On some invariants associated to simple group representations

Ph D Thesis

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Abstract

In this thesis we use Brauer quotients to try to establish an explicit bijection between simple modules and weights (in the sense of Alperin) for the symmetric groups. We present experimental evidence – in the form of tables of Brauer quotients of simple modules with respect to weight subgroups – that we have gathered using the computer algebra systems SYMMETRICA and GAP. We describe the weight subgroups of certain symmetric groups with respect to which the Brauer quotients of the simples $D^{(n-1,1)}$ and $D^{(n-2,2)}$ are simple and projective. We prove that by appending triangular partitions to a trivial partition of even size we can create Brauer quotients that have simple projective summands. We define 2-stable partitions, and prove a property they have which relates to the type of explicit correspondence between simple modules and weights under consideration. We also state several conjectures that we have drawn in the course of our work.

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Chapter 1

Introduction

In this paper we explore ways to try to establish an explicit bijection between irreducible modules and weights (in the sense of Alperin) for the symmetric groups over fields of positive characteristic.

We begin with a relatively small amount of introductory material that describes the irreducible kS_n-modules using polytabloids and a bilinear form [9]. It has the great advantage that it is possible to perform explicit theoretical computations for some modules, and we use this in Chapters 4 and 5. Once we know what the irreducible modules for kS_n look like, we define the notion of weight, which is basically all we need to be able to state Alperin's Conjecture. We have irreducibles and weights, and the only thing that is missing is something that will provide a connection between the two: this is a construction frequently called the Brauer quotient.

Brauer quotients are one of the constructions coming out of the Brauer homomorphism which was used by Brauer in parameterizing blocks. We believe that Brauer quotients can be used to provide an explicit bijection between weights and irreducibles. Brauer quotients are not hard to define, and Section 2.3 has all that is needed to understand how we use them in this paper.

After the first three sections of Chapter 2, the rest of this paper consists mostly of original research aimed at showing that there is a connection between Brauer quotients, weights and irreducibles. The last section of Chapter 2 presents our original plan to give a combinatorial construction of an explicit bijection which demonstrates Alperin's Conjecture for the symmetric groups. Our goal was to establish an explicit bijection between weights and irreducibles for F_2S_n, which can be construed as a criterion that determines when a weight and an irreducible module should be called a "match", and we wanted to use Brauer quotients to do so. Weights will be explained in Chapter 2, but for the moment suffice it to say that a weight is a pair (Q, S), where Q is a p-subgroup of S_n (p is the characteristic of the field, which for our purposes will usually be 2) and S is a simple projective module for the quotient group $N_{S_n}(Q)/Q$. If V is a kS_n module, the Brauer quotient of V with respect to the subgroup Q, denoted $\overline{FP}_V(Q)$, is also a module for $N_{S_n}(Q)/Q$. Probably the most natural way to use Brauer quotients to determine whether a weight (Q, S) and a simple kS_n-module V should be a match is to ask if there is a connection between the $N_{S_n}(Q)/Q$-modules S and $\overline{FP}_V(Q)$, and the easiest question to ask is whether these two modules coincide.

If such a natural bijection were to exist, we believe that many of the properties of the irreducible modules would have weight counterparts and vice-versa. Moreover, since irreducibles are already naturally bijected with p-regular partitions of n, we should be able to discover connections between weights and p-regular partitions. We have reason to believe we have begun to unearth some interesting results about 2-regular partitions, as shown in Chapters 5 and 6. The fact that these properties came about as a consequence of the assumption of the validity of our proposed bijection makes us think that we might be on the right track.

First allow me to explain how we have been able to test our proposed bijection between weights and irreducibles. We did it with the help of two computer algebra systems: SYMMETRICA 1.0 (see [11]) and GAP (Groups, Algorithms and Programming, see [6]). We used SYMMETRICA to create a library of irreducible modules for F_2S_n, $n \leq 10$. The way to store a representation for F_2S_n is the following: first we fix a set of generators for the group S_n, and our convention is to use the permutations (1,2) and (1,2,...,n) in that order. Next we ask SYMMETRICA what the action of (1,2) is on the simple module parameterized by the 2-regular partition λ (denoted D^λ), and SYMMETRICA returns a matrix of 0's and 1's (the elements of F_2) that gives the action of (1,2) on D^λ, that is, the image of (1,2) under the corresponding irreducible representation $\rho : S_n \longrightarrow GL(m, F_2)$, where m is the dimension of D^λ over F_2. Similarly, we obtain the matrix that is the image of the other generator, $(1, 2, \ldots, n)$, under ρ. These two matrices are all we need to determine the representation ρ, because a morphism of groups is determined by its values on a set of generators.

The functions that we used to perform operations on these representations were written in the algebra package GAP, which is very similar to the programming language PASCAL, except that GAP is also capable of handling many algebraic structures. GAP already has types that correspond to the notions of groups, matrices, finite fields, morphisms of groups and permutations, but it does not have a type that models representations. We created a type in GAP to simulate representations, and what we used was a record that contained the relevant information that is needed to define a representation:

- its base field
- its dimension
- its domain (the group being represented),including a list of generators of the domain
- the corresponding list of images of the generators under the representation
- a flag that alerts us that it is indeed a representation (used to validate input type for functions)
- (some) operations that can be performed on it, for example, dual, or fixed points

As the last item shows, we created functions that performed various operations on these representations. The Appendix contains a list of the most useful functions that we wrote in GAP, but here we would like to mention two very important ones: one of them computes the Brauer quotient of a module with respect to a subgroup, and the other tests whether a module is projective or not. Using these functions we created tables of dimensions of Brauer quotients of the simple modules with respect to the weight 2-subgroups of the symmetric groups, and we identified those Brauer quotients that were simple (using one of GAP's built-in functions) and projective (using our software). These tables can be found in Section 3.4.

We were able to verify that our proposed algorithm indeed established the desired bijection between weights and simple modules for values of $n \leq 8$. In other words, given any weight (Q, S) for F_2S_n with $n \leq 8$, there is a unique simple F_2S_n-module V such that $\overline{FP}_V(Q) \cong S$ as $N_{S_n}(Q)/Q$-modules, and given any simple module V there is a unique weight (Q, S) with $\overline{FP}_V(Q) \cong S$.

Unfortunately, it is a different story when $n = 9$. In this case, there is a unique weight for S_9, whose weight subgroup is $E_2 \wr E_4$, such that for any non-trivial simple module V, $\overline{FP}_V(E_2 \wr E_4)$ is zero, and so is not simple and projective. On the other hand, there is also a unique simple S_9-module, namely $D^{(7,2)}$, such that for any weight subgroup Q of S_9, $\overline{FP}_{D^{(7,2)}}(Q)$ is not simple and projective. Except for these two, our algorithm establishes another explicit bijection between the remaining simples and weights for S_9. This is why we said that our algorithm failed, although someone might argue that it did not, and that instead of providing one bijection between all weights and all simples as before, the algorithm first singled out one element from each family (thereby providing a "match" for the two of them) and then bijected the remaining weights and simples as usual. Furthermore it is even the case here that the fixed points of $E_2 \wr E_4$ on $D^{(7,2)}$ contain a simple projective module as a summand; it is just that this summand does not survive in the Brauer quotient. We have also noticed that our tables of Brauer quotients can always be arranged in a triangular shape (see Section 3.4), with simple projective modules along the diagonal (except for this one 0 in the case of S_9). This fact provided the motivation to look for ways in which to modify our algorithm. Another possible way to try to establish a bijection is the following:

Conjecture 1.0.1. *The tables of Brauer quotients can always be arranged in a triangular form, with either 0's or simple projective modules along the diagonal. Moreover, this will define an explicit bijection between weights and simple modules.*

At the moment it is hard to tell if the criterion we were using will fail irrecoverably for larger values of n, or if it will work in a modified version. We need to look at more examples, but unfortunately the sizes of the irreducibles have grown beyond the capabilities of our computer, and at the moment we are unable to gather more experimental data. In my opinion, the algorithm will work without modifications for values of n that have a special congruence class, but this is just the impression that I got after working with Brauer quotients in general, which brings us to Chapter 4.

Since we were unable to continue our experimental work, we tried to prove analytically that many Brauer quotients do turn out to be simple projective modules. I am happy to say we were successful, and even though we only considered a limited class of irreducibles, there are indeed infinitely many instances of simple projective Brauer quotients. We considered first the partition $(n-1, 1)$, because its corresponding simple module is the easiest one to describe (next to the trivial module, parameterized by the trivial partition (n)). We proved that when n is odd, the Brauer quotient $\overline{FP}_{D^{(n-1,1)}}(H)$ is either 0 or simple, and that it is projective if and only if the subgroup H is conjugate to a Sylow 2-subgroup of S_{n-3}. Since this result is true for infinitely many values of n, we think that the experimental data we obtained are more than just coincidence, and that there is indeed a connection between Brauer quotients, weights and simple modules. Next we considered the partition $(n-2,2)$, and this time we had to restrict the values of n to those that are congruent to 3 modulo 4. We proved that $\overline{FP}_{D^{(n-2,2)}}(H)$ is simple and projective if and only if H is conjugate to a subgroup of the form $E_2 \times E_2 \times E_2 \times P$, where each E_2 is an elementary abelian subgroup of order 2 acting regularly, and P is a Sylow 2-subgroup of $S_{\{7,8,\ldots,n\}}$. The last result from Chapter 4 is the description of the subgroups H such that $\overline{FP}_{D^{(n-1,1)}}(H)$ is simple and projective when n is congruent to 2 modulo 4. Surprisingly enough, the answer we get is not like the one we had for the same partition $(n-1,1)$ and n odd; it is precisely what we had for the partition $(n-2,2)$ when n is congruent to 3 modulo 4, that is, the only subgroups H for which $\overline{FP}_{D^{(n-1,1)}}(H)$ is simple and projective where $n \equiv 2(\bmod 4)$ are the conjugates of $E_2 \times E_2 \times E_2 \times P$, where each E_2 is an elementary abelian subgroup of order 2 acting regularly, and P is a Sylow 2-subgroup of $S_{\{7,8,\ldots,n\}}$. Notice that since these are non-overlapping cases (n is congruent to different numbers modulo 4) this does not contradict the existence of our proposed bijection. By the way, we also characterized the subgroups H such that the Brauer quotient $\overline{FP}_{S^{(n-1,1)}}(H)$ is simple and projective, where n is congruent to 2 modulo 4 and $S^{(n-1,1)}$ is the Specht module described in Section 2.1. We did this mainly because is was an easy byproduct of the computations needed for $D^{(n-1,1)}$, and we also used it to construct an example in Section 7.2.1.

Before I can tell you what is in Chapters 5 and 6, I must go back to Section 3.5, which I skipped earlier. After obtaining explicit bijections between weights and irreducibles for $n \leq 8$, we tried to find a way to summarize this information. We noticed that in order to describe a weight in characteristic 2, it suffices to give the weight subgroup, that is, if (Q,S) and (Q,T) are two weights for the same symmetric group S_n, then they are really the same weight, regardless of what S and T might appear to be (this does not work in other characteristics; see [3]). This means that in our case we can keep only the weight subgroups and still be able to identify which weight we have.

Next we recorded the times when a subgroup of a specific form appeared as weight subgroup of S_n, and noted which partition parameterized the irreducible bijected with that weight. For example, looking at the tables in Section 3.4, we see that the subgroup E_2 appeared as a weight subgroup of S_2, S_3, S_5 and S_8, and the partitions associated with those occurrences are (2), (3), (4,1) and (5,2,1) respectively. This is how we created the table of partitions in Section 3.5. A great deal of information can be obtained from this table of partitions, and it appears to have the following properties:

1. Each weight subgroup Q appears for the first time inside a symmetric group S_n where n is such that Q has no fixed points on the set $\{1,\ldots,n\}$ After that, Q appears precisely inside those symmetric groups of the form S_{n+t} where t is a triangular number (that is, $t = \sum_{i=0}^{r} i$ for some r).

2. The trivial subgroup indexes a row that consists of all triangular partitions (we included the

triangular partitions of size 0 and 1 for completeness).

3. The first partition of every row has empty 2-core. The second partition has 2-core of size 1, the third has 2-core of size 3 and the fourth has 2-core of size 6. In other words, the 2-core of every partition along the i-th column is the i-th triangular partition (where $\emptyset$ is the first triangular partition).

4. Along every row, each partition is contained in the next one. The difference in size from the first partition and the i-th one on any given row is the i-th triangular number.

Item 1 is proved implicitly in [3]. Item 2 is just stating the well-known fact that in characteristic 2, the only symmetric groups with simple projective modules are the S_t with t a triangular number, and that such modules are parameterized by the corresponding triangular partitions. On the other hand, items 3 and 4 were a complete surprise. The arrangement of the partitions seems to follow a natural pattern, and the interesting thing is that these are now purely combinatorial issues, outside the realm of algebra.

Item 3 provided the motivation to try to prove results that related modules to partitions. One of the conjectures that first come to mind is the following:

Conjecture 1.0.2. *Let Q be a weight subgroup of S_n, where Q has no fixed points on $\{1,\ldots,n\}$, and let λ be a partition of m such that $\overline{FP}_{D^\lambda}(Q)$ is simple and projective. Then the 2-core of λ has size $m-n$.*

We have been able to prove a partial converse so far: "Let n be an even number and Q a Sylow 2-subgroup of S_n. Let t be a triangular number and λ the partition of $n+t$ obtained by adding n nodes to the first row of the triangular partition μ of size t. Then $\overline{FP}_{D^\lambda}(Q)$ has a simple projective summand of multiplicity one, given by D^μ". This result is in Chapter 5.

We also came up with the following conjecture, which is a completely combinatorial question:

Conjecture 1.0.3. *It is possible to arrange all 2-regular partitions in an infinite table satisfying the conditions 3 and 4 mentioned above.*

We have proved that if such a table can be created, there is a class of partitions, which we call 2-stable, such that if λ is 2-stable, then all the partitions to the right of λ (that is, larger than λ and in the same row) are uniquely determined, and we can construct them explicitly from λ (this is in Chapter 6). We think this is a huge step towards providing such an infinite array of partitions, because the rules for determining where a partition can go are relatively tight. By the way, the fact that the table in Section 3.5 was derived from our proposed algorithm, that the partitions exhibit the behaviour we described and that we managed to prove a partial result in that direction are all good indicators that we might be on the right track. Even the partition (7,2), which caused great commotion before, fits inside our table exactly where it is supposed to be!

In Chapter 7 we provide some examples that show that the conditions that we placed on the number n in the theorems of Chapter 4 are necessary, that is, counterexamples can be found otherwise. We also present some modifications to our algorithm, which promising though they seemed, failed to establish the desired bijection (we include them mostly as a reference for those who would like to try something similar). Finally, Section 7.3 has some suggestions for continuing our work; some of them are presented in the form of a conjecture.

It is my hope that after reading this paper Brauer quotients will seem like a natural choice to try to relate weights and simple modules.

Chapter 2

Preliminaries

In this chapter we describe the irreducible modules of the group algebras of the symmetric groups using the characteristic-free approach in [9]. We state Alperin's Conjecture, which still remains one of the most important open problems in the modular representation theory of finite groups, but which has already been proved in the case of symmetric groups (see [3]). We define Brauer quotients, a relatively new tool in representation theory, which has been successfully used by authors such as Puig and Thévenaz [15]. We end this chapter with an algorithm that we put forward as a first approximation to finding a combinatorial proof of Alperin's Conjecture for the symmetric groups.

