THE MODULE STRUCTURE OF THE COINVARIANT ALGEBRA OF A FINITE GROUP REPRESENTATION

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We take the opportunity to describe and illustrate in some special cases results which appear in [1].

1. Classical results

Let V be a finite dimensional vector space over a field k. We define a *reflection* to be a non-identity linear endomorphism $V \to V$ of finite order which fixes a hyperplane. Such an endomorphism must be diagonalizable when k is the complex numbers, but in positive characteristic this need not be so and other examples are possible, such as transvections. A group $G \subseteq GL(V)$ of linear automorphisms of V is a *reflection group* if it is generated by the reflections it contains. We let $S(V)$ be the ring of polynomial functions on V (the *symmetric algebra* on V^*), and S^G the ring of invariants. The *coinvariant algebra* is

$$S_G := S \otimes_{S^G} k = S/(S \cdot S^G_+)$$

where S^G_+ is the set of elements of S_G which have zero constant term, and $S \cdot S^G_+$ is the ideal of S which they generate. We first state the classical results which have motivated us.

Theorem 1. *(Shephard-Todd* [6]*, Chevalley* [2]*) If $|G|$ is invertible in k then G is a reflection group if and only if S^G is a polynomial ring. These conditions imply that $S_G \cong kG$.*

Weakening the invertibility condition, we have the following.

Theorem 2. *(Serre* [5]*, Mitchell* [3]*) Even when $|G|$ is not invertible in k, if S^G is polynomial then G is a reflection group and furthermore kG and S_G have the same composition factors.*

Our goal has been to extend these results in various ways, by

(1) allowing any group over any field, not just groups for which the invariants are polynomial,
(2) describing the structure of $S^H \otimes_{S^G} k$ when H is a subgroup of G, as well as more general constructions using relative invariants which have the form $(U \otimes_k S)^G \otimes_{S^G} k$ where U is a kG-module.
(3) incorporating the action of a 'regular' group element, extending the work in [4].

We will indicate a way in which the first two of these may be done, but omit the third since it takes a little longer to describe. This account announces results

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which appear in [1] and should be taken as an illustration only of the more general statements which appear there.

We first present two examples of rings of invariants and coinvariant algebras to show the kind of thing that can happen. The first is an example which fits the context of Theorems 1 and 2 with polynomial invariants, while in the second example the invariants are not polynomial.

Example 1. We let $G = C_2$ act on $V = k^2$ by interchanging the basis elements x and y, so that V is the regular representation of G over the arbitrary field k. In fact G may be regarded as the symmetric group on two symbols, and it is well known that the invariants are a polynomial ring in the elementary symmetric polynomials $x + y$ and xy. Basis elements for the various constructions we have defined are given in Table 1. Observe that if all monomials of a certain degree lie in $S \cdot S^G_+$ then all higher degree monomials lie in this ideal, and so the coinvariant algebra is zero in this and higher degrees. The module structure of $S/(S \cdot S^G_+)$ in this example is that it is the trivial representation τ in degree 0 and the sign representation ϵ in degree 1, so that the composition factors of $S/(S \cdot S^G_+)$ are the same as the regular representation.

degree:	0	1	2
S	1	x, y	x^2, xy, y^2
S^G	1	$x + y$	$xy, (x + y)^2$
S^G_+	0	$x + y$	$xy, (x + y)^2$
$S \cdot S^G_+$	0	$x + y$	$x(x + y), y(x + y), xy$
$S/(S \cdot S^G_+)$	$\bar{1}$	$\bar{x}$	

TABLE 1. Basis elements in each degree for the regular action of C_2

Example 2. Again let $G = C_2$ and let the non-identity element of G act on $V = k^2$ via the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We assume that the characteristic of k is not 2. Using this action G is not a reflection group. Bases for the invariants and coinvariant algebra are presented in Table 2, the invariants being spanned by the monomials in even degree. This time the composition factors of the coinvariant algebra $S/(S \cdot S^G_+)$ are one copy of the trivial representation and two copies of the sign representation, so that we get more than the composition factors of the regular representation.

degree:	0	1	2	3
S	1	x, y	x^2, xy, y^2	x^3, x^2y, xy^2, y^3
S^G	1		all	
S^G_+			all	
$S \cdot S^G_+$			all	all
$S/(S \cdot S^G_+)$	$\bar{1}$	$\bar{x}, \bar{y}$		

TABLE 2. Basis elements in each degree for the -1 action of C_2

2. Two theorems

We work with subgroups $H \subseteq G \subseteq GL(V)$ and let $H\backslash G$ denote the set of right cosets Hg of H in G. This acquires an action of the normalizer $N_G(H)$ from the left ($n \cdot Hg := Hng$), and we let $kH\backslash G$ denote the corresponding permutation $kN_G(H)$-module.

For any finite group Γ we let $G_0(k\Gamma)$ be the Grothendieck group of finitely generated $k\Gamma$-modules. If M is a finitely generated $k\Gamma$-module we let $[M]$ denote the element of $G_0(k\Gamma)$ which M represents, so that two modules M and M' have the same composition factors if and only if $[M] = [M']$. We put $[M] \geq [M']$ if and only if every composition factor of M' occurs with multiplicity at least as great in M.

