,y}$, we let z denote the unique Coxeter generator contained in J according to Proposition 6.4. The action of z as the transformation θ, together with the action of $g = xy$, determines a rational representation ρ_J of D_{2m}. Each subrepresentation

$\hat{\rho}_H$ is invariant under the action of θ, with θ mapping each ζ_m^s-eigenspace to the ζ_m^{-s}-eigenspace. The matrices of θ on these subrepresentations are still rational, so this determines a D_{2m}-representation ρ_J extending $\hat{\rho}_H$. This is irreducible since the restricted representation $\hat{\rho}_H$ is irreducible.

3. The number of rational irreducible representations of a finite group equals the number of conjugacy classes of its cyclic subgroups, as a consequence of Artin's induction theorem (see [2], Exercise 15.4). For D_{2m}, the cyclic subgroups not contained in C_m are generated by reflections. When m is odd, the reflections are all conjugate and there is one additional conjugacy class. When m is even, there are two conjugacy classes which contain the two Coxeter generators. Thus the total number of rational irreducible characters of D_{2m} is $\tau(m) + 1$ if m is odd and $\tau(m) + 2$ if m is even, where $\tau(m)$ is the number of divisors of m.

For all m, the trivial character arises as $\phi_{D_{2m}}$. When m is odd and $d \mid m$ is a proper divisor, the characters $\phi_{D_{2d}}$ all have degree equal to $\phi(d)$, and this is greater than 1 when $d \notin \{1, 2\}$. These characters are distinct since they have distinct C_m-eigenspaces. The ϕ_J, along with the sign character, constitute $\tau(m) + 1$ distinct rational irreducible characters.

When m is even and $d \mid m$ is a divisor with $d < \frac{m}{2}$, then the characters $\phi_{D_{2d,x}}$ all have degree greater than 1, and again they are distinct since they have distinct C_m-eigenspaces. In the case of $d = \frac{m}{2}$, we have that $\mathbb{Q} \uparrow_{\langle g^2 \rangle}^{C_m} = \hat{\rho}_{C_m} + \hat{\rho}_{\langle g^2 \rangle}$. If $J = D_{m,x}$, then the action of x as θ, on this coset space of dimension 2, is trivial, so the action of y equals the action of xy and has a (-1)-eigenspace. Thus, $\phi_{D_{m,x}} = \phi_{1b}$. Likewise, $\phi_{D_{m,y}} = \phi_{1a}$. We have identified $\tau(m) + 1$ distinct characters of the form ϕ_J when m is even, and these along with the sign character constitute $\tau(m) + 2$ distinct rational irreducible characters.

4. When $d > 2$ this complex character is the character of the irreducible representation ρ_H constructed in Part 2, corresponding to ϕ_H, where $|C_m : H| = d$. When $d > 2$, the claim follows from Corollary 6.3. □

Remark 6.6. The proof of Part 3 shows that, when m is even, the irreducible rational characters of the form $\phi_{D_{2d,x}}$ with $d < \frac{m}{2}$ are uniquely determined by their C_m-eigenspaces. It follows that $\phi_{D_{2d,x}} = \phi_{D_{2d,y}}$ whenever the degrees of these characters are greater than 1.

Corollary 6.7. *1. Each irreducible rational representation of D_{2m} restricts to an irreducible representation of the cyclic subgroup C_m.*

2. *The above establishes a bijection between the irreducible rational representations of D_{2m} of degree at least 2, and the irreducible rational representations of C_m of degree at least 2.*

3. *For each irreducible rational representation ϕ of D_{2m} of degree at least 2, $\phi = \phi \otimes \phi_\epsilon$.*

4. *The trivial representation and sign representation of D_{2m} both restrict to the trivial representation of C_m. When m is even, the remaining two 1-dimensional representations of D_{2m} both restrict to the non-trivial 1-dimensional representation of C_m.*

Proposition 6.8. *Let J be a dihedral subgroup of D_{2m} and write $H = J \cap C_m$. Then, to within 1-dimensional representations,*

$$\mathbb{Q} \uparrow_J^{D_{2m}} \equiv \epsilon \uparrow_J^{D_{2m}} \equiv \sum_{J'} \phi_{J'} \quad \textit{(modulo characters of degree 1)},$$

where the sum is taken over dihedral subgroups J' of D_{2m} containing J, one of each possible order. The value of $\phi_{\mathrm{Const}}(J, J)$ is equal to the above sum, without characters of degree 1.