2.1 Some important kS_n-modules

We define the modules M^λ, S^λ and D^λ following James [9], and prove some results that we shall need later. The simple kS_n-modules, as is well known, can be parameterized by certain partitions of n called p-regular, where p is the characteristic of the field k. Moreover, it is possible to construct each simple module from its associated partition, and although in general this can be a cumbersome process, there are some partitions whose irreducible modules can be readily described using this method. Furthermore, in some cases it is also possible to find a formula that describes a simple module as a virtual difference of permutation modules, which will make the simple modules easier to manipulate. There is yet another advantage arising from the use of partitions to parameterize simple kS_n-modules: partitions are very visual objects, and it is far easier to discover a pattern by looking at a table of partitions like the one at the end of this chapter than by studying their corresponding simple modules.

In this section n is a natural number, k is a field of characteristic $p > 0$ (unless otherwise stated) and λ is a partition of n.

Definition 2.1.1. A λ-*tableau* is one of the $n!$ arrays of integers obtained by replacing each node in the partition λ by one of the integers $1, 2, \ldots, n$, allowing no repeats. If t is a tableau, its *row stabilizer*, R_t, is the subgroup of S_n consisting of the elements which fix all rows of t setwise. The *column stabilizer* of t, denoted C_t, is the subgroup of S_n consisting of the elements which fix all columns of t setwise. The *signed column sum* of t, denoted κ_t, is the element of kS_n given by

$$\kappa_t := \sum_{\pi \in C_t} (-1)^{\mathrm{sign}(\pi)} \pi.$$

We define an equivalence relation on the set of λ-tableaux by $t_1 \sim t_2$ if and only if $\pi t_1 = t_2$ for some $\pi \in R_{t_1}$. The *tabloid*, $\{t\}$ containing t is the equivalence class of t under this relation. The kS_n-module $M^\lambda = M_k^\lambda$ is the vector space over k whose basis elements are the various λ-tabloids. The *polytabloid*, e_t, associated with the tableau t is given by

$$e_t := \kappa_t \{t\}.$$

The *Specht module*, $S^\lambda = S_k^\lambda$ for the partition λ is the submodule of M^λ spanned by polytabloids (this is indeed a kS_n-module).

We also define an S_n-invariant, symmetric, non-singular *bilinear form* $<,>$ on M^λ, whose values on pairs of tabloids is given by

$$< t_1, t_2 > := \begin{cases} 1 & \text{if } t_1 = t_2, \\ 0 & \text{if } t_1 \neq t_2. \end{cases}$$

The partition λ is *p-singular* if it has at least p rows of the same size; otherwise, λ is *p-regular*. The module $D^\lambda = D_k^\lambda$ is defined as

$$D^\lambda := S^\lambda / (S^\lambda \cap S^{\lambda \perp})$$

where λ is a p-regular partition.

Theorem 2.1.2. *(James) As λ varies over p-regular partitions of n, D^λ varies over a complete set of inequivalent irreducible kS_n-modules. Each D^λ is self-dual and absolutely irreducible. Every field is a splitting field for S_n.*

For a proof of this result, see [9].

Remark 2.1.3. For the remainder of this chapter, k will be the field of two elements.

Example 2.1.4. Let $\lambda = (n)$ be the partition with just one row of length n. Then all λ-tableaux are row equivalent, so there is only one λ-tabloid, and $M^{(n)} = k$ is the one dimensional trivial module. We also have that $S^{(n)} = D^{(n)} = k$.

Of all the other M^λ modules, the easiest to deal with are the ones given by two-part partitions. The following well-known result provides a way of visualizing these modules.

Lemma 2.1.5. *The module $M^{(n-i,i)}$ is isomorphic to the kS_n-permutation module of all subsets of $\{1, 2, \ldots, n\}$ of size i.*

Proof. The isomorphism is given by sending an $(n-i, i)$-tabloid to the set of numbers in its second row. ∎

In the very special case when $\lambda = (n-1, 1)$ (and only for the field of two elements) we additionally have

Lemma 2.1.6. *The module $M^{(n-1,1)}$ is isomorphic to the module consisting of all subsets of $\{1, \ldots, n\}$, where addition is given by the symmetric difference of sets, and where S_n acts by permuting the elements of each set. The isomorphism takes $S^{(n-1,1)}$ to the family of subsets of $\{1, \ldots, n\}$ of even cardinality. If n is odd, then $S^{(n-1,1)} = D^{(n-1,1)}$. If n is even, then $D^{(n-1,1)}$ is isomorphic to the quotient of the family of subsets of even cardinality modulo the equivalence relation that pairs a set with its complement.*

Proof. Notice that the family of all subsets of $\{1, \ldots, n\}$ is a vector space over the field of two elements, and that the singletons are a basis permuted by S_n. The isomorphism between $M^{(n-1,1)}$ and this module is given (as before) by sending the basis of $(n-1, 1)$-tabloids to their respective second rows. Note that the elements of $S^{(n-1,1)}$ are precisely the linear combinations of an even number of tabloids, which correspond to the subsets of even cardinality. Furthermore, the bilinear form on the subsets A and B now is given by the cardinality of $A \cap B$ modulo 2. The only non-empty subset that can be orthogonal to all subsets of even cardinality is $\{1, \ldots, n\}$, which is in $S^{(n-1,1)}$ if and only if n is an even number. Thus $D^{(n-1,1)} = S^{(n-1,1)}$ if n is odd, and is as described in the statement of the Lemma when n is even. $\square$

We now record a number of well-known results which can be found either explicitly or implicitly in [9].

Corollary 2.1.7. *If n is odd then*

$$M^{(n-1,1)} = D^{(n-1,1)} \oplus k$$

Proof. We already know that $D^{(n-1,1)} = S^{(n-1,1)}$ is a submodule of codimension 1. The complement is generated by the vector $\{1, \ldots, n\}$, where we have identified $M^{(n-1,1)}$ with the power set of $\{1, \ldots, n\}$. $\square$

Now we proceed to derive a similar formula for $M^{(n-2,2)}$.

Lemma 2.1.8. *The image of the map*

$$\varphi : M^{(n-2,2)} \longrightarrow M^{(n-1,1)}, \quad \{\alpha, \beta\} \mapsto \{\alpha\} + \{\beta\}$$

is $S^{(n-1,1)}$. The kernel of φ contains $S^{(n-2,2)}$ as a submodule of codimension one.

Proof. Now we switch to the notation suggested by Lemma 2.1.5, not by Lemma 2.1.6. We can see that φ is a well-defined morphism of kS_n-modules. The image of φ is generated by all elements of the form $\{\alpha\} + \{\beta\}$ (with $\alpha \neq \beta$), which also generate $S^{(n-1,1)}$ (they are the polytabloids). There are module generators (polytabloids) of $S^{(n-2,2)}$ of the form $\{2, \beta\} + \{1, \beta\} + \{2, 3\} + \{1, 3\}$ or of the form $\{\alpha, \beta\} + \{1, \beta\} + \{\alpha, 2\} + \{1, 2\}$; both of them will go to zero under φ. Finally, notice that the kernel of φ has dimension $\frac{n(n-1)}{2} - (n-1) = \frac{n(n-3)}{2} + 1$, which is one more than the dimension of $S^{(n-2,2)}$ (see the Hook Formula, result 20.1 in [9] for the dimension of the Specht modules). $\square$

For the rest of this section n is congruent to 3 modulo 4.

Lemma 2.1.9. *When $n \equiv 3(\bmod 4)$, we have that $S^{(n-2,2)} = D^{(n-2,2)}$.*

Proof. This follows from Theorem 23.13 in [9]. $\square$

Now we can quickly get the composition factors of $M^{(n-2,2)}$.

Corollary 2.1.10. *When $n \equiv 3(\bmod 4)$ we have that the composition factors of the module $M^{(n-2,2)}$ are k, $D^{(n-1,1)}$ and $D^{(n-2,2)}$.*

Proof. The image of φ is $S^{(n-1,1)} \cong D^{(n-1,1)}$ (because n is odd) and the kernel has k and $S^{(n-2,2)} \cong D^{(n-2,2)}$ (because n is congruent to 3 modulo 4) as composition factors. $\square$

Our next result is a general statement about modules for arbitrary rings

Proposition 2.1.11. *Let R be a ring, M an R-module, S a simple R-module which is a composition factor of M of multiplicity 1. Then S is a direct summand of M if and only if there exist non-zero maps $\theta : S \longrightarrow M$ and $\phi : M \longrightarrow S$. Moreover, in the latter case, such maps must necessarily split.*

Proof. Necessity: Take the usual inclusion and projection. Sufficiency: We must show that both θ and ϕ are split. Notice that $\phi\theta \neq 0$ (otherwise the series $0 < Im(\theta) \leq Ker(\phi) < M$ would give two composition factors isomorphic to S), so $\phi\theta$ must be an automorphism of S. This proves that ϕ is a split epimorphism and θ is a split monomorphism. $\square$

We can combine the previous results to get

Proposition 2.1.12. *When n is congruent to 3 modulo 4 we have*

$$M^{(n-2,2)} \cong D^{(n-2,2)} \oplus k \oplus D^{(n-1,1)}.$$

Proof. We know that $M^{(n-2,2)}$ has three different composition factors, so we can apply the previous proposition. The map φ gives rise to a morphism from $M^{(n-2,2)}$ onto $D^{(n-1,1)}$, and both modules are self-dual, so there is a non-zero map from $D^{(n-1,1)}$ to $M^{(n-2,2)}$. By the proposition φ splits, so $M^{(n-2,2)}$ is the direct sum of $D^{(n-1,1)}$ and the kernel of φ. Similarly, $D^{(n-2,2)}$ is a submodule of the kernel of φ, so $D^{(n-2,2)}$ is embedded in $M^{(n-2,2)}$ and both modules are self-dual, so there exists an epimorphism from $M^{(n-2,2)}$ onto $D^{(n-2,2)}$. By the proposition the inclusion of $D^{(n-2,2)}$ into $M^{(n-2,2)}$ is split, so in particular the inclusion of $D^{(n-2,2)}$ into the kernel of φ is split, with k as a direct complement. $\square$

Now this decomposition theorem follows easily.

Corollary 2.1.13. *There is a decomposition*

$$M^{(n-2,2)} \cong D^{(n-2,2)} \oplus M^{(n-1,1)}.$$

Proof. It follows from the previous proposition and the fact that $M^{(n-1,1)} \cong D^{(n-1,1)} \oplus k$ because n is odd. $\square$

2.2 Alperin's Conjecture

We give the definition of weight and formulate Alperin's Conjecture in its most general form. We mention some classes of groups for which it is known to be valid (including the symmetric groups) and we note the possible advantages of a combinatorial proof, that is, an explicit bijection between weights and irreducible modules.

Throughout this section, G will be a finite group, p a prime number, and k a splitting field for G in characteristic p. All our modules will be finite dimensional over k.

Definition 2.2.1. A *weight* for G is a pair (Q, S) where Q is a p-subgroup and S is a simple module for $k[N_G(Q)]$ which is projective when regarded as a module for $k[N_G(Q)/Q]$.

Remark 2.2.2. Since S is $k[N(Q)]$-simple and Q is a p-subgroup of $N_G(Q)$, it follows that Q acts trivially on S, so S is also a $k[N_G(Q)/Q]$-module and the definition makes sense. Moreover, S is $k[N_G(Q)/Q]$-simple as well.

Remark 2.2.3. If we replace S by an isomorphic $k[N_G(Q)]$-module we consider this the same weight, and we make the same identification when we replace Q by a conjugate subgroup (so that the normalizers will be conjugate, too).

Now we can formulate the main problem that we shall discuss in this section.

Theorem 2.2.4. *(*Alperin's Conjecture*) The number of weights for G equals the number of simple kG-modules.*

A stronger version of the preceding statement is that there is a bijection within each block of the group algebra.

Definition 2.2.5. If (Q, S) is a weight for G, then S belongs to a block b of $N_G(Q)$ and this block corresponds with a block B of G via the Brauer correspondence; hence we can say that the weight (Q, S) *belongs* to the block B of G so the weights are partitioned into blocks.

Theorem 2.2.6. *(*Alperin's Conjecture, Block Form*) The number of weights in a block of G equals the number of simple modules in the block.*

This version of the conjecture implies the original one, as it can be obtained by summing the equalities from the stronger conjecture over the blocks. This stronger conjecture has been proved when G is a:

- Finite group of Lie type and characteristic p (Cabanes, [4]).
- Solvable group (Okuyama, [13]).
- Symmetric group (Alperin and Fong, [3]).
- $GL(n, q)$ and p does not divide q (Alperin and Fong, [3]).

Alperin and Fong's proof in the case of symmetric groups was just an observation of a numerical equality which did not suggest a deeper reason for the relationship. For finite groups in general one does not expect to have any canonical bijection between weights and simple modules; as a matter of fact, Alperin himself says this is unlikely (see [2], p 369). For groups of Lie type in their defining characteristic there is a canonical bijection (described in [2]). Since symmetric groups and groups of Lie type have such strong connections in their representation theory, it is reasonable to ask whether there is some canonical bijection in the case of symmetric groups.

If true, Alperin's conjecture would imply a number of known results, until now unrelated [2]. It is also reasonable to expect that if an explicit bijection can be given to prove it, this may reveal new connections between simple kG-modules and weights; there are many results known about the former, and the latter are related to the blocks of defect zero, which are not as easy to deal with as the simple modules. In fact, this is really the true importance of Alperin's conjecture in that it provides a connection between the blocks of defect zero and the set of all simple modules. More specifically, Alperin's conjecture has been shown by Robinson [12] to be equivalent to a statement which expresses the number of blocks of defect zero of a group in terms of the number of p-modular irreducibles of sections of the group of the form $N_G(P)/P$, $P \leq G$ a p-subgroup. These latter numbers are easy to compute, since by a theorem of Brauer the number of p-modular irreducibles of a group equals the number of p-regular conjugacy classes.

2.3 Brauer quotients

We define Brauer quotients and prove some of their properties, which we shall later use in our calculations. In this section k is an arbitrary field, F_q is the field with q elements, G an arbitrary finite group, H a subgroup of G, and V a kG-module. We denote by V^G the fixed points of V under G.

Definition 2.3.1. The map $tr_H^G : V^H \longrightarrow V^G$ given by

$$m \mapsto \left(\sum_{i=1}^{l} g_i \right) m,$$

where $G = \sqcup_{i=1}^{l} g_i H$, is called the *relative trace* from H to G. The *Brauer quotient* of V with respect to H is defined as

$$\overline{FP}_V(H) := V^H / \sum_{K<H} tr_K^H(V^K).$$

This is a $k[N_G(H)]$-module, where H acts trivially, so it is a $k[N_G(H)/H]$-module. The preceding definition is a particular example of the Brauer quotient of a Mackey functor: in our case we are using the fixed points Mackey functor. Constructions such as this appear in recent work by various authors such as Puig and Thévenaz, see [14].

Example 2.3.2. Let $H = \{1\}$. Then H has no proper subgroups, so we have that $\overline{FP}_V(\{1\}) = V^H = V$ for all kG-modules V.

