An Introduction to the Representations and Cohomology of Categories

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1. Introduction

Let $\mathcal{C}$ be a small category and R a commutative ring with a 1. A *representation* of $\mathcal{C}$ over R is a functor $F : \mathcal{C} \to R$-mod where R-mod is the category of R-modules. This general concept includes as special cases representations of groups and monoids, representations of quivers, representations of partially ordered sets and generic representation theory. We are perhaps even more interested in representations of categories which do not fit into these special cases and are important in applications, such as the decomposition of classifying spaces of groups and the study of p-locally determined properties of group representations. In this survey we will describe the foundations of the general theory, emphasizing the common features of the different examples, and providing a basis for deeper study. One of the striking things to be observed is that features which are familiar in special cases often have a counterpart which holds in generality.

Why are group representations a special case of representations of categories? Given a group C, we may regard it as a category $\mathcal{C}$ with one object $*$ in which every morphism is invertible. A representation of $\mathcal{C}$ is a functor $F : \mathcal{C} \to R$-mod, namely the specification of an R-module $F(*) = V$ and for each morphism g an endomorphism $F(g)$ of V. Because all morphisms in $\mathcal{C}$ are invertible, the endomorphism must be an automorphism. The functor axioms say exactly that we have a group homomorphism $C \to GL(V)$, that is, a representation of C.

Each result we state here is, in particular, a statement about group representations, and also about representations of quivers, and about representations of partially ordered sets, and so on, but when viewed in this way the results are usually either very well known or obvious. It is not our expectation that results we prove about representations of categories in general will say anything new when specialized to groups or the other extensively studied structures. We do, however, expect to learn something when representing less familiar categories. There is also the possibility to prove something new about group representations by applying representations of other categories in a novel way.

There has been considerable interest in the use of categories to determine and calculate properties of the representations and cohomology of a finite group. Early examples of this are the stable element method of Cartan and Eilenberg in group cohomology, and also Quillen's stratification theorem, which relates group cohomology to a certain limit taken over a category whose objects are the elementary abelian p-subgroups of the group. More recently a theory has been developed whereby the classifying space of the group may be approximated (in several ways) as a homotopy colimit of a functor defined on one of a number of categories constructed from the group. This is surveyed in [9], [23] and discussed further in [15] and [24], for example. The computation of the cohomology of the homotopy colimit may be done in terms of the derived functors of the limit functor on the category, and the study of such 'higher limits' is a question to do with the representation theory of the category. We will describe the basic properties of higher limits in this survey, but leave the reader to consult sources such as [15], [22], [24], [32], [34] for more advanced techniques in computing them.

On the algebraic side there has been a trend to study properties of group representations which are determined by associated categories such as the Frobenius category and orbit category, and generalizations of these. We cite the work of Puig (see [35] for an account) in block theory, and important conjectures of Alperin and of Broué in this area. A better understanding of the representations of these categories is important for the theoretical development, and we will present the basic facts about the simple and projective representations as well as the relationship with the corresponding representations of subcategories. We mention [27], [28], [34] and [38] for more advanced reading in this direction.

Let us describe more fully some of the other examples of categories (apart from groups) mentioned earlier whose representations have been extensively studied. It will be explained in the next section that representations of a quiver are the same thing as representations of the free category associated to the quiver, so that the study of representations of quivers is the same thing as the study of representations of free categories. The incidence algebra of a poset will also be defined in the next section, and its modules are the same thing as representations of the poset. Generic representation theory is the study of representations over a field of the category of vectors spaces over the same field, or in other words functors from the category of vector spaces over a field to the same category of vector spaces. One finds this mentioned in the early paper of Higman [18], and we may consult [26] and [10] for more recent overviews. The functorial formalism provides a way to study the representations of all general linear groups over the field at the same time, and there is great interest in this approach from the point of view of topology because of applications there.

This survey is organized as follows. In Section 2 we set up the language we shall use, describing the category algebra and the constant functor, and indicating ways in which these things behave well as well as giving some warnings about bad behaviour. Section 3 is about the relationship between representations of a category and its full subcategories, and it is used in Section 4 where the projective and simple representations are discussed. The rest of the article is really about cohomology of categories. We present in Section 5 a theorem which identifies the cohomology of a category (defined by Ext groups) as the cohomology of the nerve of the category. In Sections 6 and 7 we discuss identifications of the low-dimensional cohomology groups, spending some time with the theory of extensions of categories.

I should emphasize that in putting this survey together most of what I have done is to gather material from various sources; very little is original. I have tried to give attributions but it is not always easy to know where a result first appeared. If I have attributed results incorrectly the fault is mine and I offer my apologies.

I wish to express my gratitude to Fei Xu for the regular enlightening discussions I have had with him, in which he has brought many points to my attention in connection

with the subject matter of this survey. I am also grateful to Nadia Mazza and Jacques Thévenaz for pointing out a number of errors.

2. The category algebra and some preliminaries

As mentioned in the introduction, when $\mathcal{C}$ is a small category and R is a commutative ring with a 1 we define a *representation* of $\mathcal{C}$ over R to be a functor $F : \mathcal{C} \to R$-mod where R-mod is the category of R-modules. Such a representation may be identified as a module for a certain algebra which we now introduce. We define the *category algebra* $R\mathcal{C}$ to be the free R-module with the morphisms of the category $\mathcal{C}$ as a basis. The product of morphisms α and β as elements of $R\mathcal{C}$ is defined to be

$$\alpha\beta = \begin{cases} \alpha \circ \beta & \text{if } \alpha \text{ and } \beta \text{ can be composed} \\ 0 & \text{otherwise} \end{cases}$$

and this product is extended to the whole of $R\mathcal{C}$ using bilinearity of multiplication. We have constructed an associative algebra which can be found in Section 7 of [31] (where the approach is to pass through an intermediate step in which we first 'linearize' $\mathcal{C}$). Our convention is that we compose morphisms on the left, so that if the domain $\text{dom}(\alpha)$ equals the codomain $\text{cod}(\beta)$ then we obtain a composite $\alpha \circ \beta$. Because of this we will work almost entirely with left modules when we come to consider modules for the category algebra.