Proof. When V is $\mathbb{Q}$ or ϵ we have

$$V \uparrow_J^{D_{2m}} \downarrow_{C_n}^{D_{2m}} \cong \bigoplus_{C_m \backslash G/J} V \downarrow_{C_m \cap J}^{J} \uparrow_{C_m \cap J}^{C_m} = \mathbb{Q} \uparrow_H^{C_m} = \bigoplus_{C_m \geq K \geq H} \hat{\phi}_K$$

noting that V restricts to the trivial module on C_m and there is only one double coset. Since each $\hat{\phi}_K$ is the restriction of $\phi_{J'}$ whenever $\phi_{J'}$ has degree at least 2 and $K = A(J')$, the first claim follows.

The characters of degree 1 occurring in $\mathbb{Q}\uparrow_J^{D_{2m}}$ are $\phi_1 = \phi_{D_{2m}}$ and, if m is even, one of $\phi_{1b} = \phi_{D_{m,x}}$ or $\phi_{1a} = \phi_{D_{m,y}}$. The characters of degree 1 occurring in $\epsilon\uparrow_J^{D_{2m}}$ are the tensor products of these characters with ϵ, so they are distinct and do not occur as common constituents. □

Remark 6.9. This result also holds, with the same proof, when J is a rank one reflection subgroup.

We come now to one of the main results of this section, which says that the irreducible rational representations of dihedral groups can be parametrized by pairs of reflection subgroups.

Theorem 6.10. *Property 2.16 holds for the Coxeter groups of type I. That is, there is a set of ordered pairs of reflection subgroups of D_{2m}, in bijection with the irreducible rational characters of D_{2m}, and an ordering on these pairs and on the irreducible rational characters, so that the multiplicity matrix of the characters $\phi_{\mathrm{Const}}(P, Q)$ is unitriangular. Moreover, whenever (P, Q) corresponds to ϕ in the parametrization then (Q, P) corresponds to $\phi \otimes \phi_\epsilon$.*

Proof. To verify Property 2.16 when m is odd, we may choose the following collection of subgroup pairs:

- The pair $(D_{2m}, 1)$ with common constituents character equal to ϕ_1.
- The pair (D_2, D_2). By Proposition 6.8 and the following remark, the common constituents are equal to the sum of one copy of each irreducible of the form ϕ_J of degree greater than 1.
- The pairs (D_{2d}, D_{2d}) for $1 < d < m$, ordered increasing in d. The common constituents character equals the sum of characters of degree greater than 1 described in the statement of Proposition 6.8, each occurring with multiplicity 1.

- The pair $(1, D_{2m})$ with common constituents character equal to ϕ_ϵ.

When m is even, we take the following collection of pairs:

- The pair $(D_{2m}, 1)$ with common constituents character equal to ϕ_1.
- The pair $(D_{m,x}, D_{m,y})$. By the proof of Proposition 6.5, Part 3, the permutation character on $D_{m,x}$ decomposes as $\phi_1 + \phi_{1b}$. By an analogous construction, the permutation character on $D_{m,y}$ decomposes as $\phi_1 + \phi_{1a}$, so the signed permutation character is $\phi_\epsilon + \phi_{1b}$. Thus the common constituents are irreducible, equal to ϕ_{1b}.
- The pair $(D_{2,x}, D_{2,x})$ with common constituents again equal to the sum of one copy of each irreducible of the form ϕ_J of degree greater than 1, by Proposition 6.8 and the following remark.
- The pairs $(D_{2d,x}, D_{2d,x})$ for $1 < d < \frac{m}{2}$, ordered increasing in d. The common constituents are again equal to the sum of characters of degree greater than 1 described in the statement of Proposition 6.8, each irreducible summand occurring with multiplicity 1.
- The pair $(D_{m,y}, D_{m,x})$ which, analogously to the pair $(D_{m,x}, D_{m,y})$, has ϕ_{1a} as its common constituents.
- The pair $(1, D_{2m})$ with common constituents character equal to ϕ_ϵ.

Immediately from the above descriptions of the characters, we have that the multiplicity matrix is lower unitriangular. Moreover, it is clear by our choice of pairs that (P, Q) corresponds to ϕ if and only if (Q, P) corresponds to $\phi \otimes \epsilon$. □

Remark 6.11. For dihedral groups, we may consider the effect on parametrizations of an additional dual operation given by the nontrivial automorphism of the Dynkin diagram of type $I_2(m)$. The corresponding outer automorphism of D_{2m} swaps the generators x and y. Suppose m is even. If we take the parametrization from the proof of Theorem 6.10 and replace every subgroup $D_{2d,x}$ occurring in each subgroup pair with

$D_{2d,y}$, and vice-versa, the resulting collection of subgroup pairs gives another (distinct) parametrization of the irreducible rational characters. This parametrization also satisfies Property 2.16, and the order of the characters ϕ_{1a} and ϕ_{1b} is swapped.