Example 2.3.3. Let k be a field of characteristic p, G any finite group, H a subgroup of G, and $V = k$ the trivial kG-module. If H is a p-subgroup of G, then as is well known all the relative traces from proper subgroups of H to H are identically zero, and $\overline{FP}_k(H) = k$. On the other hand, if H is not a p-subgroup, the relative trace from a Sylow p-subgroup of H to H is not zero, and $\overline{FP}_k(H) = 0$.

For convenience we state some well-known properties of Brauer quotients that will allow us to simplify our computations. The proof of the first one follows from the fact that both fixed points and relative traces preserve direct sums. The second proposition is result (27.6) in [15].

Proposition 2.3.4. *Brauer quotients preserve direct sums of kG-modules, that is,*

$$\overline{FP}_{V_1 \oplus V_2}(H) \cong \overline{FP}_{V_1}(H) \oplus \overline{FP}_{V_2}(H)$$

Proposition 2.3.5. *Let V be a kG-module that is also an H permutation module, i.e. there exists a basis X of V over k that is also an H-set, and let Y denote the fixed points of H on the set X. Then the normalizer of H in G acts on the set Y, and there is an isomorphism of $k[N_G(H)]$-modules*

$$\overline{FP}_V(H) \cong kY$$

where kY denotes the permutation module on the set Y.

We can combine the preceding results to obtain effortlessly the Brauer quotients of many modules.

Example 2.3.6. If $H \neq \{1\}$, then $\overline{FP}_{kG}(H) = 0$ since H does not fix any elements of G, so $\overline{FP}_{(kG)^n}(H) = 0$ and $\overline{FP}_V(H) = 0$ for any projective kG-module V. We shall later study in greater detail other connections between Brauer quotients and projectivity.

2.4 Proposed Algorithm

Alperin's Conjecture has been proved for all symmetric groups, so we know that the number of weights for S_n equals the number of simple kS_n-modules. However, no explicit bijection between those two sets is known yet. We are about to present an algorithm that establishes an explicit bijection between weights and irreducibles for F_2S_n when $n \leq 8$. The idea is to use Brauer quotients to determine when a weight and an irreducible F_2S_n-module are "compatible". The procedure was suggested by Webb.

Consider a simple F_2S_n-module V, and for each weight (Q, S) of F_2S_n compute the Brauer quotient of V with respect to the weight subgroup Q. More often than not the Brauer quotient will turn out to be 0; however, there are times when a weight (Q, S) has the property that the Brauer quotient of V with respect to Q is precisely the $N_{S_n}(Q)/Q$-module S. Furthermore, for all values of $n \leq 8$, given any irreducible module V there is exactly one weight (Q, S) such that the Brauer quotient of V with respect to the subgroup Q is isomorphic to the $N_{S_n}(Q)/Q$-module S, and given any weight (Q, S), there is a unique irreducible module V such that $\overline{FP}_V(Q) \cong S$. This assignment defines an explicit bijection between irreducibles and weights for F_2S_n, $n \leq 8$.

The previous fact was proved using mostly the computer software that we wrote (described in Chapter 3). The empirical evidence that we gathered provided the motivation to determine other instances when the Brauer quotient of a simple F_2S_n-module is simple and projective (Chapters 4 and 5). The careful scrutiny of the table of partitions in Section 3.5 led us to the discovery of 2-stable partitions and their fundamental property (described in Chapter 6). Much to our chagrin, our algorithm failed to provide a match between an irreducible and a weight for F_2S_9, as shown in Section 3.4. However, even in this case there is an obvious bijection described between weights and irreducibles, and we think that after some modifications, a similar procedure may be used to provide a combinatorial proof of Alperin's Conjecture. We suggest ways in which this might be achieved in the Epilogue, along with other conjectures that have arisen from our work.

Chapter 3

Computer Assisted Methods

We created computer software in GAP [6] to prove that the algorithm provides an explicit bijection between weights and irreducibles for S_n with $n \leq 8$, and in a modified form with $n = 9$. For $n \geq 10$ some of the representations are quite large and take a lot of computer time to handle. The software that we created was the following:

- A library of weight subgroups of S_n for the prime 2.
- A library of irreducible modules of S_n (using SYMMETRICA 1.0 [11]).
- The subroutines which perform the operations on modules that are required to establish the desired correspondence.

We now explain what each of the preceding items consists of.

3.1 Library of weight subgroups of S_n for the prime 2.

Alperin gives a description of the weight subgroups of S_n in [3]. We start with some relevant definitions.

Definition 3.1.1. Let G be a finite group, Q a p-subgroup of G. We say that Q is a *p-radical* subgroup of G if $Q = O_p(N_G(Q))$, that is, if Q is the largest normal p-subgroup of its normalizer in G.

Definition 3.1.2. Let k be a field of characteristic p, G a finite group, Q a p-subgroup of G. We call Q a *weight subgroup* of G if there is a simple $k[N_G(Q)]$-module S which is $k[N_G(Q)/Q]$-projective, that is, if (Q, S) is a weight for G.

The last two concepts are related. Our next result gives a necessary (though not sufficient) condition for a subgroup to be a weight subgroup.

Proposition 3.1.3. *[3] If Q is a weight p-subgroup of G, then Q is p-radical.*

A further criterion that can be used when $G = S_n$ is the following.

Theorem 3.1.4. *[3] If Q is a weight subgroup of S_n for the prime p, then Q is a direct product of wreath products of elementary abelian groups acting regularly.*

We give counterexamples to the converses of Proposition 3.1.3 and Theorem 3.1.4 in Example 3.1.5. However, these results are very useful because they help us narrow our search for weight subgroups of S_n.

From now on, the only prime number that we shall be interested in is $p = 2$, and E_{2^n} will denote the elementary abelian group of order 2^n. For each value of $n \leq 8$, we created a list of subgroups of S_n which were direct products of wreath products E_{2^m}'s acting regularly, and it is easy to verify – either theoretically or using GAP – which ones are 2-radical. The following example might help to illustrate this technique.

The weight subgroups of S_7: By Theorem 3.1.4, we know that up to conjugacy, the first candidates for weight subgroups of S_7 for the prime 2 are the following direct products of wreath products acting regularly:

2-subgroups	Generators
(i) $\{1\}$	()
(ii) E_2	(1,2)
(iii) E_4	(1,2)(3,4), (1,3)(2,4)
(iv) $E_2 \wr E_2$	(1,2), (1,3)(2,4)
(v) $E_2 \times E_2$	(1,2), (3,4)
(vi) $E_2 \times E_2 \times E_2$	(1,2), (3,4), (5,6)
(vii) $E_2 \times E_4$	(1,2), (3,4)(5,6), (3,5)(4,6)
(viii) $E_2 \times (E_2 \wr E_2)$	(1,2), (3,4), (3,5)(4,6)

Note that these can all be embedded in S_6, because $[S_7 : S_6] = 7$ is not divisible by 2. However, the $O_p(N)$ will change with the embedding.

Notice that although (iii) and (v) are isomorphic as groups, they are treated as different candidates. The reason for this is that their embeddings in S_7 using the regular action are essentially different: E_4 acts regularly as the group generated by the permutations $(1,2)(3,4)$ and $(1,3)(2,4)$, whereas for E_2 we need the permutation $(1,2)$, so $E_2 \times E_2$ is generated by the permutations $(1,2)$ and $(3,4)$. These subgroups are not conjugate in S_7 (their cyclic structures are different).

Recall that if A and B are groups acting on a and b points respectively, then $A \times B$ acts on $a+b$ points, and $A \wr B$ acts on ab points. From this and the fact that the E_{2^m} have to act regularly, we conclude that the list we proposed is complete.

Using GAP we can verify that the 2-radical subgroups are (i), (ii), (iii), (iv), (vi), (vii) and (viii). Notice that (iii) is 2-radical, but (v) is not!

Finally, for each of these subgroups H we explicitly compute the quotient $N_{S_7}(H)/H$, to see which ones admit a simple projective module. In this way we eliminate two more subgroups, which we present in the next example.

Example 3.1.5. The subgroups $\{1\}$ and E_2 of S_7 are 2-radical but are not weight 2-subgroups. Their respective quotients $N_{S_7}(H)/H$ are S_7 and S_5 (neither 7 nor 5 is a triangular number).

3.2 Library of irreducible modules of S_n.

We used SYMMETRICA 1.0 [11] to obtain matrices representing the action of the permutations $(1,2)$ and $(1,2,\ldots,n)$ on the irreducible modules of S_n. We then converted the output files from SYMMETRICA into a form which can be read by GAP.

We defined a data structure in GAP that corresponds to the notion of representation. It consists of a record whose fields include: the group represented, the dimension of the representation, a list of generators of the group and a list of matrices that give the action of the generators on the underlying vector space.

An example of an irreducible representation for S_7 is the following:

```
gap> M:=ModularIrreducible([1,2],2);
Representation( Group( (1,2), (1,2,3) ), Images
[ [  1,  1 ],
  [  0,  1 ] ]
[ [  1,  1 ],
  [  1,  0 ] ]
 )
gap> RecFields(M);
[ "group", "genimages", "field", "dimension",
"isRepresentation", "operations" ]
```

The last line shows the fields of the record M, that is, the different objects associated to the representation M as it is stored in the computer.

3.3 Library of subroutines.

We have available within GAP a library of subroutines which we use to handle modular representations. As well as standard operations such as fixed points, dual, tensor product, submodule and quotient module, we also have implemented routines which compute with the relative trace, culminating in the computation of the Brauer quotient and a test for projectivity. The Appendix has a list of the most useful routines, with some brief descriptions of what they do.

3.4 Tables of dimensions of Brauer quotients

These are the tables with the dimensions of the Brauer quotients of the irreducible F_2S_n-modules with respect to the weight subgroups of S_n. The rows of the tables are labelled by weight subgroups,

Table 3.1: Dimensions of Brauer quotients for S_2

	[2]
E_2	$\boxed{1}$

Table 3.2: Dimensions of Brauer quotients for S_3

	[3]	[2,1]
E_2	$\boxed{1}$	0
$\{1\}$	1	$\boxed{2}$

and the columns by irreducible F_2S_n-modules (which are denoted by the 2-regular partitions which parameterize them). The bijection we proposed is indicated by putting a box around the place where there is a weight. We should like to remark that the boxed entries represent the only nontrivial projective modules which appear in these tables, and all of the boxed entries are blocks of defect zero. Most of the entries were obtained using our computer software.

3.5 Table of partitions

After defining explicit bijections between weights and irreducibles, the next natural question to ask is whether there is a pattern hidden in this construction. The following table of partitions is an attempt at unraveling this pattern. Notice that since the characteristic is 2, each weight (Q, S) for S_n is determined by its weight subgroup Q (using the fact proved in [3] that the quotient $N_{S_n}(Q)/Q$ is a direct product of $GL(m_i, 2)$, and that each $GL(m, p)$ has exactly $p-1$ simple projective modules). Each weight subgroup of S_n is used to index one row of the table; the entries along that row are the partitions that parameterize the irreducibles assigned to the weight using the proposed algorithm.

Table 3.3: Dimensions of Brauer quotients for S_4

	[4]	[3,1]
$E_2 \wr E_2$	$\boxed{1}$	0
E_4	1	$\boxed{2}$

Table 3.4: Dimensions of Brauer quotients for S_5

	[5]	[4,1]	[3,2]
$E_2 \wr E_2$	$\boxed{1}$	0	0
E_2	1	$\boxed{2}$	0
E_4	1	0	$\boxed{2}$

Table 3.5: Dimensions of Brauer quotients for S_6

	[6]	[5,1]	[4,2]	[3,2,1]
$E_2 \times (E_2 \wr E_2)$	$\boxed{1}$	0	0	0
$E_2 \times E_2 \times E_2$	1	$\boxed{2}$	0	0
$E_2 \times E_4$	1	0	$\boxed{2}$	0
$\{1\}$	1	6	6	$\boxed{16}$

Table 3.6: Dimensions of Brauer quotients for S_7

	[7]	[6,1]	[5,2]	[4,2,1]	[4,3]
$E_2 \times (E_2 \wr E_2)$	$\boxed{1}$	0	0	0	0
$E_2 \wr E_2$	1	$\boxed{2}$	0	0	0
$E_2 \times E_2 \times E_2$	1	0	$\boxed{2}$	0	0
$E_2 \times E_4$	1	0	0	$\boxed{2}$	0
E_4	1	2	0	2	$\boxed{4}$

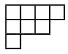
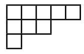
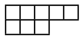
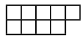
Table 3.7: Dimensions of Brauer quotients for S_8

	[8]	[7,1]	[6,2]	[5,3]	[5,2,1]	[4,3,1]
$(E_2 \wr E_2) \wr E_2$	$\boxed{1}$	0	0	0	0	0
$E_2 \wr E_4$	1	$\boxed{2}$	0	0	0	0
$(E_2 \wr E_2) \times E_4$	1	0	$\boxed{2}$	0	0	0
$E_4 \wr E_2$	1	0	0	$\boxed{2}$	0	0
E_2	1	4	6	0	$\boxed{16}$	0
E_8	1	3	0	3	0	$\boxed{8}$

Table 3.8: Dimensions of Brauer quotients for S_9

	[9]	[8,1]	[7,2]	[5,4]	[5,3,1]	[4,3,2]	[6,2,1]	[6,3]
$(E_2 \wr E_2) \wr E_2$	$\boxed{1}$	0	0	0	0	0	0	0
$(E_2 \wr E_2) \times E_2$	1	$\boxed{2}$	0	0	0	0	0	0
$E_2 \wr E_4$	1	0	0	0	0	0	0	0
$E_4 \wr E_2$	1	0	0	$\boxed{2}$	0	0	0	0
E_8	1	0	0	4	$\boxed{8}$	0	0	0
$E_2 \times E_4$	1	2	0	0	0	$\boxed{4}$	0	0
$(E_2 \wr E_2) \times E_4$	1	0	3	0	0	0	$\boxed{2}$	0
$E_2 \times E_2 \times E_2$	1	2	2	0	0	0	2	$\boxed{4}$

Table 3.9: Two-regular Partitions

E_8:		
$E_4 \wr E_2$:		

Notice the following facts about this table of partitions:

1. Each weight subgroup Q appears for the first time inside a symmetric group S_n where n is such that Q has no fixed points on the set $\{1, \ldots, n\}$ After that, Q appears precisely inside those symmetric groups of the form S_{n+t} where t is a triangular number (that is, $t = \sum_{i=0}^{r} i$ for some r).

2. The trivial subgroup indexes a row that consists of all triangular partitions (we included the triangular partitions of size 0 and 1 for completeness).

3. The first partition of every row has empty 2-core. The second partition has 2-core of size 1, the third has 2-core of size 3 and the fourth has 2-core of size 6. In other words, the 2-core of every partition along the i-th column is the i-th triangular partition (where $\emptyset$ is the first triangular partition).

4. Along every row, each partition is contained in the next one. The difference in size from the first partition and the i-th one on any given row is the i-th triangular number.

Item 1 is proved implicitly in [3]. Item 2 is just stating the well-known fact that in characteristic 2, the only symmetric groups with simple projective modules are the S_t with t a triangular number, and that such modules are parameterized by the corresponding triangular partitions. On the other hand, items 3 and 4 were a complete surprise. The arrangement of the partitions seems to follow a natural pattern, and the interesting thing is that these are now purely combinatorial issues, outside the realm of algebra.