If $\mathcal{C}$ happens to be a group, that is a category with one object in which every morphism is invertible, then a representation of $\mathcal{C}$ is the same thing as a representation of the group in the usual sense, namely a group homomorphism from the group to the group of automorphisms of an R-module, and the category algebra $R\mathcal{C}$ is the *group algebra*. It is a familiar fact that group representations may be regarded as the same thing as modules for the group algebra, and we will see that something similar holds with categories in general. One of the themes of this account is that representations of categories share a number of the properties of group representations.

When the category happens to be a partially ordered set the category algebra $R\mathcal{C}$ is known as the *incidence algebra* of the poset. Indeed, this may be taken as a definition of the incidence algebra. There is a sizable literature to do with incidence algebras of posets and their representations, and we mention [6, 13, 21] as a sample.

The third example is that of representations of a quiver ([1, Sec. III.1]). A *quiver* Q is a directed graph, and given such data we may form the free category FQ on Q ([30, p. 48]), which is the category whose objects are the vertices of Q and whose morphisms are all the possible composites of the arrows in Q (including for each object a composite of length zero which is the identity morphism at that object). The category algebra RFQ is the same as the path algebra of Q, and it is well known that, provided Q has finitely many vertices, modules for the path algebra may be identified with representations of the quiver.

It may be helpful to point out for comparison some constructions which appear to be representations of categories, but which do not fit into the framework we are describing. Mackey functors may be defined as R-linear functors from a certain category defined by Lindner [29] to R-modules. This construction is also described in [36], where an algebra is constructed, called the Mackey algebra, with the property that Mackey functors may be identified with modules for this algebra (see also [37]). In this definition a Mackey functor is indeed a representation of Lindner's category, but not all representations are Mackey functors because the condition of R-linearity may fail, and with it the Mackey decomposition axiom. Similar comments can be made about the functors called 'globally defined Mackey functors' in [37] which may be defined as R-linear functors on a linearized category, but which do not constitute all of the representations of this category.

Our first result says that representations of $\mathcal{C}$ are the same thing as $R\mathcal{C}$-modules in general, at least when $\mathcal{C}$ has finitely many objects.

(2.1) PROPOSITION ([31], Theorem 7.1). *Let $\mathcal{C}$ be a small category, let $(R\text{-mod})^{\mathcal{C}}$ be the category of representations of $\mathcal{C}$ and let $R\mathcal{C}$-mod be the category of $R\mathcal{C}$-modules. There are functors $r : (R\text{-mod})^{\mathcal{C}} \to R\mathcal{C}\text{-mod}$ and $s : R\mathcal{C}\text{-mod} \to (R\text{-mod})^{\mathcal{C}}$ with the properties that*
(1) $sr \cong 1_{(R\text{-mod})^{\mathcal{C}}}$, and
(2) r embeds $(R\text{-mod})^{\mathcal{C}}$ as a full subcategory of $R\mathcal{C}$-mod, and if $\mathcal{C}$ has finitely many objects then $rs \cong 1_{R\mathcal{C}\text{-mod}}$.
Thus if $\mathcal{C}$ has finitely many objects the representations of $\mathcal{C}$ over R may be identified with $R\mathcal{C}$-modules.

Proof. The idea is the same as the identification of group representations with modules for the group algebra, with an extra ingredient. Given a representation $M : \mathcal{C} \to R$-mod we obtain an $R\mathcal{C}$-module $r(M) = \bigoplus_{x\in \mathrm{Ob}\,\mathcal{C}} M(x)$ where the action of a morphism $\alpha : y \to z$ on an element $u \in M(x)$ is to send it to $M(\alpha)(u)$ if $x = y$ and zero otherwise (applying morphisms from the left.) Conversely, given an $R\mathcal{C}$-module U, for each $x \in \mathrm{Ob}\,\mathcal{C}$ let 1_x denote the identity morphism at x and define a functor $M = s(U) : \mathcal{C} \to R$-mod by $M(x) = 1_xU$. If $\alpha : x \to z$ is a morphism in $\mathcal{C}$ and $u \in 1_xU$ we define $M(\alpha)(u) = \alpha u$. The two functors r and s evidently have the properties claimed, and in case $\mathcal{C}$ has finitely many objects they give an equivalence of categories between representations of $\mathcal{C}$ over R and $R\mathcal{C}$-modules. □

The $R\mathcal{C}$-modules which arise in the image of r are the ones whose elements u have the property that $1_xu \neq 0$ for only finitely many objects x. Thus when $\mathcal{C}$ has finitely many objects we may talk about representations of $\mathcal{C}$ over R and $R\mathcal{C}$-modules interchangeably, and in practice this is a useful thing to be able to do. All the examples of categories which we will explicitly consider here do have finitely many objects, except for the case of generic representation theory. To simplify matters we will assume in our arguments that $\mathcal{C}$ does have finitely many objects, but often the arguments can be made to work equally well without this assumption. We leave it to the reader to extend them if necessary. Observe that $\mathcal{C}$ has finitely many objects if and only if $R\mathcal{C}$ has an identity element, namely $\sum_{x\in \mathrm{Ob}(\mathcal{C})} 1_x$ where 1_x is the identity endomorphism of x.