Theorem 6.12. *For dihedral groups $I_2(m)$, $m > 1$, Property 2.13 holds. That is, the maximal subgroup pairs in the support of the function ϕ_{Const} are all irreducible.*

Proof. The maximal elements of the support of ϕ_{Const} when m is odd are the following pairs:

- The pairs $(D_{2m}, 1)$ and $(1, D_{2m})$. For any finite Coxeter group W, the only reflection subgroup Q such that $\epsilon \uparrow_Q^1$ contains the irreducible constituent $1 = 1 \uparrow_W^W$ is $Q = 1$. Thus, the pair $(W, 1)$ is maximal for every Coxeter group W, and $\phi_{\text{Const}}(W, 1) = \phi_1$ is irreducible. Likewise, $(1, W)$ is always maximal and $\phi_{\text{Const}}(1, W) = \phi_\epsilon$ is irreducible.

- The pairs $(D_{2\frac{m}{p}}, D_{2\frac{m}{p}})$, for each prime divisor p of m. By Proposition 6.8,

$$\phi_{\text{Const}}\left(D_{2\frac{m}{d}}, D_{2\frac{m}{d}}\right) = \sum_{1<d'\leq d} \phi_{D_{2\frac{m}{d'}}}$$

  with the summands $\phi_{D_{2\frac{m}{d'}}}$ all irreducible characters of degree greater than 1. There is a unique summand if and only if d is prime.

When m is even, the maximal pairs are as follows:

- The pairs $(D_{2m}, 1)$ and $(1, D_{2m})$ as in the odd case.

- The pairs $(D_{m,x}, D_{m,y})$ and $(D_{m,y}, D_{m,x})$. In the proof of Theorem 6.10, the common constituents of these pairs were determined as ϕ_{1b} and ϕ_{1a} respectively.

- The pairs $(D_{2\frac{m}{p},x}, D_{2\frac{m}{p},x})$ and $(D_{2\frac{m}{p},y}, D_{2\frac{m}{p},y})$, for each odd prime p dividing $\frac{m}{2}$. As with the m odd case, the common constituents are irreducible equal to $\phi_{D_{2\frac{m}{p},x}} = \phi_{D_{2\frac{m}{p},y}}$.

- The pairs $(D_{\frac{m}{2},x}, D_{\frac{m}{2},x})$ and $(D_{\frac{m}{2},y}, D_{\frac{m}{2},y})$ in the case that $4 \mid m$. As with the previous pairs, the common constituents are irreducible equal to $\phi_{D_{\frac{m}{2},x}} = \phi_{D_{\frac{m}{2},y}}$.

These are all of the maximal pairs of the support of ϕ_{Const}, and we see that the common constituents on these pairs all consist of a single irreducible module. □

6.2 Examples

We present several computations that show the parametrization of irreducible rational representations of dihedral groups by pairs of reflection subgroups.

Example 6.13. We consider the 2-group D_{16}. We write $\phi_2 = \phi_{4,x} = \phi_{4,y}$ and $\phi_4 = \phi_{2,x} = \phi_{2,y}$. Table 6.2 gives the direct sum decompositions of all generalized common constituent characters of reflection subgroup pairs.

	1	$D_{2,x}$	$D_{4,x}$	$D_{8,x}$	$D_{2,y}$	$D_{4,y}$	$D_{8,y}$	D_{16}
D_{16}	ϕ_1	0	0	0	0	0	0	0
$D_{8,y}$	0	ϕ_{1b}	ϕ_{1b}	ϕ_{1b}	0	0	0	0
$D_{4,y}$	0	$\phi_{1b} + \phi_2$	$\phi_{1b} + \phi_2$	ϕ_{1b}	ϕ_2	ϕ_2	0	0
$D_{2,y}$	0	$\phi_{1b} + \phi_2 + \phi_4$	$\phi_{1b} + \phi_2$	ϕ_{1b}	$\phi_2 + \phi_4$	ϕ_2	0	0
$D_{8,x}$	0	0	0	0	ϕ_{1a}	ϕ_{1a}	ϕ_{1a}	0
$D_{4,x}$	0	ϕ_2	ϕ_2	0	$\phi_{1a} + \phi_2$	$\phi_{1a} + \phi_2$	ϕ_{1a}	0
$D_{2,x}$	0	$\phi_2 + \phi_4$	ϕ_2	0	$\phi_{1a} + \phi_2 + \phi_4$	$\phi_{1a} + \phi_2$	ϕ_{1a}	0
1	0	0	0	0	0	0	0	ϕ_ϵ

Table 6.2: Table of common constituents for all reflection subgroup conjugacy class pairs for $W = D_{16}$.

We see from this table that the maximal elements of the support of ϕ_{Const} are as follows:

$$(D_{16}, 1),\ (D_{8,x}, D_{8,y}),\ (D_{8,y}, D_{8,x}),\ (D_{4,x}, D_{4,x}),\ (D_{4,y}, D_{4,y}),\ (1, D_{16}).$$

In each case the common constituents consist of a single irreducible representation. Although there are 6 maximal elements and 6 irreducible rational characters, the mapping $(P, Q) \mapsto \phi_{\text{Const}}(P, Q)$ does not give a parametrization as it is not a bijection.