Item 3 provided the motivation to try to prove results that related modules to partitions. One of the conjectures that first come to mind is the following:

Conjecture 3.5.1. *Let Q be a weight subgroup of S_n, where Q has no fixed points on $\{1, \ldots, n\}$, and let λ be a partition of m such that $\overline{FP}_{D^\lambda}(Q)$ is simple and projective. Then the 2-core of λ has size $m - n$.*

We have been able to prove a partial converse so far: "Let n be an even number and Q a Sylow 2-subgroup of S_n. Let t be a triangular number and λ the partition of $n + t$ obtained by adding n nodes to the first row of the triangular partition μ of size t. Then $\overline{FP}_{D^\lambda}(Q)$ has a simple projective summand of multiplicity one, given by D^μ". This result is in Chapter 5.

We also came up with the following conjecture, which is a completely combinatorial question:

Conjecture 3.5.2. *It is possible to arrange all 2-regular partitions in an infinite table satisfying the conditions 3 and 4 mentioned above.*

Remark 3.5.3. It is easy to construct a table satisfying almost all of these requirements except that we cannot guarantee that each partition be contained in the next one along its row.

We have proved that if such a table as the one described in Conjecture 3.5.2 can be created, there is a class of partitions, which we call 2-stable, such that if λ is 2-stable, then all the partitions to the right of λ (that is, larger than λ and in the same row) are uniquely determined, and we can construct them explicitly from λ (this is in Chapter 6). We think this is a huge step towards providing such an infinite array of partitions, because the rules for determining where a partition can go are relatively tight. By the way, the fact that the table in Section 3.5 was derived from our proposed algorithm, that the partitions exhibit the behaviour we described and that we managed to prove a partial result in that direction are all good indicators that we might be on the right track. Even the partition (7,2), which caused great commotion before, fits inside our table exactly where it is supposed to be!

Chapter 4

Brauer quotients

In this chapter we answer the question "when is the Brauer quotient of a module V with respect to a subgroup H of S_n both simple and projective as an $N_{S_n}(H)/H$-module?", in the cases when $V = D^{(n-1,1)}$ with n odd, $V = D^{(n-2,2)}$ with $n \equiv 3(\bmod 4)$, $V = S^{(n-1,1)}$ with $n \equiv 2(\bmod 4)$ and V (or rather, W) $= D^{(n-1,1)}$ with $n \equiv 2(\bmod 4)$ and the field is F_2. The conditions imposed on the number n are necessary so that we can use the results from Section 2.1; the Epilogue has some counterexamples for other values of n.

4.1 Preliminaries

We remind the reader of some useful facts about the representations of Cartesian products of groups.

Proposition 4.1.1. *Let k be a splitting field for the finite groups R and S, let U, T be finite dimensional modules for kR and kS respectively, and let $k(R \times S)$ act on $U \otimes_k T$ via $(r,s)(u \otimes t) = ru \otimes st$. Then*
(i) $U \otimes_k T$ is a simple $k[R \times S]$-module if and only if U is a simple kR-module and T is a simple kS-module.
(ii) $U \otimes_k T$ is a projective $k[R \times S]$-module if and only if U is a projective kR-module and T is a projective kS-module.

Proof. (i) A proof of this result can be found in [5], Theorem (10.33).
(ii) It is clear that the tensor product of two projective modules is projective. Assume that $U \otimes T$ is a projective module for the group $R \times S$. Then its restriction to the subgroup $R = R \times \{1\}$ is projective, and this is isomorphic to several copies of the R-module U (as many copies as the dimension of T over k), which proves that U must be projective. A similar argument proves that T is a projective kS-module. $\square$

This situation arises naturally when a subgroup H of S_n has fixed points on the set $\{1, \ldots, n\}$.

Lemma 4.1.2. *Let H be a subgroup of the symmetric group S_M with fixed points F and let Θ be* *the complement of F in M. Then*

$$N_{S_M}(H)/H = (N_{S_\Theta}(H)/H) \times S_F$$

Proof. One containment is immediate. Now let $\tau \in N_{S_M}(H)$. Then τ permutes the fixed points F of H, so $\tau = \alpha\beta$ with $\alpha \in S_\Theta$ and $\beta \in S_F$. It follows that α is in $N_{S_\Theta}(H)$. □

As a result, a module for the quotient group $N_{S_X}(H)/H$ is really a module for the product $(N_{S_\Theta}(H)/H) \times S_F$. In this context, if we refer to a kS_F-module U as a $k[N/H]$-module, we mean $k \otimes_k U$, that is, $N_{S_\Theta}(H)/H$ acts trivially on U. Notice that the $k[N/H]$-module $k \otimes_k U$ will be simple and projective if and only if both k and U are simple and projective modules for $N_{S_\Theta}(H)/H$ and S_F respectively, that is, if and only if H is a Sylow 2-subgroup of S_Θ and U is a simple and projective kS_F-module.

4.2 Brauer quotient of $D^{(n-1,1)}$, n odd

We prove that the Brauer quotient of $D^{(n-1,1)}$ is either 0 or simple, and describe the subgroups H of S_n with respect to which the Brauer quotient is also projective as a module for $N_{S_n}(H)/H$. Note that the number of fixed points of any 2-subgroup H on the set $\{1, \dots, n\}$ must be odd because n is odd.

Theorem 4.2.1. *(*Webb, private communication*) Let k be the field of two elements, n an odd number greater than 1, H a 2-subgroup of S_n whose fixed points on the set $\{1, \dots, n\}$ are precisely $\{1, \dots, r\}$, $N = N_{S_n}(H)$ and V the simple module $D^{(n-1,1)}$. If $r = 1$ then $\overline{FP}_V(H) = 0$. If $r > 1$ then the natural quotient $N/H \longrightarrow S_r$ splits, and the action of N/H on $\overline{FP}_V(H)$ is isomorphic to that of S_r on $D^{(r-1,1)}$.*

Proof. We have that $M^{(n-1,1)} = D^{(n-1,1)} \oplus k$, so

$$\overline{FP}_{M^{(n-1,1)}}(H) \cong \overline{FP}_V(H) \oplus \overline{FP}_k(H)$$

But $M^{(n-1,1)}$ is an H-permutation module with permutation basis the set $\{1, \dots, n\}$ (as in Lemma 2.1.5), so its Brauer quotient with respect to H is the permutation module on the set $\{1, \dots, r\}$ (where N/H acts as S_r) by Proposition 2.3.5. If $r = 1$ then cancellation gives $\overline{FP}_V(H) = 0$. If $r > 1$ then the previous permutation module is isomorphic to $M^{(r-1,1)}$. This gives the formula

$$M^{(r-1,1)} \cong \overline{FP}_V(H) \oplus k$$

Since n is odd and H is a 2-subgroup of S_n it follows that r is odd, so by Corollary 2.1.7 we get $M^{(r-1,1)} = D^{(r-1,1)} \oplus k \cong \overline{FP}_V(H) \oplus k$ and cancellation gives $\overline{FP}_V(H) \cong D^{(r-1,1)}$. □

The following well-known result can be proved for all n using the Nakayama Conjecture, or by looking at powers of 2 dividing the dimension of $D^{(n-1,1)}$ and $n!$, but it is also a consequence of our previous theorem when n is odd.

Corollary 4.2.2. *Let k be the field of two elements and let n be odd. The module $D^{(n-1,1)}$ is a projective kS_n-module if and only if $n = 3$.*

Proof. We know that $D^{(2,1)}$ is the two dimensional simple projective module of kS_3 (see [1]). If $n \geq 5$, then the 2-subgroup H generated by the transposition (1,2) has at least 3 fixed points, so by the theorem we have $\overline{FP}_{D^{(n-1,1)}}(H) \neq 0$, so $D^{(n-1,1)}$ cannot be projective (see Example 2.3.6). □

Since any 2-subgroup of S_n is conjugate to an H as described in Theorem 4.2.1, the condition that the fixed points be exactly $\{1, \ldots, r\}$ is not necessary.

Corollary 4.2.3. *With the notation from Theorem 4.2.1, $\overline{FP}_{D^{(n-1,1)}}(H)$ is either 0 or a simple N/H-module. It is projective if and only if H is conjugate to a Sylow 2-subgroup of S_{n-3}.*

Proof. The Brauer quotient is simple because $D^{(r-1,1)}$ is a simple S_r-module, and it is projective if and only if the kernel of the map $N/H \longrightarrow S_r$ has no elements of order 2 and $D^{(r-1,1)}$ is a projective S_r-module (since $N/H \cong \big(N_{S_{n-r}}(H)/H\big) \times S_r$). We know that $N_{S_{n-r}}(H)/H$ has even order if and only if H is not a Sylow 2-subgroup of S_{n-r} by a theorem about normalizers in p-groups, and that $D^{(r-1,1)}$ is projective precisely when $r = 3$. □

4.3 Brauer quotient of $D^{(n-2,2)}$, $n \equiv 3 \pmod 4$

We describe all the subgroups H of S_n such that the Brauer quotient of $D^{(n-2,2)}$ is both simple and projective as a module for $N_{S_n}(H)/H$.

In this section n is congruent to 3 modulo 4 and $n \geq 7$, k is the field of two elements, V is the simple module $D^{(n-2,2)}$, H is a 2-subgroup of S_n whose fixed points on $\{1, \ldots, n\}$ are precisely $\{p_1, \ldots, p_r\}$, $N = N_{S_n}(H)$, A is the family of H-orbits on $\{1, \ldots, n\}$ of size 2, B and C are the families of subsets of size 2 and 1 respectively of the fixed points of H on $\{1, \ldots, n\}$. As usual, if X is a set where a group acts, kX denotes the permutation module on X. Recall that $N/H \cong \big(N_{S_{n-r}}(H)/H\big) \times S_r$.

Proposition 4.3.1. *We have that*

$$\overline{FP}_V(H) \oplus kC \cong kA \oplus kB$$

as N/H-modules.

Proof. We compute $\overline{FP}_{M^{(n-2,2)}}(H)$ in two different ways. From

$$M^{(n-2,2)} = D^{(n-2,2)} \oplus M^{(n-1,1)}$$

we get

$$\overline{FP}_{M^{(n-2,2)}}(H) \cong \overline{FP}_V(H) \oplus \overline{FP}_{M^{(n-1,1)}}(H) \cong \overline{FP}_V(H) \oplus kC$$

On the other hand, using the fact that $M^{(n-2,2)}$ is an H-permutation module we get that $\overline{FP}_{M^{(n-2,2)}}(H) \cong kA \oplus kB$, since the fixed points of H acting on the subsets of $\{1, \ldots, n\}$ of size 2 are made up of subsets fixed pointwise by H, and subsets forming a single H-orbit of size 2. □

This is not an explicit description of $\overline{FP}_V(H)$, but it is enough to determine when the Brauer quotient is a simple projective module.

Corollary 4.3.2. *If $r \neq 1$ then $\overline{FP}_V(H)$ is not a simple and projective $k[N/H]$-module. If $r = 1$, then $\overline{FP}_V(H)$ is a direct summand of kA of codimension 1 and the number of orbits of size 2 is odd.*

Proof. If $r \geq 5$ then $kB \cong M^{(r-2,2)}$, $kC \cong M^{(r-1,1)}$, and by Lemma 2.1.8, if $\overline{FP}_V(H)$ were simple and projective (counting composition factors) we would have $\overline{FP}_V(H) \cong D^{(n-2,2)}$ as N/H-modules, and $D^{(r-2,2)}$ (or rather, $k \otimes_k D^{(n-2,2)}$) is not a simple projective $k[N/H]$-module. If $r = 3$ then $kB \cong kC$ (subsets of size 2 and 1 will give isomorphic families), so $\overline{FP}_V(H)$ is isomorphic to the permutation module kA, which can be simple only if it is one-dimensional, which implies that H has exactly 3 fixed points and one orbit of size 2 on $\{1, \ldots, n\}$, so n is congruent to 1 modulo 4, contradicting our hypothesis. If $r = 1$, then $kC = k$, $kB = 0$, and the number of orbits of size 2 must be odd because $n \equiv 3(\bmod 4)$. □

Remark 4.3.3. For the remainder of this section we shall assume that $r = 1$ and that H has at least 3 orbits of size 2 (if it had only 1 then kA would have dimension 1 and $\overline{FP}_V(H) = 0$).

We must determine what the action of N/H on kA is, that is, how N/H acts on the orbits of size 2 of H on $\{1, \ldots, n\}$. Without loss of generality, let $\{i, l+i\}_{i=1}^{l}$ be the orbits of H of size 2 (where l is the number of such orbits).

Lemma 4.3.4. *If kA has a projective non-trivial summand (as a module for N/H) then H contains the transpositions $(i, l+i)$, $1 \leq i \leq l$ and a Sylow 2-subgroup of $S_{\{2l+1,\ldots,n\}}$.*

Proof. If $(i, l+i) \notin H$, then it represents an element of order 2 in N/H that acts trivially on the set A, hence on the module kA, hence on its direct summands, which cannot be projective. A similar argument proves that H must contain a Sylow 2-subgroup of $S_{\{2l+1,\ldots,n\}}$. □

Corollary 4.3.5. *If kA has a projective non-trivial summand (as a module for N/H), then H is the internal direct product of the elementary abelian subgroup generated by the transpositions $(i, l+i)$ $1 \leq i \leq l$ and a Sylow 2-subgroup of $S_{\{2l+1,\ldots,n\}}$.*

Proof. The lemma proves one containment. The other follows from the fact that H is a 2-subgroup and it is therefore contained in the direct product of some Sylow 2-subgroups of the symmetric groups on its orbits, which in this case are generated by the elements mentioned. □

Proposition 4.3.6. *Let m be a natural number, E the subgroup of S_m generated by the disjoint transpositions $\{(a_i, b_i) \mid 1 \leq i \leq l\}$, and X the complement of the previous $2l$ points. Let Λ be a subgroup of S_X, put $Q = E \cdot \Lambda$ and let $N = N_{S_m}(Q)$. Then the action of N on the set of l pairs $\{a_i, b_i\}$ induces a surjective morphism $\phi : N \longrightarrow S_l$ whose kernel is $E \cdot N_{S_X}(\Lambda)$.*

Proof. The morphism ϕ induced by the action of N on the l pairs is well defined because N sends Q-orbits to Q-orbits. In order to prove that ϕ is surjective, it suffices to cover all transpositions in

S_l. If $\{a, b\}$ and $\{c, d\}$ are two of the l pairs, then $(a, c)(b, d)$ is an element in N whose action on the pairs is that of the transposition we wanted. Now let us determine the kernel of ϕ. We have that that $E \cdot N_{S_X}(\Lambda)$ is in the kernel of ϕ. Now let σ be in the kernel of ϕ. Then σ preserves the l pairs, so $\sigma = \varepsilon\delta$ with $\varepsilon \in E$ and $\delta \in S_X$, so $\delta = \varepsilon^{-1}\sigma \in N \cap S_X \subset N_{S_X}(\Lambda)$. □

Corollary 4.3.7. *We have that*

$$N/H \cong S_l$$

where l is the number of H-orbits of size 2.