We use the language of module theory when discussing representations of a category, and we mention some aspects of this now. Thus the category of representations of $\mathcal{C}$ over R is abelian. We may form subrepresentations, and quotient representations of a representation by a subrepresentation. Sometimes we will say that 'u is an element of a representation M of $\mathcal{C}$' and this will mean that u is an element of the module which corresponds to M, namely $\bigoplus_{x\in \mathrm{Ob}\,\mathcal{C}} M(x)$. We may speak of the subfunctor generated by a set of elements of M, and this means the intersection of all the subfunctors which contain these elements. Thus we may say that a functor is 'generated by its value at an object x', for example, to mean that it is the smallest subfunctor whose value at x is the given functor. Observe that a sequence of $R\mathcal{C}$-modules is exact if and only if every sequence obtained by evaluating on objects of $\mathcal{C}$ is exact.

Very often in considering the representations of a category it is convenient to replace the category by a skeletal subcategory, namely a full subcategory containing precisely one representative of each isomorphism class. Working with a skeletal subcategory does not change the representations.

(2.2) PROPOSITION. *Let $\mathcal{C}$ and $\mathcal{D}$ be equivalent categories and R a commutative ring. Then the categories of representations of $\mathcal{C}$ and of $\mathcal{D}$ over R are also equivalent. Thus when $\mathcal{C}$ and $\mathcal{D}$ have finitely many objects, $R\mathcal{C}$ and $R\mathcal{D}$ are Morita equivalent.*

Proof. The equivalence of categories means there are functors $A : \mathcal{C} \to \mathcal{D}$ and $B : \mathcal{D} \to \mathcal{C}$ so that AB and BA are naturally isomorphic to the identity functors. Now if $F : \mathcal{C} \to R$-mod is a representation of $\mathcal{C}$ we obtain a representation FB of $\mathcal{D}$ and if G is a representation of $\mathcal{D}$ we obtain a representation GA of $\mathcal{C}$. On composing the natural isomorphism of BA and the identity with F we obtain a natural isomorphism between the functor $F \mapsto FBA$ and the identity functor $F \mapsto F$, and similarly the functor $G \mapsto GAB$ is isomorphic to $G \mapsto G$. This demonstrates the equivalence of categories of representations. □

There is a product defined on representations of a category which yields another representation of the category. Thus if M and N are representations of $\mathcal{C}$ over R we may define $M \boxtimes N$ to be the functor $M \boxtimes N(x) = M(x) \otimes_R N(x)$ on objects, and on morphisms $\alpha : x \to y$ it is $M(\alpha) \otimes_R N(\alpha)$. This product is the Kronecker product in the case of group representations, but in other contexts it is less used, and we will not use it in this account.

There is a always a distinguished representation of any category $\mathcal{C}$, namely the *constant functor* $\underline{R}$. This is defined to be $\underline{R}(x) = R$ for every object x, and applied to every morphism of $\mathcal{C}$ it yields the identity morphism. In the case of group representations it is the trivial representation and is of great importance to the theory. The same is true here and we devote a section to the study of limits, to which it is closely connected. For now we mention that the constant functor has the property that it acts as the identity with respect to the product we have defined: $M \boxtimes \underline{R} \cong M$ always.

3. Restriction and induction of representations

Given a functor between two small categories $F : \mathcal{D} \to \mathcal{C}$ we obtain a functor $F^* : R\mathcal{C}\text{-mod} \to R\mathcal{D}\text{-mod}$ obtained by composition with F, and this is restriction along F. It has a left adjoint and a right adjoint given by left and right Kan extensions as described in [19, Theorem IX.5.1]. In certain circumstances these operations can be interpreted usefully in terms of the category algebra, but not always, as we now point out.

(3.1) PROPOSITION (Xu [39]). *Let $F : \mathcal{D} \to \mathcal{C}$ be a functor between small categories. Then F induces a (non-unital) algebra homomorphism $R\mathcal{D} \to R\mathcal{C}$ given by $\alpha \mapsto F(\alpha)$ if and only if F is one-to-one on the objects of $\mathcal{D}$.*

Proof. The specification $\alpha \mapsto F(\alpha)$ always is a homomorphism of R-modules $R\mathcal{D} \to R\mathcal{C}$, and the issue is whether or not it is multiplicative, or in other words $F(\alpha\beta) = F(\alpha)F(\beta)$ for all morphisms $\alpha : w \to x$ and $\beta : y \to z$ in $\mathcal{D}$. Here the juxtaposition of morphisms indicates their product in the category algebra, which equals their composite if they can be composed, but not otherwise. Suppose that x and y are objects for which $F(x) = F(y)$. Then $F(1_x) = F(1_y) = 1_{F(x)} = F(1_x)F(1_y) \neq 0$ but on the other hand $1_x 1_y = 0$ so that this product maps to zero. This shows that if we obtain an algebra homomorphism then F must be one-to-one on objects. Conversely the only circumstance in which $F(\alpha)F(\beta)$ is non-zero is when the domain $F(w)$ of $F(\alpha)$ equals the codomain $F(z)$ of $F(\beta)$, and if F is one-to-one on objects then this implies that $w = z$. In this situation it is true that $F(\alpha\beta) = F(\alpha)F(\beta)$. □