However, there are many possibilities for parametrizing the irreducible representations by pairs of reflection subgroups in order to obtain a unitriangular multiplicity matrix. Table 6.3 gives the parametrization discussed in the proof of Theorem 6.10.

Example 6.14. We generalize the previous example to all dihedral 2-groups. Let $W =$

	$(D_{16}, 1)$	$(D_{8,x}, D_{8,y})$	$(D_{2,x}, D_{2,x})$	$(D_{4,x}, D_{4,x})$	$(D_{8,y}, D_{8,x})$	$(1, W)$
ϕ_1	1	0	0	0	0	0
ϕ_{1b}	0	1	0	0	0	0
ϕ_4	0	0	1	0	0	0
ϕ_2	0	0	1	1	0	0
ϕ_{1a}	0	0	0	0	1	0
ϕ_ϵ	0	0	0	0	0	1

Table 6.3: Multiplicity matrix of selected common constituents for $W = D_{16}$, invariant under simultaneously interchanging P with Q and ϕ with $\phi \otimes \epsilon$.

D_{2^k}, $k > 2$. For $1 \leq i \leq k-2$, denote by ϕ_{2^i} the irreducible character $\phi_{D_{2^{k-1-i},x}} = \phi_{D_{2^{k-1-i},y}}$. Table 6.4 displays the parametrization discussed in Theorem 6.10.

	$(D_{2^k}, 1)$	$(D_{2^{k-1},x}, D_{2^{k-1},y})$	$(D_{2,x}, D_{2,x})$	$(D_{4,x}, D_{4,x})$	$\cdots$	$(D_{2^{k-2},x}, D_{2^{k-2},x})$	$(D_{2^{k-1},y}, D_{2^{k-1},x})$	$(1, W)$
ϕ_1	1	0	0	0	$\cdots$	0	0	0
ϕ_{1b}	0	1	0	0	$\cdots$	0	0	0
$\phi_{2^{k-2}}$	0	0	1	0	$\cdots$	0	0	0
$\phi_{2^{k-3}}$	0	0	1	1	$\cdots$	0	0	0
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
ϕ_2	0	0	1	1	$\cdots$	1	0	0
ϕ_{1a}	0	0	0	0	$\cdots$	0	1	0
ϕ_ϵ	0	0	0	0	$\cdots$	0	0	1

Table 6.4: Multiplicity matrix of selected common constituents for $W = D_{2^k}$, invariant under simultaneously interchanging P with Q and ϕ with $\phi \otimes \epsilon$.

In the case of dihedral 2-groups, we see that this multiplicity matrix satisfies a further additional property: it is symmetric about the matrix antidiagonal. However, the next example shows that this is not always possible for dihedral groups.

Example 6.15. Let $W = D_{30}$. We write $\phi_2 = \phi_{D_{10}}$, $\phi_4 = \phi_{D_6}$, and $\phi_8 = \phi_{D_2}$. The table of common constituents and the multiplicity matrix are given in Tables 6.5 and 6.6.

We observe from Table 6.5 that the subgroup pair (D_2, D_2) must occur in any parametrization, as only its common constituents character has ϕ_8 as a summand. To have a multiplicity matrix symmetric about the matrix antidiagonal, there should exist a common constituent character equal to the sum of two of the terms appearing in

$\phi_{\text{Const}}(D_2, D_2)$, but such a character does not exist. Although we do not have symmetry about the antidiagonal, note that the invariance given in Property 2.16 still holds.

	1	D_2	D_6	D_{10}	D_{30}
D_{30}	ϕ_1	0	0	0	0
D_{10}	0	ϕ_2	0	ϕ_2	0
D_6	0	ϕ_4	ϕ_4	0	0
D_2	0	$\phi_2 + \phi_4 + \phi_8$	ϕ_4	ϕ_2	0
1	0	0	0	0	ϕ_ϵ

Table 6.5: Table of common constituents for all reflection subgroup conjugacy class pairs for $W = D_{30}$.

	$(D_{30}, 1)$	(D_2, D_2)	(D_6, D_6)	(D_{10}, D_{10})	$(1, D_{30})$
ϕ_1	1	0	0	0	0
ϕ_8	0	1	0	0	0
ϕ_4	0	1	1	0	0
ϕ_2	0	1	0	1	0
ϕ_ϵ	0	0	0	0	1

Table 6.6: Multiplicity matrix of selected common constituents for $W = D_{30}$, invariant under simultaneously interchanging P with Q and ϕ with $\phi \otimes \epsilon$.

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