Proof. By Proposition 4.3.6, $\phi : N \longrightarrow S_l$ is surjective, and since $N_{S_X}(P) = P$ where P is a Sylow 2-subgroup of $S_{\{2l+1,\dots,n\}}$ contained in H, then the kernel of ϕ is H. □

Corollary 4.3.8. *Let l be the number of H-orbits of size 2. Then $kA \cong M^{(l-1,1)}$ and $\overline{FP}_V(H) \cong D^{(l-1,1)}$ as S_l-modules.*

Proof. The morphism $\phi : N \longrightarrow S_l$ that induces the isomorphism between N/H and S_l takes the permutation module of H-orbits of size 2 (i.e. kA) to the permutation module arising from the set $\{1, \dots, l\}$, (i.e., $M^{(l-1,1)}$). We also know that l is odd, so $M^{(l-1,1)} \cong D^{(l-1,1)} \oplus k$. Now apply Corollary 4.3.2. □

Corollary 4.3.9. *We have that $\overline{FP}_V(H)$ is simple and projective if and only if H is conjugate to a subgroup of the form $E \cdot P$ where E is generated by $(1, 2), (3, 4), (5, 6)$, and P is a Sylow 2-subgroup of $S_{\{7,\dots,n\}}$. In this case, $N/H \cong S_3$ and $\overline{FP}_V(H) \cong D^{(2,1)}$.*

Proof. The module $D^{(l-1,1)}$ is S_l-projective if and only if $l = 3$. This and Corollary 4.3.5 determine the structure of H. In this case, from Corollary 4.3.7 it follows that $N/H \cong S_3$, and by Corollary 4.3.8 we have that $\overline{FP}_V(H) \cong D^{(2,1)}$ is simple and projective. □

4.4 $\overline{FP}_{S^{(n-1,1)}}$ and $\overline{FP}_{D^{(n-1,1)}}$, $n \equiv 2 \pmod 4$

We describe the subgroups H of S_n (when n is congruent to 2 modulo 4) with respect to which the Brauer quotient of $S^{(n-1,1)}$ is simple and projective as a module for $N_{S_n}(H)/H$, and similarly for $D^{(n-1,1)}$. Let k be the field with two elements, H a 2-subgroup of S_n, N the normalizer of H in S_n, K a maximal subgroup of H, V the Specht module $S^{(n-1,1)}$ and W the irreducible module $D^{(n-1,1)}$. From now on we shall use the characterization of V and W as families of subsets under symmetric differences (see Lemma 2.1.6). Note that $V/X \cong W$ where X be the vector subspace of V spanned by the vector $\{1, \dots, n\}$.
In order to obtain the Brauer quotient, first we need to describe the fixed points V^H and W^H.

Proposition 4.4.1. *The fixed points V^H of H on V are the H-invariant subsets of $\{1, \dots, n\}$ of even cardinality. The fixed points W^H of H on W are the classes of elements in V^H modulo X.*

Proof. The statement for V^H follows from the definition of fixed points and the characterization of V. Now let A be a subset representing an element in W that is fixed by all the elements in H, and let $h \in H$. Then $h(A)$ is either A or its complement. Suppose $h(A) = A^c$. Then both subsets have the same cardinality, namely $n/2$, which is an odd number (because n is congruent to 2 modulo 4), and this contradicts the fact that A has even cardinality (since it is in V). Thus $h(A) = A$, which proves the assertion about the fixed points of H on W. □

Large orbits (of size 4 or larger) are in the image of relative traces from proper subgroups, so they will not contribute towards the Brauer quotient.

Proposition 4.4.2. *Let $\Omega \in V^H$ be an H-orbit whose size is divisible by 4. Then there is a maximal subgroup K of H and $\Lambda \in V^K$ such that $tr_K^H(\Lambda) = \Omega$. If $[\Omega]$ and $[\Lambda]$ are the corresponding classes in W^H and W^K, then $tr_K^H([\Lambda]) = [\Omega]$.*

Proof. We have that Ω is isomorphic to an H-set of the form H/L where L is a subgroup whose index in H is at least 4. Let K be any maximal subgroup of H containing L. Then Ω is a union of two K-orbits of equal size, say Λ and $h\Lambda$, where $h \in H$ but $h \notin K$. Since Ω has size divisible by 4, we have that Λ has even size, so $\Lambda \in V^K$. Furthermore, $tr_K^H(\Lambda) = \Lambda + h\Lambda = \Lambda \triangle h\Lambda = \Lambda \coprod h\Lambda = \Omega$. We also have $tr_K^H([\Lambda]) = [tr_K^H(\Lambda)] = [\Omega]$. □

Remark 4.4.3. Since all H-orbits of size divisible by 4 are in the image of a relative trace map from a proper subgroup, the Brauer quotient (of V and W) with respect to H is a quotient of the space spanned by (classes of) fixed points and orbits of size 2 of H. Now we proceed to determine which of these orbits will also be in the image of a relative trace from a proper subgroup. Since $tr_K^H \circ tr_L^K = tr_L^H$, it suffices to consider maximal subgroups of H.

Lemma 4.4.4. *Let A be the set of K-fixed points that are not fixed by H, and let h be an element in H that is not in K. Then h acts on A as a product of disjoint transpositions. These transpositions are independent of the choice of h, and the pairs they determine are H-orbits of size 2.*

Proof. We have that K is normal in H, so h permutes the K fixed points without fixing any of them (otherwise that point would also be fixed by H), but $h^2 \in K$, so h^2 acts trivially on A, and this implies that h acts on A as a product of disjoint transpositions. The rest follows from the fact that K has index 2 in H, so any other representative is of the form hk with $k \in K$. □

Definition 4.4.5. Let B and C be H-orbits. We say they are *glued* if for every h in H, h fixes B pointwise if and only if h fixes C pointwise.

Lemma 4.4.6. *Gluing is an equivalence relation in the set of all H-orbits of size 2.*

Proof. This is immediate from the definition. □

We also know that the normalizer of H sends glued orbits to glued orbits.

Lemma 4.4.7. *Let D, E be glued orbits of H, and let $n \in N$. Then nD, nE are also glued H-orbits. In particular, N permutes the equivalence classes of orbits of H under gluing, and it preserves the cardinality of each equivalence class.*

Proof. Let $h \in H$, Then h fixes D pointwise if and only if $hd = d$ for all $d \in D$, if and only if $nhn^{-1}(nd) = nd$ for all $d \in D$, if and only if nhn^{-1} fixes nD pointwise. In other words, nhn^{-1} fixes nD pointwise if and only if h fixes D pointwise, if and only if h fixes E pointwise, if and only if nhn^{-1} fixes nE pointwise, so nD and nE are glued nHn^{-1}-orbits. Since $n \in N$, we have the desired result. □

Remark 4.4.8. The H-orbits of size 2 described in Lemma 4.4.4 are glued. Moreover, they form a single equivalence class. The following result proves that this is always how these equivalence classes arise.

Lemma 4.4.9. *Let A be the union of an equivalence class of H-orbits of size 2 under gluing. Let L be the pointwise stabilizer of A in H. Then L is maximal in H.*

Proof. Let x be any element in H that is not in L. Then x must act on A as a product of disjoint transpositions without fixed points. If y is another element in H that is not in L, then y also acts as the same product of transpositions, so xy acts as the identity on those points, and it is in L. This proves that L has index 2 in H. □

Corollary 4.4.10. *The maximal subgroups of H which have strictly more fixed points than H are precisely the ones described in Lemma 4.4.9.*

Proof. This follows from Lemmas 4.4.4 and 4.4.9. □

Next we prove that the Brauer quotients of V and W cannot be simple and projective unless H acts fixed-point freely on the set $\{1, \ldots, n\}$.

Theorem 4.4.11. *Assume that H has at least one fixed point on the set $\{1, \ldots, n\}$, say a. Then the images of the relative traces from proper subgroups of H are the subspace of V^H generated by all the orbits of H on $\{1, \ldots, n\}$ of even cardinality. The images of the relative traces in W^H are the classes of such orbits of even cardinality modulo X.*

Proof. By Proposition 4.4.2, every orbit of H on $\{1, \ldots, n\}$ whose size is divisible by 4 will be in the image of a relative trace from a proper subgroup. Let $\{\alpha, \beta\}$ be an H-orbit of size 2. Since we are assuming that a is a fixed point of H, we must have $\alpha \neq a \neq \beta$. Let L be the maximal subgroup of H fixing α and β (see Lemma 4.4.9), and let $h \in H$, $h \notin L$, so by Lemma 4.4.4 $h\alpha = \beta$. We have that $\{a, \alpha\} \in V^L$, and $tr_L^H(\{a, \alpha\}) = \{a, \alpha\} + \{a, \beta\} = \{\alpha, \beta\}$. In W, we have that $[\{a, \alpha\}] \in W^L$ and $tr_L^H([\{a, \alpha\}]) = [\{\alpha, \beta\}]$. The other containment follows from the fact that if L is maximal in H, $\Omega \in V^H$ and x is a point fixed by H with $x \in \Omega$, then $x \in h\Omega$ for every $h \in H$, so $x \notin tr_L^H(\Omega)$, and the image of a relative trace from a maximal subgroup cannot contain any points fixed by H. If $[\Omega] \in W^L$ with $\Omega \in V^L$ then $tr_L^H([\Omega]) = [tr_L^H(\Omega)]$, so $tr_L^H(W^L)$ consists of the classes of the elements in $tr_L^H(V^L)$. □

Corollary 4.4.12. *Assume that the fixed points of H on $\{1,\ldots,n\}$ are precisely $\{1,\ldots,r\}$. Then $\overline{FP}_V(H) \cong S^{(r-1,1)}$ and $\overline{FP}_W(H) \cong D^{(r-1,1)}$, where N/H acts via the canonical quotient map $N/H \longrightarrow S_r$. In particular, neither $\overline{FP}_V(H)$ nor $\overline{FP}_W(H)$ is a simple projective module.*

Proof. Note that r is even because n is even and H is a 2-group. The Brauer quotient $\overline{FP}_V(H)$ is the family of H-invariant subsets of $\{1,\ldots,n\}$ of even cardinality modulo the span of the orbits of H of even cardinality. Let $\varphi : V^H \longrightarrow M^{(r-1,1)}$ be the morphism of kS_r-modules sending a set Ω to the set of fixed points of H in Ω (we are again using the description of $M^{(r-1,1)}$ given in terms of subsets of $\{1,\ldots,n\}$ from Lemma 2.1.6). Then the kernel of φ consists of the H-invariant subsets of even cardinality with no fixed points, which is the same as the vector subspace spanned by the H-orbits of even cardinality, so $\overline{FP}_V(H) \cong \varphi(V^H)$, and since an H-invariant subset of even cardinality must have an even number of fixed points, we have $\varphi(V^H) = S^{(r-1,1)}$. This in turn induces an isomorphism between $\overline{FP}_W(H) \cong \overline{FP}_V(H)/ < [\{1,\ldots,n\}] >$ and $S^{(r-1,1)}/ < \varphi(\{1,\ldots,n\}) >= S^{(r-1,1)}/ < \{1,\ldots,r\} >= D^{(r-1,1)}$. Since r is even, $(r-1,1)$ cannot be a triangular partition, so neither $S^{(r-1,1)}$ nor $D^{(r-1,1)}$ is a simple and projective module for kS_r. □

This means that if we want a simple projective Brauer quotient, we must only consider subgroups H with no fixed points on $\{1,\ldots,n\}$.

Theorem 4.4.13. *Assume H has no fixed points. Then the images of the relative traces generate the same module as the (classes of) orbits of size divisible by 4 together with the sums of pairs of glued H-orbits of size 2 (i.e. subsets $\{a,b,c,d\}$ where $\{a,b\}$, $\{c,d\}$ are glued H-orbits of size 2).*

Proof. By Proposition 4.4.2, any orbit whose size is divisible by 4 will be in the image of a relative trace. Each element in V^K is a K-invariant subset of $\{1,\ldots,n\}$, and can be written as a disjoint union of K-orbits. If Ω is a K-orbit with $|\Omega| \geq 2$ and $h \in H$, $h \notin K$, then $tr_K^H(\Omega) = \Omega + h\Omega = \Omega \triangle h\Omega$. Let $w \in \Omega$, so $\Omega = Kw$, and $h\Omega = hKw = hKh^{-1}hw = K(hw)$ is another K-orbit of the same size, so either $h\Omega = \Omega$ (and $\Omega \triangle h\Omega = \emptyset$) or $\Omega \cap h\Omega = \emptyset$ (and $\Omega \triangle h\Omega$ is an H-orbit of size $2|\Omega|$). We see that the only possible orbits of size 2 in the images of the relative traces are the images of these maps on subsets of the K-fixed points of maximal subgroups K. The only possible fixed points for K will not be fixed by H, so they come from glued H-orbits of size 2 (see Remark 4.4.8). Recall that the subsets of even cardinality are generated by all possible pairs. A pair of fixed points under K is either an H-orbit, or they lie in distinct H-orbits which are glued. Let $\{a,b\}$ and $\{c,d\}$ be glued H-orbits, and let K be their corresponding maximal subgroup. If they are different orbits, then $tr_K^H(\{a,c\}) = \{a,c,b,d\}$, but using the same orbit gives $tr_K^H(\{a,b\}) = 0$, so the 2-orbits in the image of this relative trace are the sums of pairs of different glued H-orbits of size 2. □

Now we know that the Brauer quotients depend on the glued orbits of size 2. We use this to give another description of the Brauer quotients in terms of certain subspaces of the fixed points.

Lemma 4.4.14. *Assume that H has no fixed points on $\{1,\ldots,n\}$. Recall that X denotes the subspace of V generated by the vector $\{1,\ldots,n\}$, so X is also a subspace of V^H. Let Y be the subspace of V^H spanned by the H-orbits of size 2, and let Z be the subspace of Y spanned by all*

sums of pairs of glued H-orbits of size 2. Let Λ be the vector subspace of V^H spanned by all the H-orbits of size greater than or equal to 4, and let w be the sum of all the orbits of size 2. We have that:
(i) The dimension of Y equals the number of H-orbits of size 2.
(ii) The codimension of Z in Y equals the number of equivalence classes of orbits of size 2 under gluing.
(iii) $V^H = Y \oplus \Lambda$
(iv) $W^H = V^H/X = (Y \oplus \Lambda)/X$
(v) $\sum tr_K^H(V^K) = Z \oplus \Lambda$
(vi) $\sum tr_K^H(W^K) = (Z + \Lambda + X)/X$
(vii) $\overline{FP}_V(H) \cong Y/Z$ as $k[N/H]$-modules.
(viii) $\overline{FP}_W(H) \cong Y/(Z+ < w >)$ as $k[N/H]$-modules.

Proof. (i) is immediate, and (ii) follows from the fact that if $v_1, \ldots, v_t$ is a basis of a vector space, the subspace generated by all the sums $v_i + v_j$ with $i \neq j$ has codimension 1. (iii) follows from the fact that an H-invariant subset of $\{1, \ldots, n\}$ can be written uniquely as a disjoint union of orbits, and (iv) is the description of W^H from Proposition 4.4.1. (v) and (vi) are given by Theorem 4.4.13 (notice that we do not know whether $\{1, \ldots, n\} \in Z \oplus \Lambda$ or not, so the sum $Z + \Lambda + X$ may not be direct). We have $\overline{FP}_V(H) = V^H / \sum tr_K^H(V^K) \cong (Y \oplus \Lambda)/(Z \oplus \Lambda) \cong Y/Z$, which proves (vii). We use one of the isomorphism theorems to get $\overline{FP}_W(H) = W^H / \sum_K^H(W^K) \cong [(Y \oplus \Lambda)/X] \,/\, [(Z + \Lambda + X)/X] \cong (Y \oplus \Lambda)/(Z + \Lambda + X)$. Finally, consider the composition of morphisms of $k[N/H]$-modules $Y \longrightarrow Y \oplus \Lambda \longrightarrow (Y \oplus \Lambda)/(Z + \Lambda + X)$. Note that this is surjective, and that its kernel is $Y \cap (Z + \Lambda + X) = Z+ < w >$, so we have (viii). □

We proceed to study the modules V and W separately.