In the situation when F is one-to-one on objects we can regard the restriction functor $R\mathcal{C}\text{-mod} \to R\mathcal{D}\text{-mod}$ as restriction along the algebra homomorphism $R\mathcal{D} \to R\mathcal{C}$. This is the case in the important situation when $\mathcal{D}$ is a subcategory of $\mathcal{C}$ and F is the inclusion functor. Assume furthermore that $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ and that $\mathcal{D}$ has finitely many objects. In this situation the category algebra of $\mathcal{D}$ is a subalgebra of the category algebra of $\mathcal{C}$, and in fact $R\mathcal{D} = 1_{R\mathcal{D}} R\mathcal{C} 1_{R\mathcal{D}}$. Restriction from $\mathcal{C}$ to $\mathcal{D}$ is a functor which sends an $R\mathcal{C}$-module M to $M \downarrow^{\mathcal{C}}_{\mathcal{D}} = 1_{R\mathcal{D}} M$. Its left adjoint is $N \mapsto N \uparrow^{\mathcal{C}}_{\mathcal{D}} = R\mathcal{C} \otimes_{R\mathcal{D}} N$, and the right adjoint of restriction is $N \mapsto \mathrm{Hom}_{R\mathcal{D}}(R\mathcal{C}, N)$.

The relationship between algebras A and eAe where e is an idempotent in A has been well studied in a variety of contexts, including that of algebras which have a stratification (see for example [14, Sec. 6.2], [7, Sec. 2], [38]). Here are the immediate properties of this relationship, expressed in the language of representations of categories.

(3.2) PROPOSITION (see [14, 6.2]). *Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ and suppose that $\mathcal{D}$ has finitely many objects. Let M be a representation of $\mathcal{D}$.*

(1) Induction $\uparrow^{\mathcal{C}}_{\mathcal{D}}$ sends projective objects to projective objects. If $\mathcal{E}$ is a set of objects of $\mathcal{D}$ and M is generated by its values on $\mathcal{E}$ then $M \uparrow^{\mathcal{C}}_{\mathcal{D}}$ is also generated by its values on $\mathcal{E}$. Furthermore $M \uparrow^{\mathcal{C}}_{\mathcal{D}} \downarrow^{\mathcal{C}}_{\mathcal{D}} = M$.

(2) Restriction $\downarrow_{\mathcal{D}}^{\mathcal{C}}$ is an exact functor which sends each simple $R\mathcal{C}$-module either to a simple $R\mathcal{D}$-module or to zero. Every simple $R\mathcal{D}$-module arises in this way, and there is a bijection given by restriction between the simple $R\mathcal{C}$-modules which are non-zero on $\mathcal{D}$, and the simple $R\mathcal{D}$-modules.

Proof. (1) Since $R\mathcal{D}\uparrow_{\mathcal{D}}^{\mathcal{C}}= R\mathcal{C}\otimes_{R\mathcal{D}} R\mathcal{D}\cong R\mathcal{C}$ is projective it follows that the induction of an arbitrary projective is projective. The property of generation also follows from the tensor product description of induction since if A is a subset of M for which $M = R\mathcal{D}\cdot A$ then

$$R\mathcal{C}(1_{R\mathcal{D}}\otimes_{R\mathcal{D}} A) = R\mathcal{C}\otimes_{R\mathcal{D}} (R\mathcal{D}\cdot A) = R\mathcal{C}\otimes_{R\mathcal{D}} M = M\uparrow_{\mathcal{D}}^{\mathcal{C}} .$$

We have

$$M\uparrow_{\mathcal{D}}^{\mathcal{C}}\downarrow_{\mathcal{D}}^{\mathcal{C}}= 1_{R\mathcal{D}}(R\mathcal{C}\otimes_{R\mathcal{D}} M) = 1_{R\mathcal{D}}R\mathcal{C}1_{R\mathcal{D}}\otimes_{R\mathcal{D}} M = R\mathcal{D}\otimes_{R\mathcal{D}} M\cong M.$$

(2) Exactness of a sequence of functors is detected by evaluating the functors at objects and if the evaluations are exact on all objects of $\mathcal{C}$, they are also exact on all objects of $\mathcal{D}$. If T is a simple $R\mathcal{C}$-module and $x = 1_{R\mathcal{D}}x$ any non-zero element of $T\downarrow_{\mathcal{D}}^{\mathcal{C}}$ then $T = R\mathcal{C}1_{R\mathcal{D}}x$ by simplicity so $T\downarrow_{\mathcal{D}}^{\mathcal{C}}= 1_{R\mathcal{D}}R\mathcal{C}1_{R\mathcal{D}}x = R\mathcal{D}x$, from which it follows that $T\downarrow_{\mathcal{D}}^{\mathcal{C}}$ is simple since it is generated by any non-zero element.