Theorem 3. *For any field k, and finite groups $H \subset G \subset GL(V)$ as above, we have in $G_0(kN_G(H))$ the inequality*

$$[S^H \otimes_{S^G} k] \geq [kH\backslash G],$$

with equality if and only if S^H is a free S^G-module. When S^H is a free S^G-module, putting $K = \mathrm{Frac}(S^G)$, there is a filtration of $KH\backslash G$ by $KN_G(H)$-submodules so that counting from the bottom, the factor in position j is isomorphic as a $kN_G(H)$-module to the jth homogeneous component $K \otimes_k (S^H \otimes_{S^G} k)_j$.

We see this result illustrated in the second example of Section 1 where we take $H = 1$ and find that the coinvariant algebra $S_G = S \otimes_{S^G} k$ has at least the composition factors of the regular representation kG. In fact it has an extra sign representation as a composition factor, indicating (according to the theorem) that S is not free as a S^G-module.

We now show how to improve the inequality to an equality, even when S^H is not free as a S^G-module. Given a finite group Γ, a *(non-negatively) graded* $k\Gamma$-module is one with a direct sum decomposition $M = \oplus_{d \geq 0} M_d$ in which each M_d is a finite-dimensional $k\Gamma$-module. Such an M gives rise to an element $[M](t) := \sum_d [M_d] t^d$ in the formal power series ring

$$G_0(k\Gamma)[[t]] := \mathbb{Z}[[t]] \otimes_{\mathbb{Z}} G_0(k\Gamma).$$

The situation where we wish to consider this arises as follows. We let R be a finitely generated graded, connected, commutative k-algebra and let M be a finitely generated graded $R\Gamma$-module where the elements of Γ are taken to be in degree 0. In this situation the groups $\mathrm{Tor}_i^R(M, k)$ are all graded $k\Gamma$-modules with the functorial action of Γ, as may be seen in computing Tor by taking a graded resolution of M by graded $R\Gamma$-modules which are free as R-modules. We may see further that in each degree j there are only finitely many i for which the component $\mathrm{Tor}_i^R(M, k)_j$ is non-zero. Thus it makes sense to define

$$[\mathrm{Tor}^R(M, k)](t) := \sum_{i \geq 0} (-1)^i \ [\mathrm{Tor}_i^R(M, k)](t)$$

as an element of $G_0(k\Gamma)[[t]]$. In the next result we let $\mathbb{Q}(t)$ denote the field of rational functions in the indeterminate t.

Theorem 4. *Let k be any field, and consider finite groups $H \subseteq G \subseteq GL(V)$. Then the element*

$$[\mathrm{Tor}^{S^G}(S^H, k)](t)$$

lying in $G_0(kN_G(H))[[t]]$ actually lies in the subring $\mathbb{Q}(t)\otimes_{\mathbb{Z}} G_0(kN_G(H))$, and has a well-defined limit as t approaches 1, namely

$$\lim_{t\to 1}\,[\mathrm{Tor}^{S^G}(S^H,k)](t) = [kH\backslash G].$$

3. Examples

3.1. When S^G is polynomial (so G is a reflection group by Theorem 2) then S is free as an S^G-module, since S is Cohen Macaulay, hence free over any homogeneous system of parameters. Thus we recover the second conclusion of Theorem 2. It is furthermore the case that whatever subgroup H of G we take, S^H always has a finite projective resolution over S^G and so the element $[\mathrm{Tor}^{S^G}(S^H,k)](t)$ is in fact a polynomial in t.

3.2. Let $G = C_2$ be cyclic of order 2, acting on a 2-dimensional vector space with the -1 action as in Example 2 of Section 1. We take $H = 1$. Here $S = S^G \oplus S^-$ where S^- is the linear span of monomials of odd degree and we readily verify that we have a minimal resolution

$$S = S^G \oplus S^- \xleftarrow{d_0} S^G \oplus (S^G)^2[1] \xleftarrow{d_1} (S^G)^2[3] \xleftarrow{d_2} (S^G)^2[5] \xleftarrow{d_3} \cdots$$

where

$$d_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & y \end{pmatrix}, \quad d_1 = \begin{pmatrix} 0 & 0 \\ xy & -y^2 \\ -x^2 & xy \end{pmatrix}, \quad d_2 = \begin{pmatrix} xy & y^2 \\ x^2 & xy \end{pmatrix}$$

and $[n]$ means the degree is shifted by n. Here G acts as -1 on all terms except the first two copies of S^G, where the action is trivial. Let us write τ for the trivial kG-module and ϵ for the 1-dimensional sign representation. We calculate that

$$\begin{aligned}[\mathrm{Tor}^{S^G}(S,k)](t) &= \tau + 2\epsilon t - 2\epsilon t^3 + 2\epsilon t^5 - \cdots \\ &= \tau + \frac{2t}{1+t^2}\epsilon \\ &\to \tau + \epsilon \text{ as } t \to 1\end{aligned}$$

giving the same composition factors as kG, as predicted by Theorem 4.

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