Theorem 4.4.15. *Assume H has no fixed points. Let $A_1, \ldots, A_r$ be the equivalence classes of 2-orbits under gluing. Then N permutes the A_i, and the Brauer quotient of V with respect to H is isomorphic to the permutation module on $A_1, \ldots, A_r$.*

Proof. We use the notation from Lemma 4.4.14. Let φ be the map from the permutation module of the A_i into the Brauer quotient given by sending A_i to the class of any of its representatives. This map is well defined, because the difference of two representatives is in Z. It is also surjective, because any orbit of size 2 lies in one of the A_i. By Lemma 4.4.14 (ii) and (vii), the dimension of $\overline{FP}_V(H)$ is r, so the domain and image of φ have the same dimension, so φ is an isomorphism. □

Theorem 4.4.16. *We have that $\overline{FP}_V(H)$ is simple and projective if and only if H is a Sylow 2-subgroup of S_n. In this case, $N = H$, and $\overline{FP}_V(H)$ is the 1-dimensional trivial module.*

Proof. If H is a Sylow 2-subgroup then it has no fixed points and it has a unique orbit of size 2 (because $n \equiv 2(\bmod 4)$, and we have the desired result. Now suppose H is a 2-subgroup with simple projective Brauer quotient. By Corollary 4.4.12 H cannot have any fixed points, and by

Theorem 4.4.15 the Brauer quotient is a permutation module, which is simple only when 1 dimensional trivial, and this is in turn projective only for a group of odd order, so H must be a Sylow 2-subgroup of S_n. $\square$

Now for $W = D^{(n-1,1)}$.

Theorem 4.4.17. *Assume H has no fixed points. Let $A_1, \dots, A_r$ be the equivalence classes of 2-orbits under gluing. Then N permutes the A_i, and the Brauer quotient $\overline{FP}_W(H)$ is isomorphic to the quotient of the permutation module on $A_1, \dots, A_r$ by the submodule spanned by the element $\sum_{i=1}^r |A_i| A_i$.*

Proof. Once again we use the notation from Lemma 4.4.14. Let φ be the map from the permutation module of the A_i into the module $Y/(Z+ < w >)$ given by sending A_i to the class of any of its representatives. This map is well defined, because the difference of two representatives is in Z. It is also surjective, because any orbit of size 2 lies in one of the A_i. We can also see that the element $f := \sum_{i=1}^r |A_i| A_i$ is in the kernel of φ because $\varphi(f) = \sum_{i=1}^r |A_i| \varphi(A_i) = w$. Note that a sum of 2-orbits is in Z if and only if it has an even number of elements in each A_i, so $w \in Z$ if and only if $f = 0$. By Lemma 4.4.14 (ii) and (viii), the dimension of $\overline{FP}_W(H)$ is either r if $w \in Z$, or $r-1$ if $w \notin Z$. There are two cases:
Case 1: $w \in Z$. Then $f = 0$, the dimension of the Brauer quotient is r and φ is an isomorphism, so its kernel is spanned by f and the result holds.
Case 2: $w \notin Z$. Then $f \neq 0$, the dimension of the Brauer quotient is $r-1$ and the kernel of φ has dimension 1, so it is spanned by f, and the result holds. $\square$

Theorem 4.4.18. *We have that $\overline{FP}_W(H)$ is simple and projective if and only if H is conjugate to a subgroup of the form $< (1,2), (3,4), (5,6) > \times P$, where P is a Sylow 2-subgroup of $S_{\{7,\dots,n\}}$. In this case, N/H is isomorphic to S_3, and $\overline{FP}_W(H)$ is the 2-dimensional projective.*

Proof. It is immediate from Theorem 4.4.17 that the Brauer quotient of W with respect to the subgroup $H =< (1,2), (3,4), (5,6) > \times P$ is isomorphic to $M^{(2,1)}/k \cong D^{(2,1)}$ and that $N/H \cong S_3$. Suppose, conversely, that $\overline{FP}_W(H)$ is simple and projective. By Corollary 4.4.12, H cannot have any fixed points, so the conditions for Theorem 4.4.17 are satisfied. We also claim that no two different H-orbits of size 2 can be glued (if $\{a, b\}$ is glued to $\{c, d\}$, then $(a, c)(b, d)$ is in N but not in H, has order 2, and acts trivially on $\overline{FP}_W(H)$, which cannot be N/H-projective). The subgroup H must be contained in a subgroup of the form $E \cdot P$ where E is the subgroup generated by the transpositions (a_i, b_i) for all 2-orbits of H, and P is a Sylow 2-subgroup of the symmetric group on the remaining points. Note that $E \cdot P$ is a 2-subgroup containing H that acts trivially on the Brauer quotient. If H were properly contained in $E \cdot P$, then there would be an element in N/H of order 2 acting trivially on a projective module, so $H = E \cdot P$. Let $A_1, \dots, A_r$ be the H-orbits of size 2. The action of N on the A_i induces a surjective morphism of groups to S_r whose kernel is H (see Proposition 4.3.6). Then the quotient N/H is isomorphic to S_r where r is the number of H-orbits of size 2, and the isomorphism is given by the action of N on such orbits. Furthermore,

$\overline{FP}_W(H)$ is, as an N/H-module, isomorphic to the quotient of the permutation module $k\{1, \ldots, r\}$ modulo the sum of the basic elements, and this module is $D^{(r-1,1)}$. This is kS_r-projective if and only if $r = 3$, so H and $\overline{FP}_W(H)$ have the desired form. $\square$

Chapter 5

Triangular partitions and weights

In this chapter we study what happens when triangular partitions are appended to other partitions to create weights. Unless otherwise stated, k is the field of two elements, t is a triangular number, μ the triangular partition of t, n a natural number, and λ the partition obtained by adjoining n nodes to the first row of μ. Section 4.1 will be very useful throughout this chapter.

5.1 Simple projective modules

First we recall the fundamental property of triangular partitions. For a proof of this result, see [10].

Lemma 5.1.1. *We have that $S^\mu = D^\mu$ is simple and projective as a kS_t-module. Furthermore, kS_n has a simple projective module if and only if n is a triangular number, and in this case there is only one simple projective module.*

Combining Lemma 5.1.1 and what we know from Section 4.1 we have the following result.

Corollary 5.1.2. *Let H be a weight subgroup of S_n, with no fixed points on $\{1, \ldots, n\}$. For each $q > n$, regard S_n as the subgroup of S_q that fixes the points $x > n$. Then H is a weight subgroup of S_q if and only if $q - n$ is a triangular number. In this case, the number of isomorphism classes of simple projective $N_{S_n}(H)/H$-modules equals the number of isomorphism classes of simple projective $N_{S_q}(H)/H$-modules.*

Proof. The group $N_{S_q}(H)/H$ is isomorphic to $(N_{S_n}(H)/H) \times S_{q-n}$, which has a simple projective module if and only if $q - n$ is a triangular number. The last part follows from the fact that the simple projective modules for a product of groups are the tensor products of their respective simple projective modules. $\square$

Remark 5.1.3. From now on, we shall concentrate on the trivial partition (one row of size n) and the trivial weight (P, k), where P is a Sylow 2-subgroup of S_n and k is the one dimensional trivial module. We shall thus analyze what happens when a triangular partition is appended to the trivial partition of size n.

Lemma 5.1.4. *Let ν be any partition, with rows $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_s$. Remove n symbols and push up, obtaining a composition, $\hat{\nu}$ (the rows of $\hat{\nu}$ need not be in descending order). Rearrange the rows of $\hat{\nu}$ to get a partition $\tilde{\nu}$. Then $\tilde{\nu}$ "fits inside" ν, that is, $\tilde{\nu}_i \leq \nu_i$ for all i.*

Proof. Induction on n. Let $n = 1$, and let j be the index of the row where a node is deleted. Then the j-th row of the composition $\hat{\nu}$ is $\nu_j - 1$, but all the other rows coincide with those of ν. Let a be the smallest number such that $\nu_j > \nu_a$ (possible, since $\nu_j > 0$). Then we have $\nu_j = \nu_{j+1} = \cdots = \nu_{a-1} > \nu_a$, so $\nu_j - 1 \geq \nu_a$, and rearranging the rows of $\hat{\nu}$ in decreasing order we get that $\tilde{\nu}_i = \nu_i$ if $i \neq a-1$, and $\tilde{\nu}_{a-1} = \nu_{a-1} - 1$. Thus $\tilde{\nu}$ fits inside ν.
Now assume the result holds for n symbols. Removing n symbols from ν gives a partition ψ that fits inside ν, and removing one symbol from ψ gives a partition ϕ that fits inside ψ, so ϕ fits inside ν as well. □

Remark 5.1.5. When we remove one node, it suffices to take it from a row that is strictly larger than the next (no shifting will occur). Note also that both $\hat{\nu}$ and $\tilde{\nu}$ depend on the choice of the rows from which the n symbols are removed.

Lemma 5.1.6. *Let $\tilde{\lambda}$ be any partition obtained from λ by removing n nodes as above. Then $\tilde{\lambda}$ dominates μ, and $\tilde{\lambda} = \mu$ if and only if the n nodes were removed from the first row.*

Proof. To show $\sum_{i=1}^{r} \tilde{\lambda}_i \geq \sum_{i=1}^{r} \mu_i$ for all $r \geq 1$, we show that $\sum_{i>r} \tilde{\lambda}_i \leq \sum_{i>r} \mu_i$, and this follows from the previous lemma. The only way to obtain μ is to remove n nodes from the first row. □

Proposition 5.1.7. *The module M^{λ} restricted to S_t is a direct sum of M^{ν}, where each ν is obtained from λ as $\tilde{\lambda}$, removing the nodes with the symbols $t+1, \ldots, t+n$ from each λ-tabloid. The module M^{μ} occurs precisely once.*

Proof. The module M^{λ} is the permutation module with λ-tabloids as a basis. Restricting this to S_t we get again a permutation module. Let α be a λ-tabloid. We claim that the orbit of α under S_t gives rise to a permutation module $M^{\tilde{\lambda}}$, where $\tilde{\lambda}$ is obtained by removing n nodes from λ as described in Lemma 5.1.4, the n nodes being deleted from the rows where the numbers $t+1, \ldots, t+n$ are placed in the λ-tabloid α. These two kS_t-permutation modules are isomorphic because their underlying S_t-sets are isomorphic, which in turn follows from the fact that the stabilizer in S_t of α coincides with the stabilizer of the $\tilde{\lambda}$-tabloid obtained from α by deleting the n nodes from the rows where the symbols $t+1, \ldots, t+n$ are. The only time that we get M^{μ} is when we remove all n nodes from the first row, which can be done in exactly one way, so M^{μ} appears exactly once as a summand (α is a tabloid, so the order of the symbols within each row is irrelevant). □

Corollary 5.1.8. *The module M^{λ} restricted to S_t has exactly one composition factor isomorphic to D^{μ}.*

Proof. It follows from the previous proposition and the fact (see [9]) that the composition factors of M^{ν} are all of the form D^{α}, where α dominates ν (and, when ν is 2-regular, exactly one composition factor is isomorphic to D^{ν}). □

We shall prove that as a kS_t-module, $\overline{FP}_{D^\lambda}(H)$ has D^μ as a summand for any subgroup H of S_n. We start with the fixed points.

Proposition 5.1.9. *Let $\varphi = \varphi_{t,n} : M^\mu \longrightarrow M^\lambda$ be the map given by adjoining the numbers $t+1, t+2, \ldots, t+n$ to the first row of each μ-tabloid (and extending by linearity). Then φ is a monomorphism of kS_t-modules. Moreover, φ sends S^μ into S^λ, and if $\pi : S^\lambda \longrightarrow D^\lambda$ is the natural quotient map, then $\pi \circ \varphi$ sends S^μ isomorphically into a summand (as a kS_t-module) of the fixed points of D^λ under the subgroup $S_{\{t+1,t+2,\ldots,t+n\}}$. In particular, if K is any subgroup of S_n, there is a monomorphism of S_t- modules $D^\mu \longrightarrow (D^\lambda)^K$.*

Proof. We see that φ is a monomorphism of S_t-modules. Let α be a μ-tableau, and let β be the tableau obtained from α by adjoining the numbers $t+1, t+2, \ldots, t+n$ to the first row. Then the column stabilizer in S_t of α is the same as the column stabilizer of β in S_{t+n}, so the polytabloid generated by α will be sent by φ to the polytabloid generated by β. Since this holds for any tableau α, we have that φ sends S^μ into S^λ. The map φ also preserves the standard bilinear form on M^μ, and since $S^\mu = D^\mu$, we have that $\varphi(S^\mu) \cap (S^\lambda)^\perp$ is contained in $\varphi(S^\mu) \cap (\varphi(S^\mu))^\perp = \varphi(S^\mu \cap (S^\mu)^\perp) = 0$, so the composition $\pi \circ \varphi$ is injective. The rest follows from the fact that S^μ is S_t-projective. □

Corollary 5.1.10. *The module S^λ restricted to S_t has exactly one composition factor isomorphic to D^μ, and so do D^λ and $(D^\lambda)^K$ for any subgroup K of S_n.*

Proof. The module $S^\lambda \downarrow_{S_t}$ is a submodule of $M^\lambda \downarrow_{S_t}$, so it has at most one D^μ as composition factor (see Corollary 5.1.8). But $D^\lambda \downarrow_{S_t}$ is a quotient module of $S^\lambda \downarrow_{S_t}$, and $(D^\lambda)^K$ is a submodule of $D^\lambda \downarrow_{S_t}$, so these also have at most one D^μ as a composition factor. Finally, Proposition 5.1.9 states that $(D^\lambda)^K$ has at least one composition factor isomorphic to D^μ, hence so do $D^\lambda \downarrow_{S_t}$ and $S^\lambda \downarrow_{S_t}$. □

Corollary 5.1.11. *Let H be a 2-subgroup of S_n, and K a proper subgroup of H. Let $\psi : D^\mu \longrightarrow (D^\lambda)^K$ be the monomorphism of kS_t-modules defined in Proposition 5.1.9. Then $tr_K^H \circ \psi = 0$.*

Proof. We have that $\psi(D^\mu)$ is contained in $(D^\lambda)^H$, so $tr_K^H(\psi(v)) = [H : K]\psi(v) = 0$. □

We analyze what the relative traces do to this copy of D^μ in the fixed points $(D^\lambda)^H$.

Lemma 5.1.12. *If $K \leq H \leq S_n$ then*

$$tr_K^H : (D^\lambda)^K \longrightarrow (D^\lambda)^H$$

is a morphism of kS_t-modules, where S_t is regarded as a subgroup of the quotient $N_{S_{t+n}}(H)/H$.