Now let S be a simple $R\mathcal{D}$-module, and consider $N = \{x\in S\uparrow_{\mathcal{D}}^{\mathcal{C}}\big|\ 1_{R\mathcal{D}}R\mathcal{C}x = 0\}$. This is the largest $R\mathcal{C}$-submodule of $S\uparrow_{\mathcal{D}}^{\mathcal{C}}$ which is zero on $\mathcal{D}$. Observe that $S\uparrow_{\mathcal{D}}^{\mathcal{C}}$ is generated by any element which is non-zero on $\mathcal{D}$, since such an element generates the restriction to $\mathcal{D}$ by simplicity of S, and this generates $S\uparrow_{\mathcal{D}}^{\mathcal{C}}$ by part (1). It follows that N is the unique maximal submodule of $S\uparrow_{\mathcal{D}}^{\mathcal{C}}$, since if $x\notin N$ then $R\mathcal{C}x = S\uparrow_{\mathcal{D}}^{\mathcal{C}}$ as just observed. In particular $\hat{S} := S\uparrow_{\mathcal{D}}^{\mathcal{C}}/N$ is a simple $R\mathcal{C}$-module. Since $N\downarrow_{\mathcal{D}}^{\mathcal{C}}= 0$ we have $\hat{S}\downarrow_{\mathcal{D}}^{\mathcal{C}}= S\uparrow_{\mathcal{D}}^{\mathcal{C}}\downarrow_{\mathcal{D}}^{\mathcal{C}}= S$ by part (1). This shows that every simple $R\mathcal{D}$-module arises as the restriction of a simple $R\mathcal{C}$-module. If T is a simple $R\mathcal{C}$-module for which $T\downarrow_{\mathcal{D}}^{\mathcal{C}}\cong S$ then by adjointness we have a non-zero homomorphism $S\uparrow_{\mathcal{D}}^{\mathcal{C}}\to T$ from which it follows that T is a simple quotient of $S\uparrow_{\mathcal{D}}^{\mathcal{C}}$, and hence $T\cong\hat{S}$. This completes the proof. □

It is a remarkable and useful property that every simple module defined on $\mathcal{D}$ extends to a simple module defined on $\mathcal{C}$. More can be said about this relationship, and we mention that a theory of relative projectivity, vertices and sources inspired by Green's theory for group representations is developed in the thesis of Xu [39].

4. Parametrization of simple and projective representations

We start by parametrizing the simple representations of a category $\mathcal{C}$. It is the case that they are naturally defined over a field R, and we could make the assumption that R is a field without loss of generality if we wish. In fact it does not seem to make a difference to the first results of this section.

We start by repeating Proposition 3.2 in a special case.

(4.1) PROPOSITION. *Let S be a simple representation of $\mathcal{C}$ over R.*

(1) For every full subcategory $\mathcal{D}$ of $\mathcal{C}$ with finitely many objects the restriction $S\downarrow^{\mathcal{C}}_{\mathcal{D}}$ is either a simple $R\mathcal{D}$-module or zero.

(2) For every object x of $\mathcal{C}$ the evaluation $S(x)$ is a simple $R\operatorname{End}_{\mathcal{C}}(x)$-module.

(3) If T is another simple representation of $\mathcal{C}$ over R and x is an object of $\mathcal{C}$ for which $T(x)\cong S(x)$ as $R\operatorname{End}_{\mathcal{C}}(x)$-modules, and $T(x)\neq 0$, then $S\cong T$ as representations of $\mathcal{C}$.

Proof. The result merely restates and interprets Proposition 3.2, and (1) is nothing more than this. For (2) and (3) we apply Proposition 3.2 in the case of the full subcategory which has x as its only object. Here the category algebra is $R\operatorname{End}_{\mathcal{C}}(x)$ and the statements follow immediately. □

Consider the set of pairs (x,V) where x is an object of $\mathcal{C}$ and V is a simple $R\operatorname{End}_{\mathcal{C}}(x)$-module. We will write $(x,V)\sim(y,W)$ if and only if there is a simple $R\mathcal{C}$-module S with $S(x)\cong V$ and $S(y)\cong W$. Certainly if x and y are isomorphic in $\mathcal{C}$ and $V\cong W$ as $R\operatorname{End}_{\mathcal{C}}(x)$-modules, where the action of $\operatorname{End}_{\mathcal{C}}(x)$ on W is transported via an isomorphism between x and y, then $(x,V)\sim(y,W)$, but this property may arise in other circumstances as well, as we will illustrate by example after the next result, which follows immediately from Proposition 4.1.

(4.2) COROLLARY.

(1) The relation $\sim$ is an equivalence relation on the set of pairs (x,V) where x ranges through objects of $\mathcal{C}$ and V ranges through simple $R\operatorname{End}_{\mathcal{C}}(x)$-modules.

(2) The isomorphism classes of simple representations of $\mathcal{C}$ are in bijection with the equivalence classes of pairs (x,V), the bijection sending a simple module S to the equivalence class of $(x,S(x))$, where x is any object of $\mathcal{C}$ for which $S(x)\neq 0$.

Example. Let $\mathcal{C}$ be a category with two objects, x and y, and with morphisms 1_x, 1_y, $\alpha:x\to y$, $\beta:y\to x$ and $\gamma:y\to y$, satisfying $\beta\alpha=1_x$ and $\alpha\beta=\gamma$. From this it follows that $\gamma^2=\gamma$, $\beta\gamma=\beta$ and $\gamma\alpha=\alpha$. We see that $\mathbb{Q}\operatorname{End}_{\mathcal{C}}(x)=\mathbb{Q}$ has one simple module and

$$\mathbb{Q}\operatorname{End}_{\mathcal{C}}(y)\cong\mathbb{Q}[c]/(c^2-c)\cong\mathbb{Q}[c]/(c)\oplus\mathbb{Q}[c]/(c-1)\cong\mathbb{Q}\oplus\mathbb{Q}$$

has two simple modules, giving rise to pairs $(x,\mathbb{Q}),(y,\mathbb{Q}_0),(y,\mathbb{Q}_1)$, where γ acts on $\mathbb{Q}_0$ and $\mathbb{Q}_1$ as multiplication by 0 and 1, respectively. We see that

$$\mathbb{Q}\mathcal{C} = \mathbb{Q}\langle 1_x,\alpha\rangle \oplus \mathbb{Q}\langle \gamma,\beta\rangle \oplus \mathbb{Q}\langle 1_y-\gamma\rangle$$

as $\mathbb{Q}\mathcal{C}$-modules, and that the three submodules in the decomposition are simple and $\mathbb{Q}\langle 1_x,\alpha\rangle \cong \mathbb{Q}\langle \gamma,\beta\rangle$. In fact the first two modules, when regarded as functors, both take the value $\mathbb{Q}$ on x and y, and every morphism acts as the identity morphism – they are the constant functor. They are simple because they are generated by any non-zero vector which they contain. Thus $\mathbb{Q}\mathcal{C}$ has two simple modules, $S_{x,\mathbb{Q}} \cong S_{y,\mathbb{Q}_1}$ and $S_{y,\mathbb{Q}_0}$. Under the equivalence relation $\sim$ the equivalence classes are $\{(x,\mathbb{Q}),(y,\mathbb{Q}_1)\}$ and $\{(y,\mathbb{Q}_0)\}$.

Sometimes the equivalence relation on pairs (x,V) has the form $(x,V)\sim(y,W)$ if and only if $x\cong y$ and $V\cong W$ (where the action of $\mathrm{End}_{\mathcal{C}}(x)$ on W is obtained by transport along an isomorphism from x to y). In such a situation the simple modules are described as $S_{x,V}$ where both x and V are taken up to isomorphism. This is the case with representations of quivers which have no oriented cycles, see [1, III.1]. It is also the case with EI-categories, namely categories in which every endomorphism is an isomorphism. Many categories which arise in the context of the p-local structure of groups (such as the Frobenius category, the orbit category, etc. [4]) are EI-categories, and the basics of their representation theory are described in [8].

(4.3) PROPOSITION. *(Lück [8]) Let $\mathcal{C}$ be an EI-category. Then the simple $R\mathcal{C}$-modules are parametrized as $S_{x,V}$ where x is an object of $\mathcal{C}$ taken up to isomorphism, and V is a simple $R\,\mathrm{Aut}(x)$-module, taken up to isomorphism.*

Proof. For each pair (x,V) we may construct a representation $S_{x,V}$ of $\mathcal{C}$ by fixing for each object x' isomorphic to x an isomorphism $x'\to x$ and defining $S_{x,V}(x')=V$ where the action of $\mathrm{Aut}(x')$ is obtained by transporting the action of $\mathrm{Aut}(x)$ along the isomorphism. We define $S_{x,V}(y)=0$ if y is not isomorphic to x. This does define a representation of $\mathcal{C}$, and it is simple since it is generated by each non-zero vector in it. From the previous discussion of the parametrization of simple modules, there can be no more than the ones we have constructed. □

We now turn to the projective representations of $\mathcal{C}$, and for this we will assume that R is a field or a complete discrete valuation ring and that $\mathcal{C}$ is finite, so that $R\mathcal{C}$ is an R-algebra of finite rank and the Krull-Schmidt theorem holds. Each simple representation $S_{x,V}$ has a projective cover $P_{x,V}$, and this gives a parametrization of the indecomposable projective representations by the equivalence classes of pairs (x,V).

Let us examine the structure of the indecomposable projectives a little further. For each object $x\in\mathrm{Ob}(\mathcal{C})$ we may construct a *linearized representable functor* $F_x:\mathcal{C}\to R$-mod defined by $F_x(y)=R\,\mathrm{Hom}_{\mathcal{C}}(x,y)$, the free R-module with the elements of $\mathrm{Hom}_{\mathcal{C}}(x,y)$ as a basis.

(4.4) PROPOSITION. *Let x be an object of $\mathcal{C}$*

(1) (Yoneda's lemma) Let M be a representation of $\mathcal{C}$. Then $\mathrm{Hom}_{R\mathcal{C}}(F_x, M) \cong M(x)$.

(2) The $R\mathcal{C}$-module F_x is projective and generated by its value at x.

(3) Regarded as an $R\mathcal{C}$-module, $F_x \cong R\mathcal{C}1_x$.

(4) Let $\mathcal{D}$ be any full subcategory of $\mathcal{C}$ which contains x. Then $F_x \cong F_x^{\mathcal{D}} \uparrow_{\mathcal{D}}^{\mathcal{C}}$ where $F_x^{\mathcal{D}} = F_x \downarrow_{\mathcal{D}}^{\mathcal{C}}$ is the functor F_x constructed for $\mathcal{D}$.

Proof. (1) We define $\alpha : \mathrm{Hom}_{R\mathcal{C}}(F_x, M) \to M(x)$ by $\alpha(\eta) = \eta_x(1_x)$. In the opposite direction we define $\beta : M(x) \to \mathrm{Hom}_{R\mathcal{C}}(F_x, M)$ as follows: if $u \in M(x)$ and $\sum_{\gamma:x\to y} \lambda_\gamma \gamma \in F_x(y) = R\,\mathrm{Hom}_{\mathcal{C}}(x, y)$ we put $\beta(u)_y(\sum_{\gamma:x\to y} \lambda_\gamma\gamma) = \sum_{\gamma:x\to y} \lambda_\gamma M(\gamma)(u)$. We verify in the usual way that α and β are mutually inverse isomorphisms.