Proof. Let $y \in S_t$. Then for any $x \in H$ we have $yx = xy$, so $ytr_K^H(v) = y \sum x_i v = \sum y x_i v = \sum x_i(yv) = tr_K^H(yv)$, where $yv \in (D^\lambda)^K$. □

Corollary 5.1.13. *If $K \leq H \leq S_n$ then $Im(tr_K^H)$ is an S_t-submodule of $(D^\lambda)^H$.*

Lemma 5.1.14. *Let $V_1, \ldots, V_r$ be a family of submodules of a module V. Then $\sum V_i$ is a quotient module of $\oplus V_i$.*

Proof. The map $(v_1, \dots, v_s) \mapsto v_1 + \dots + v_n$ is surjective. $\square$

Corollary 5.1.15. *Let H be a 2-subgroup of S_n, and K a proper subgroup of H. Then*
(i) The kernel of $tr_K^H : (D^\lambda)^K \longrightarrow (D^\lambda)^H$ has D^μ as a composition factor.
(ii) The image of tr_K^H does not have D^μ as a composition factor.
(iii) No composition factor of $\sum_{K<H} Im(tr_K^H)$ is isomorphic to D^μ.
(iv) As a kS_t-module (and not necessarily as a $k[N_{S_n}(H)/H]$-module), the Brauer quotient $\overline{FP}_{D^\lambda}(H)$ has D^μ as a composition factor of multiplicity one.

Proof. (i) follows from Corollary 5.1.11; (ii) follows from Corollary 5.1.10 and (i); (iii) follows from (ii) and a Lemma 5.1.14; (iv) follows from (iii) and Corollary 5.1.10. $\square$

Now we prove that there are no simple projective kS_d-modules arising from a triangular number d greater than t.

Lemma 5.1.16. *Let λ be a partition of m, μ a 2-regular partition of t, $t < m$. If D^μ is a composition factor of $M^\lambda \downarrow_{S_t}$, then there exists a partition ν of t such that ν fits inside λ and μ dominates ν.*

Proof. By Proposition 5.1.7, $M^\lambda \downarrow_{S_t}$ is a direct sum of M^ν, where each ν fits inside α. If one of the M^ν has D^μ as a composition factor, we know (see [9]) that μ dominates ν. $\square$

Lemma 5.1.17. *Let t be a triangular number, μ the triangular partition of size t, λ the partition obtained by adding n nodes to the first row of μ. Let ν be a subpartition of λ (so $\mu_i \geq \nu_i$ for all $i \geq 2$) and α a triangular partition with $|\alpha| = |\nu|$ and that dominates ν. Then $\mu_1 \geq \alpha_1$, and hence $|\mu| \geq |\alpha|$.*

Proof. Suppose $\alpha_1 > \mu_1$. Then α is a larger triangle than μ, so there exists r such that $\alpha_r > 0 = \mu_r$, so $|\alpha| \geq \sum_{i=1}^r \alpha_i > \alpha_1 + \sum_{i=2}^r \mu_i \geq \nu_1 + \sum_{i=2}^r \nu_i = |\nu|$, contradicting the fact that α and ν had the same size. $\square$

Corollary 5.1.18. *Let H be a Sylow 2-subgroup of S_n. Then $N_{S_{n+t}}(H)/H \cong S_t$, and $\overline{FP}_{D^\lambda}(H)$ has a simple projective summand of multiplicity one, given by D^μ. If K is a proper subgroup of H, then $\overline{FP}_{D^\lambda}(K)$ contains no simple projective summands.*

Proof. The first part follows from Corollary 5.1.15. Let r be the number of points in $\{1, \dots, n\}$ that K moves. We have that $N_{S_{n+t}}(K) = S_{n+t-r} \times N_{S_r}(K)$, and $N_{S_{n+t}}(K)/K \cong S_{n+t-r} \times N_{S_r}(K)/K$. Suppose that $\overline{FP}_{D^\lambda}(K)$ has a simple projective summand. Then $n + t - r$ is a triangular number, and its triangular partition α is such that D^α is a composition factor of $M^\lambda \downarrow_{S_{|\alpha|}}$. By Lemma 5.1.16, there exists a subpartition ν of λ such that $|\alpha| = |\nu|$ and α dominates ν. By Lemma 5.1.17, $|\alpha| \leq |\mu|$, i.e. $n + t - r \leq t$, so $n = r$.
We have then that $N_{S_{n+t}}(K) = S_t \times N_{S_n}(K)$, so $N_{S_{n+t}}(K)/K \cong S_t \times N_{S_n}(K)/K$. A simple projective module for the latter group must be of the form $D^\mu \otimes B$, where B is a simple projective $N_{S_n}(K)/K$-module and D^μ is the only simple projective S_t-module. If $\overline{FP}_{D^\lambda}(K)$ had such a summand, then restricting to S_t the Brauer quotient would have a summand $(D^\mu \otimes B) \downarrow_{S_t} \cong$

$(D^\mu)^{\dim(B)}$, so D^μ would have multiplicity $\dim(B)$, and since D^μ appears only once, B must have dimension 1. Thus B cannot be projective for $N_{S_n}(K)/K$ if K is a proper subgroup of H since then $N_{S_n}(K)/K$ has order divisible by 2. $\square$

Chapter 6

Stable partitions

This chapter is of a more combinatorial nature. Our main goal is to prove that many entries in the table of partitions in Section 3.5 are uniquely determined if certain conditions are imposed. We define the concept of a 2-stable partition, (very similar to James' notion of alternating partition, see [8]), and use it to describe the above-mentioned uniquely determined partitions.

6.1 2-stability

We remind the reader of some of the concepts we shall use. For a more detailed description see [10].

Definition 6.1.1. Let λ be a partition. A *skew-hook* is a connected part of the rim of λ which can be removed to leave a proper diagram. The *r-core* of λ is the partition obtained by removing all possible skew-hooks of size r from λ (this is a well-defined partition, that is, the order in which we remove the skew-hooks does not matter). A *brick* is a skew-hook of size 2. Recall that if $\lambda = (\lambda_1, \dots, \lambda_t)$ is a 2-regular partition, then its rows must be of different sizes, i.e. $\lambda_1 > \lambda_2 > \cdots > \lambda_t$.

We define the main concept of this chapter.

Definition 6.1.2. Let λ be a partition. We call λ *2-stable* if it has the same number of rows as its 2-core.

Proposition 6.1.3. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_t)$ be a partition. The following are equivalent:*
(i) λ is 2-stable.
(ii) $\lambda_t \equiv 1 (\bmod\ 2)$ and $\lambda_i \not\equiv \lambda_{i+1} (\bmod\ 2)$ for all $i = 1, \dots, t-1$.
(iii) $\lambda_i \equiv t - i + 1 (\bmod\ 2)$ for all $i = 1, \dots, t$.
(iv) λ is obtained from its 2-core by adjoining horizontal bricks to the non-empty rows of the core.

Proof. (i) implies (ii): If λ_t were even, then the t-th row would be a string of horizontal bricks, so we could remove it and the 2-core would have at most $t-1$ rows. Thus λ_t must be odd. If λ_{t-1} were also odd, then we would be able to remove all nodes but one from λ_t, all nodes but one from its neighbour and then remove a vertical brick, which means the 2-core would have at most $t-2$ rows.

Now assume that the parities of $\lambda_t, \lambda_{t-1}, \ldots, \lambda_{i+1}$ alternate. We must show that $\lambda_i \not\equiv \lambda_{i+1} (\bmod 2)$. Without loss of generality, we may assume that $\lambda_t = 1, \lambda_{t-1} = 2, \ldots, \lambda_{i+1} = t - i$ (by removing all possible horizontal bricks from the bottom row λ_t and working our way up). Note that the 2-core of λ must have t rows, so it must be the triangular partition $(t, t-1, \ldots, 1)$, and in particular, its $i+1$ row has size $t - i$. If λ_i had the same parity as λ_{i+1}, then we would be able to remove horizontal bricks from λ_i until we have $t - i$ nodes left, and then we would be able to remove a vertical brick, so that the row $i + 1$ of the 2-core of λ would have at most $t - i - 1$ nodes, contradicting the fact that it had exactly $t - i$ nodes.
(ii) implies (iii): We have $\lambda_t \equiv 1 (\bmod 2)$, so (iii) holds when $i = t$. Now use induction going down from $i = t$ to $i = 1$.
(iii) implies (iv): Since $\lambda_t \equiv 1 (\bmod 2)$, we can remove horizontal bricks from the last row to leave one node, then proceed to remove horizontal bricks from the previous row to leave two nodes, and work our way up until we get the triangular partition $(t, t - 1, \ldots, 1)$, which is the 2-core of λ.
(iv) implies (i): If we remove the horizontal bricks that were adjoined we shall obtain the 2-core of λ, so both partitions must have the same number of rows (no bricks were added to form new rows). □

Corollary 6.1.4. *If λ is 2-stable, then it is also 2-regular.*

Proof. Since $\lambda_i \not\equiv \lambda_{i+1} (\bmod 2)$, no two consecutive rows can have the same size. □

Corollary 6.1.5. *If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ is 2-stable, then the partition given by $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_t + 1, 1)$ is also 2-stable.*

Proof. This follows from Proposition 6.1.3, part (ii). □

Remark 6.1.6. Note that the rows of a 2-stable partition have the same parity as the rows of its 2-core. A possible way to measure how far a partition is from being 2-stable is to count the number of its "mismatched rows", that is, the rows that have a different parity from the corresponding rows of the 2-core. The following lemma gives an estimate of how many of these rows an arbitrary partition can have.

Lemma 6.1.7. *Let $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ be a partition (not necessarily 2-regular) with 2-core $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$. Let $\Lambda = \{i \mid \gamma_i \not\equiv \mu_i (\bmod 2),\ 1 \leq i \leq k\}$ be the set of* "mismatched" *rows of μ. Then $s \geq k + 2|\Lambda|$.*

Proof. We use induction on $|\mu|$. If $|\mu| = 0$ or 1 then μ is its own 2-core and $\Lambda = \emptyset$. Now assume the result holds for all partitions of size smaller than $|\mu|$. If μ is a 2-core, then once again $\Lambda = \emptyset$ and the result holds. If μ is not a 2-core, let ν be any partition obtained from μ by removing a brick, so $|\nu| < |\mu|$ and the 2-core of ν is also γ. Let s_1 be the number of rows of ν, and $\Lambda_1 = \{i \mid \gamma_i \not\equiv \nu_i (\bmod 2),\ 1 \leq i \leq k\}$ the set of mismatched rows of ν. By the induction hypothesis, $s_1 \geq k + 2|\Lambda_1|$. There are two cases:

Case 1: We removed a horizontal brick to obtain ν from μ. Then $\Lambda_1 = \Lambda$, so $s \geq s_1 \geq k + 2|\Lambda_1| = k + 2|\Lambda|$.
Case 2: We removed a vertical brick to obtain ν from μ. Let $i, i+1$ be the rows where the vertical brick was removed. Then $\nu_i = \nu_{i+1}$. If $\nu_{i+1} > \nu_{i+2}$, then continue to remove all possible vertical bricks from rows $i, i+1$ until rows $i+1$ and $i+2$ have the same size. If $\nu_{i+2} > \nu_{i+3}$ then remove all possible vertical bricks from rows $i+1, i+2$, and continue in this manner until you reach the last two rows, $s-1, s$ (which could have been the original $i, i+1$), and simply remove them both (using vertical bricks). Let $\alpha = (\alpha_1, \ldots, \alpha_{s-2})$ be the resulting partition. Notice that α and μ have the same 2-core, $|\alpha| < |\mu|$ and α has exactly two fewer rows than μ. Let $\Lambda_2 = \{i \mid \gamma_i \neq \alpha_i (\bmod 2),\ 1 \leq i \leq k\}$. All vertical bricks removed from any of the first k rows kept the size of two consecutive rows equal, so exactly one out of each such pair contributed to the set of mismatched rows, and the number of mismatched rows remained the same. Similarly, no vertical bricks removed from any of the rows $k+1$ through s changed the size of the set of mismatched rows (because these rows do not appear in the 2-core). The only time when the number of mismatched rows could have changed was while removing vertical bricks from the rows k and $k+1$, and this number cannot have been changed by more than one unit (depending on whether the k-th row kept its mismatched status or not). This means that $||\Lambda| - |\Lambda_2|| \leq 1$, and since α satisfies the induction hypothesis, we have

$$s - 2 \geq k + 2|\Lambda_2| \geq k + 2(|\Lambda| - 1),$$

and the result is valid for μ. □

The following is our main result in this section.

Theorem 6.1.8. *Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ be a 2-stable partition, and let $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ be a 2-regular partition containing λ. If $|\mu| = |\lambda| + t + 1$ and the 2-core of μ is $(t+1, t, \ldots, 1)$, then $\mu = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_t + 1, 1)$, and μ is 2-stable.*

Proof. Since λ is a subpartition of μ, it is possible to write $\mu = (\lambda_1 + \alpha_1, \ldots, \lambda_t + \alpha_t, \alpha_{t+1}, \ldots, \alpha_s)$. It suffices to show that $\alpha_i \geq 1$ for all $1 \leq i \leq t+1$ since then the equality $|\mu| - |\lambda| = t + 1$ forces $\alpha_i = 1$ where $1 \leq i \leq t + 1 = s$. Note that $\alpha_{t+1} \geq 1$ since the 2-core of μ has $t+1$ rows. Let $\Gamma = \{i \mid \alpha_i = 0,\ 1 \leq i \leq t\}$. We must show that $\Gamma = \emptyset$. Suppose $|\Gamma| \geq 1$. Note that

$$\sum_{i=1}^{t} \alpha_i = \sum_{i \in \{1, \ldots, t\} - \Gamma} \alpha_i \geq t - |\Gamma|.$$

Since λ is 2-stable, by Proposition 6.1.3 (iii) it has no mismatched rows. However, the 2-core of μ is the next triangle, and all the rows of the smaller triangle must change parity, so all the rows of λ that kept their parity will be mismatched rows of μ, so Γ is a subset of the set Λ of mismatched rows of μ. By Lemma 6.1.7 we have $s \geq (t+1) + 2|\Lambda| \geq t + 1 + |\Gamma|$, so

$$s - t \geq |\Gamma| + 1$$

Since μ is 2-regular, $1 \leq \alpha_s < \alpha_{s-1} < \cdots < \alpha_{t+1}$, so $\sum_{i=t+1}^{s} \alpha_i \geq \sum_{i=1}^{s-t} i = \frac{(s-t)(s+1-t)}{2}$, and

$$\begin{aligned}
t+1 =& \sum_{i=1}^{s} \alpha_i = \sum_{i=t+1}^{s} \alpha_i + \sum_{i \in \{1,\ldots,t\}-\Gamma} \alpha_i \geq \frac{(s-t)(s+1-t)}{2} + t - |\Gamma| \\
\geq& t + \frac{(|\Gamma|+1)(|\Gamma|+2)}{2} - |\Gamma| = t + \frac{|\Gamma|^2 + 3|\Gamma| + 2}{2} - |\Gamma| \\
=& t + 1 + \frac{|\Gamma|^2 + |\Gamma|}{2} > t+1
\end{aligned}$$

which is a contradiction. It is now immediate that μ is 2-stable. □

The significance of this last result is that it seems possible that to each weight subgroup we may associate a sequence of partitions $\lambda^0, \lambda^1, \lambda^2, \ldots$ as in the table in Section 3.5, so that

a) the 2-core of λ^i is the i-th triangle

b) $|\lambda^i| = |\lambda^0| + t_i$, where $t_i = \sum_{j=0}^{i-1} j$ is the i-th triangular number

c) λ^i is contained in λ^{i+1} for all i

By Theorem 6.1.8 we see that once some λ^i is 2-stable then so is every λ^j with $j \geq i$, and furthermore λ^{j+1} is constructed from λ^j by adding one new node to each row, together with an extra node as a new last row.

Chapter 7

Epilogue

We have seen how Brauer quotients can be used to provide an explicit bijection between irreducibles and weights for F_2S_n, $n \leq 8$. We have proved that there are infinitely many instances of simple projective Brauer quotients. We have appended triangular partitions to trivial partitions and obtained Brauer quotients with exactly one simple projective summand. We have also studied 2-regular partitions from a more combinatorial perspective, but relying on the facts provided by our computations with Brauer quotients to guide the flow of our intuition.

In this final chapter we tie up a few loose ends. We give several counterexamples to show that the conditions on the number n that we imposed in Chapter 4 are necessary. We present our attempts at modifying the original algorithm to solve the problem for $n = 9$, explaining what worked and what did not. We conclude with a series of possible directions for further research, including some interesting conjectures.

7.1 Examples

These examples (taken from our tables in Section 3.4) show that the restrictions imposed on n that we gave in Chapter 4 are necessary. In this section, we shall always work with the field of two elements.

1. Let $n = 4$, $\lambda = (3, 1)$. Then the subgroup $H = E_4$ is such that the Brauer quotient of $D^{(3,1)}$ with respect to E_4 is simple and projective (see Table 3.3), but E_4 is not conjugate to a Sylow 2-subgroup of S_1 (that is, E_4 is not trivial), contradicting the conclusion of Corollary 4.2.3. The fact that n is even causes problems, because the group H may not have any fixed points (as is the case here).

2. Let $n = 9$, $\lambda = (7, 2)$. Then the Brauer quotient of $D^{(7,2)}$ is not projective for any weight subgroup of S_9 (see Table 3.8), contradicting the conclusion of Corollary 4.3.9. In this case it is harder to understand what the problem is (other than the fact that $n \not\equiv 3 (\bmod 4)$) or how to fix it, but this is no surprise; after all, $D^{(7,2)}$ is the irreducible for which the proposed

algorithm failed.

3. Let $n = 8$, $\lambda = (7,1)$, $H = E_2 \wr E_4$. Then the Brauer quotient of $D^{(7,1)}$ with respect to H is both simple and projective (see Table 3.7), and H is not of the form described in Theorem 4.4.18.

7.2 Modifications to the algorithm

We tried several variations on our original algorithm to provide an explicit bijection in the case $n = 9$, where there should be a match between the weight given by the subgroup $E_2 \wr E_4$ and the irreducible $D^{(7,2)}$ (their Brauer quotient is 0). In this section we present the modifications that failed, explaining why they did not work. In the next section we shall discuss what we are currently working on.

7.2.1 Using other modules instead of D^λ

We took Brauer quotients of other modules to try to establish a bijection. We wanted to get a triangular matrix of dimensions with simple projective modules represented along the diagonal. We also wanted this procedure to coincide with the bijection that we had established before, using the D^λ.

The other possible modules that we can associate to a p-regular partition λ are M^λ, Y^λ, and S^λ where Y^λ is a Young module (see [7]). All of them have failed to keep the previous bijection at other places.

- M^λ and Y^λ: Consider the match $Q = E_2 \wr E_4$ and $\lambda = (7,1)$. Since the Young module is a direct summand of M^λ, it suffices to prove that the Brauer quotient of M^λ is 0. We have that $M^{(7,1)}$ is a permutation module, so $\overline{FP}_{M^{(7,1)}}(E_2 \wr E_4)$ is the permutation module on the set of $(7,1)$-tabloids that are $E_2 \wr E_4$-stable. But our subgroup has no fixed points, so it does not fix any $(7,1)$-tabloids, so $\overline{FP}_{M^{(7,1)}}(E_2 \wr E_4) = 0$.

- S^λ: This clashes with the previous bijection. For any value of n that is congruent to 2 modulo 4, Theorem 4.4.16 says that $S^{(n-1,1)}$ has a simple and projective Brauer quotient with respect to the Sylow 2-subgroup of S_n. But our algorithm matches the Sylow 2-subgroup with the trivial partition for all values of n, no exceptions.

7.3 Directions for further research

There are many things that can be done to continue our work, ranging from module-theoretical questions to purely combinatorial problems. What follows is but a list of our preferences.

7.3.1 Between fixed points and Brauer quotients

Conjecture 7.3.1. *It is possible to establish the desired bijection using a quotient module of the fixed points by the span of (possibly) fewer images of relative traces from proper subgroups.*

The reason why our original algorithm failed was because the Brauer quotient was too small. In the particular case when $n = 9$ and $H = E_2 \wr E_4$, we had that the fixed points of $D^{(7,2)}$ did contain a simple projective summand, that is, a copy of the weight module we wanted. We claim that this is not an isolated event, and provide the analytical evidence we have gathered so far. Notice that the following results are true for all finite groups and all characteristics.

Definition 7.3.2. Let k be a field of characteristic p, G a finite group, (Q, S) a weight for G and V a kG-module. We say that V *contains* (Q, S) if V^Q has S as a direct summand (as $N_G(Q)/Q$-modules).

Proposition 7.3.3. *The regular module kG contains all weights.*

Proof. For any subgroup Q we have that kG^Q is the permutation module on the cosets of Q in G, which contains the regular module $k(N_G(Q)/Q)$. □

Proposition 7.3.4. *If V contains a weight (Q, S) and W is a submodule of V that does not contain that weight, then V/W contains that weight.*

Proof. We know that S is a composition factor of V^Q but not of W^Q, so it is a composition factor of V^Q/W^Q, which in turn is a submodule of $(V/W)^Q$. □

Corollary 7.3.5. *Let G be any finite group, k any field of characteristic p, (Q, S) a weight for G. Then there exists a kG-simple module that contains (Q, S).*

Proof. Choose a composition series of the regular kG-module, and go down the series using Proposition 7.3.4 until you find a simple kG-module that contains the weight (Q, S). □

On the other hand, the fixed points may be too large. For example, the weight (P, k), where P is a Sylow p-subgroup and k is the trivial module, is contained in every irreducible module. But something between the large fixed points and the small Brauer quotient might be what we need to establish an explicit bijection between weights and irreducibles. The problem is to determine what we should divide the fixed points by in order to get the simple projective module.

7.3.2 Triangular tables of dimensions

Conjecture 7.3.6. *The tables of Brauer quotients can always be arranged in a triangular form, with either 0's or simple projective modules along the diagonal. Moreover, this will define an explicit bijection between weights and simple modules.*

The tables of dimensions of Brauer quotients that we have obtained can always be arranged in a triangular form by permuting its rows and columns, that is, by re-arranging the order of the weight

subgroups and the partitions. Moreover, the diagonal entries in our tables are all simple projective modules, except for one 0. For every value of n that admits such a triangular table, and provided that there are not too many 0's along the diagonal, we would obtain an explicit bijection between weights for that symmetric group and its irreducibles. It might be the case that the existence of a 0 along the diagonal would tell us something special about that symmetric group, and perhaps it is possible to determine which n's will have null diagonal entries. After all, our results about Brauer quotients being simple and projective rely heavily on n having a certain congruence class. The first step in this direction would be to prove that the Brauer quotient of any non-trivial irreducible of S_n with respect to the Sylow 2-subgroup is 0; in other words, that the first row of the table has a 1 followed by 0's.
It is possible to prove that the Brauer quotient (with respect to the Sylow 2-subgroup) of an irreducible that does not lie in the principal block is 0, but we have not been able to prove that non-trivial irreducibles in the principal block also have 0 Brauer quotients with respect to the Sylow 2-subgroup.

7.3.3 Appending triangular partitions

Conjecture 7.3.7. *Let Q be a weight subgroup of S_n, where Q has no fixed points on $\{1, \ldots, n\}$, and let λ be a partition of m such that $\overline{FP}_{D^\lambda}(Q)$ is simple and projective. Then the 2-core of λ has size $m - n$.*

In the case of the trivial partitions, it is easy to see how to append every triangular partition to it (just enlarge the first row of the triangle). But even in this relatively simple case we have not been able to prove that the Brauer quotient is precisely the simple projective module, only that it has it as a summand. The above conjecture is formulated from a different perspective: given that the Brauer quotient $\overline{FP}_{D^\lambda}(Q)$ is simple and projective, what can we say about the partition λ?

7.3.4 Combinatorial implications

Conjecture 7.3.8. *It is possible to arrange all 2-regular partitions in an infinite table satisfying the following conditions:*

- *Along every row, each partition is contained in the next one.*
- *The 2-core of every partition along the i-th column is the i-th triangular partition (where $\emptyset$ is the first triangular partition).*
- *The difference in size from the first partition and the i-th one on any given row is the i-th triangular number.*

A direct consequence of Alperin's Conjecture (and also of other more elementary results) is that the number of 2-regular partitions of n with empty 2-core equals the number of 2-regular partitions of $n + t$ with 2-core of size t for any triangular number t. No explicit bijection is known, but our table of partitions indicates that (for small values of n) it is possible to define a bijection with the property that corresponding partitions are comparable (that is, the rows of one are greater than or equal to the corresponding rows of the other). If such a statement were true, our results from Chapter 6 prove that larger partitions assigned to 2-stable partitions are uniquely determined (there is only one way to "grow" out of a 2-stable partition), and it would only be a matter of a finite number of steps to go from a partition with empty 2-core to a 2-stable partition.

7.3.5 Other fields and other primes

We have worked mostly with the prime 2 and the field of order 2, but some of our results hold in greater generality. Other results will have to be modified for them to work in other characteristics or with other fields of characteristic 2, and concepts such as p-stable partitions will have to be carefully redefined. Most of our software can be used independently of the finite field, but it will be necessary to create new libraries of weight subgroups and irreducibles. It is worth noticing that in the case when $p > 2$, the sign representation is non-trivial and irreducible, but its restriction to the alternating subgroup is trivial, and so is its restriction to any p-subgroup. Therefore it has a non-zero Brauer quotient with respect to all p-subgroups of S_n, including the Sylow p-subgroup (compare this to Section 7.3.2).

Appendix

We describe the most important computer subroutines that we created. Before we discuss the subroutines, we would like to say more about the way that group representations are handled using GAP [6]. We defined a data structure in GAP that corresponds to the notion of representation. It consists of a record whose fields are: the group represented, a list of matrices (the images of the list of generators of the group under the representation), the field used, the dimension of the representation, a boolean variable (which indicates it is indeed a representation) and some of the operations which can be performed on representations.
An example of an irreducible representation for S_3 is shown in the following GAP session:

```
gap> V:=ModularIrreducible([1,2],2);
Representation( Group( (1,2), (1,2,3) ), Images
[ [  1,  1 ],
  [  0,  1 ] ]
[ [  1,  1 ],
  [  1,  0 ] ]
 )

gap> RecFields(V);
[ "group", "genimages", "field", "dimension",
  "isRepresentation", "operations" ]
gap> V.group;
Group( (1,2), (1,2,3) )
gap> V.genimages;
[ [ [ Z(2)^0, Z(2)^0 ], [ 0*Z(2), Z(2)^0 ] ],
  [ [ Z(2)^0, Z(2)^0 ], [ Z(2)^0, 0*Z(2) ] ] ]
gap> V.field;
GF(2)
gap> V.dimension;
2
gap> V.isRepresentation;
true
```

```
gap> V.operations;
rec(
  FixedPoints := function ( rep ) ... end,
  FixedQuotient := function ( rep ) ... end,
  SubFixedQuotient := function ( rep, u ) ... end,
  Dual := function ( rep ) ... end,
  TensorProduct := function ( g, h ) ... end,
  Maps := function ( g, h ) ... end,
  SubmoduleRep := function ( rep, v ) ... end,
  QuotientRep := function ( rep, v ) ... end,
  SumOfImages := function ( g, h ) ... end,
  SumOfImagesFromDual := function ( g, h ) ... end,
  Spin := function ( rep, v ) ... end,
  CoSpin := function ( rep, v ) ... end,
  BrauerQuotient := function ( rep ) ... end,
  Print := function ( M ) ... end )
```

There are more functions than the above, but they have not yet been incorporated to the "operations" extension. Here we list only the functions that are in some way related to our present work. We use the following conventions: G denotes a group, H a subgroup of G, k a field of characteristic p, L a list (of vectors, group elements, etc), P is a p-group (possibly a p-subgroup of G), V is a kG-representation.

BrauerQuotient(V) computes the dimension of the Brauer quotient of V. This routine was used to create the tables of dimensions, since it is much faster than its more complete counterpart, BrauerRep.

BrauerRep(G, P, V) computes the Brauer quotient as a module for the quotient group $N_G(H)/H$. It was used to analyze the structure of the Brauer quotient and to obtain the boxed entries in the tables of dimensions.

DirectSumRep(V_1, V_2) computes the direct sum $V_1 \oplus V_2$ of the two representations V_1 and V_2.

Dual(V) computes the dual module $V^* = Hom_k(V, k)$ of V.

FixedPoints(V) computes a basis for the the fixed points V^G of V with respect to the group G. In order to find the fixed points V^H of a representation V with respect to a proper subgroup H, you must use FixedPoints(RestrictedRep(G, H, V)).

FxdPtsForQuotient(V, G, H) computes the representation V^H as a module for the quotient group $N_G(H)/H$. It may change the generators of $N_G(H)/H$.

GeneralProjectiveDecomposition(V) computes a list L where $L[1] = V_{core}$, $L[2] = V_{proj}$ (or rather, it computes bases for the corresponding submodules). The module V_{proj} is the largest projective summand of V, and $V = V_{core} \oplus V_{proj}$.

GeneralTensorProduct(V_1, V_2) computes the tensor product $V_1 \otimes_k V_2$ of the representations V_1 and V_2 as a representation of the product $G_1 \times G_2$, where each V_i is a kG_i-module ($i = 1, 2$), and $(g_1, g_2)(u \otimes v) = g_1 u \otimes g_2 v$.

InducedRep(G, H, V) computes the induced module $V \uparrow_H^G$ from H to G.

IsProjective(p, V) tests whether the kG-module V is projective or not. It was used to determine the boxed entries in the tables of dimensions of Brauer quotients.

IsRelativelyProjective(G, H, V) tests whether the kG-module V is relatively H-projective. Although it can also be used to determine if a kG-module is projective (by setting H equal to $\{1\}$), this routine is considerably slower than IsProjective.

IsRepresentation(V) tests whether V is truly a representation, that is, whether the assignment obtained by sending `V.group.generators` to `V.genimages` gives a well-defined morphism of groups.

ModularIrreducible(partition, q) returns the unique irreducible representation associated to the given p-regular partition, where q is the cardinality of the field we are working with.

QuotientRep(V, L) computes the quotient representation of V over a submodule. The elements of L are a k-basis for the submodule of V.

Rep(G, L) creates a representation V of G such that `V.genimages`$= L$.

RestrictedRep(G, H, V) computes the restricted module $V \downarrow_H^G$ from G to H.

Spin(V, L) computes a basis for the kG-submodule of V generated by the list of vectors L.

SubmoduleRep(V, L) computes the sub-representation of V generated by the list of vectors L.

symm(n) computes the symmetric group S_n. Its generators are the ordered list [(1,2),(1,2,...,n)], which is an important convention needed to define representations of S_n.

TensorProduct(V_1, V_2) computes the tensor product $V_1 \otimes_k V_2$ of V_1 and V_2 with the diagonal action of G, that is, $g(u \otimes v) = gu \otimes gv$.

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