(2) Suppose we have an epimorphism of $R\mathcal{C}$-modules $\theta : M \to N$ and a morphism $\eta : F_x \to N$. We may find $u \in M(x)$ so that $\theta(u) = \eta_x(1_x)$. Now the morphism $\beta(u) : F_x \to M$ satisfies $\theta \circ \beta(u) = \eta$ since $\theta_x(\beta(u)_x(1_x)) = \theta_x(u) = \eta_x(1_x)$. If $\gamma : x \to y$ then $\gamma = F_x(\gamma)(1_x)$ lies in the subfunctor of F_x generated by $1_x \in F_x(x)$. This shows that F_x is generated by its value at x.

(3) From the definitions, the value of $R\mathcal{C}1_x$ at an object y is

$$1_y R\mathcal{C}1_x = R\,\mathrm{Hom}_{\mathcal{C}}(x, y) = F_x(y)$$

and this shows that $F_x \cong R\mathcal{C}1_x$ as $R\mathcal{C}$-modules.

(4) We have $F_x^{\mathcal{D}} = R\mathcal{D}1_x$ so

$$F_x^{\mathcal{D}} \uparrow_{\mathcal{D}}^{\mathcal{C}} = R\mathcal{C} \otimes_{R\mathcal{D}} R\mathcal{D}1_x = R\mathcal{C}1_x = F_x.$$

Also, if y is an object of $\mathcal{D}$ then

$$F_x \downarrow_{\mathcal{D}}^{\mathcal{C}} (y) = 1_y R\mathcal{C}1_x = \mathrm{Hom}_{\mathcal{C}}(x, y) = \mathrm{Hom}_{\mathcal{D}}(x, y) = F_x^{\mathcal{D}}(y),$$

which shows that $F_x^{\mathcal{D}} = F_x \downarrow_{\mathcal{D}}^{\mathcal{C}}$. □

(4.5) COROLLARY. *Suppose that R is a field or a complete discrete valuation ring and let P be an indecomposable projective $R\mathcal{C}$-module where $\mathcal{C}$ is a finite category. Then for some object x of $\mathcal{C}$, P is generated by its value at x and is isomorphic to a direct summand of F_x. It has the form $P \cong R\mathcal{C}e$ where e is a primitive idempotent in the monoid algebra $R\,\mathrm{End}_{\mathcal{C}}(x)$. Every primitive idempotent in $R\,\mathrm{End}_{\mathcal{C}}(x)$ remains primitive in $R\mathcal{C}$.*

Proof. By Yoneda's lemma, for each object y and element of $P(y)$ there is a homomorphism $F_y \to P$ having that element in its image and so P is a homomorphic image of a direct sum of representable functors $\bigoplus_i F_{x_i}$. Since P is indecomposable projective, the surjection must split and P is isomorphic to a direct summand of a functor F_x. Since F_x is generated by its value at x, so is P. Since by Yoneda's lemma $\mathrm{End}_{R\mathcal{C}}(F_x) \cong R\,\mathrm{End}_{\mathcal{C}}(x)$, the direct summand has the form $R\mathcal{C}1_x e = R\mathcal{C}e$ for some idempotent $e \in R\operatorname{End}_{\mathcal{C}}(x)$, and e is primitive in $R\mathcal{C}$ since P is indecomposable, hence *a fortiori* primitive in $R\operatorname{End}_{\mathcal{C}}(x)$.

Equally, if f is a primitive idempotent in $R\operatorname{End}_{\mathcal{C}}(x)$ then $R\mathcal{C}f$ is a projective $R\mathcal{C}$-module which is generated by its value at x. If $R\mathcal{C}f = M \oplus N$ is a decomposition as a direct sum of $R\mathcal{C}$-modules, then each of M and N is also generated by its value at x, and so if they are non-zero then $1_x R\mathcal{C}f = 1_x M \oplus 1_x N$ is a non-zero decomposition of the indecomposable $R\operatorname{End}_{\mathcal{C}}(x)$-module $R\operatorname{End}_{\mathcal{C}}(x)f$. Since this is not possible we deduce that $R\mathcal{C}f$ is indecomposable, and so f is primitive in $R\mathcal{C}$. $\square$

The above result shows that the indecomposable projective $R\mathcal{C}$-modules may be parametrized by a subset of the primitive idempotents of the $R\operatorname{End}_{\mathcal{C}}(x)$ as x ranges over the objects of $\mathcal{C}$, and these in turn are parametrized by the equivalence classes of pairs (x, V) where x is an object of $\mathcal{C}$ and V is a simple $R\operatorname{End}_{\mathcal{C}}(x)$-module, since we have already seen that these parametrize the simple representations.

5. The constant functor and limits

The *constant functor* $\underline{R} : \mathcal{C} \to R$-mod is the functor which takes every object to the R-module R and every morphism to the identity morphism. When $\mathcal{C}$ is a group this representation is the trivial representation. It was mentioned in Section 2 that it acts as the identity with respect to the product $\boxtimes$. In this section we will examine the role of the Ext and Tor groups with $\underline{R}$ as coefficients and see that they are of particular importance. We will define the cohomology of a category $\mathcal{C}$ with coefficients in an $R\mathcal{C}$-module M as $H^i(\mathcal{C}, M) := \operatorname{Ext}^i_{R\mathcal{C}}(\underline{R}, M)$ thus extending what happens with group cohomology, which is defined algebraically as $H^i(G, M) = \operatorname{Ext}^i_{RG}(R, M)$. With group cohomology there is an interpretation of these groups as the cohomology of a space – and Eilenberg-MacLane space $K(G, 1)$ – and so also there is an interpretation of $H^i(\mathcal{C}, M)$ as the cohomology of the nerve of $\mathcal{C}$. We explain this in this section. In the remaining sections we give interpretations of the low-dimensional cohomology groups which extend the usual results from group cohomology.

We start with the connection between $\underline{R}$ and limits. In part (2) of the next result we understand that $\underline{R}$ is regarded as a right $R\mathcal{C}$-module, that is a contravariant functor from $\mathcal{C}$ to R-modules.

(5.1) PROPOSITION. *Let M be a representation of $\mathcal{C}$.*
(1) $\operatorname{Hom}_{R\mathcal{C}}(\underline{R}, M) \cong \varprojlim M$.
(2) $\underline{R} \otimes_{R\mathcal{C}} M \cong \varinjlim M$.

Proof. (1) A natural transformation $\eta : \underline{R} \to M$ is the same as the specification of an R-module homomorphism $\eta_x : R \to M(x)$ for each object x of $\mathcal{C}$ so that the triangles

$$\begin{array}{ccc} R & & \\ \eta_x \big\downarrow & \searrow \eta_y & \\ M(x) & \xrightarrow{M(\alpha)} & M(y) \end{array}$$

always commute. By the universal property of the limit we obtain a unique homomorphism $R \to \varprojlim M$ so that everything commutes, and this determines an element $\phi(\eta) \in \varprojlim M$, namely the image of 1_R. We may reverse this procedure: for each element $u \in \varprojlim M$ we obtain a homomorphism $R \to \varprojlim M$ which sends $1_R \mapsto u$. Now composition with this homomorphism produces for each object x a homomorphism $\eta_x : R \to M(x)$, giving a natural transformation $\eta : \underline{R} \to M$. These two inverse operations give the isomorphism.

(2) Regarding $\underline{R}$ and M as R-modules we have R-module homomorphisms $M(x) \to \underline{R} \otimes_{R\mathcal{C}} M$ sending $u \mapsto 1_R \otimes u$. For each morphism α in $\mathcal{C}$ we have $M(\alpha)u \mapsto 1_R \otimes \alpha u = 1_R\alpha \otimes u = 1_R \otimes u$ and so all triangles

$$\begin{array}{ccc} M(x) & \xrightarrow{M(\alpha)} & M(y) \\ & \searrow & \big\downarrow \\ & & \underline{R} \otimes_{R\mathcal{C}} M \end{array}$$

commute. It follows from the universal property of $\varinjlim$ that we have a unique homomorphism $\varinjlim M \to \underline{R} \otimes_{R\mathcal{C}} M$. In the other direction, let $\theta_x : M(x) \to \varinjlim M$ be the standard maps to the direct limit and consider the mapping $\Theta : \underline{R} \times M \to \varinjlim M$ given by

$$\Theta\Big(\sum_{x \in \mathrm{Ob}\,\mathcal{C}} r_x, \sum_{y \in \mathrm{Ob}\,\mathcal{C}} u_y\Big) = \sum_{x \in \mathrm{Ob}\,\mathcal{C}} \theta_x(r_x u_x).$$

This satisfies $\Theta(r\alpha, u) = \Theta(r, \alpha u)$ for every morphism α in $\mathcal{C}$ and so passes to an R-module homomorphism $\underline{R} \otimes_{R\mathcal{C}} M \to \varinjlim M$ which is inverse to the previously constructed homomorphism. □

As an example of the statement of 5.1, when $\mathcal{C}$ is a group and $\underline{R}$ is the trivial module, $\mathrm{Hom}_{R\mathcal{C}}(\underline{R}, M)$ identifies as the fixed points of $\mathcal{C}$ acting on M, while $\underline{R} \otimes_{R\mathcal{C}} M$ identifies as the largest fixed quotient of M. These can thus be regarded as the $\varprojlim$ and $\varinjlim$ of the action of the group.

We see in general that $\varprojlim$ is left exact and $\varinjlim$ is right exact, because the same is true of Hom and tensor product. Furthermore we may identify the right derived functors $\varprojlim^i$ and the left derived functors $\varinjlim_i$ of these functors as Ext and Tor groups, which in case $\mathcal{C}$ is a group are the group cohomology and homology.

(5.2) COROLLARY. $\varprojlim^i M \cong \mathrm{Ext}^i_{R\mathcal{C}}(\underline{R}, M)$ *and* $\varinjlim_i M \cong \mathrm{Tor}_i^{R\mathcal{C}}(\underline{R}, M)$.

These Ext and Tor groups have a topological interpretation in terms of the nerve $|\mathcal{C}|$ of the category $\mathcal{C}$. There are many books we may consult to find out about the nerve, and one possible source is [9, 4.10]. For the reader less familiar with nerves it may help to point out that it is a generalization to arbitrary categories of the order complex of a partially ordered set. At the end of this section in an exercise we indicate an argument which shows that when $\mathcal{C}$ is a group, $|\mathcal{C}|$ is an Eilenberg-MacLane space $K(\mathcal{C}, 1)$. In any case when we discuss topological properties of a category, such as contractibility, we mean the corresponding property of the nerve.

We will make use of a construction which in the case of a discrete group plays the role of the universal cover EG of the classifying space BG. This is the functor

$$E : \mathcal{C} \to \text{small categories}$$

specified by $E(x) = \mathcal{C}\downarrow_x$, the overcategory over x (see [30, II.6]). The overcategory has as its objects the morphisms $w \to x$, and as its morphisms the commutative triangles $(w \to x) \overset{(\alpha, 1_x)}{\longrightarrow} (y \to x)$ where $\alpha : w \to y$. If $\beta : x \to z$ is a morphism in $\mathcal{C}$ we define $E(\beta) : E(x) \to E(z)$ by composition of the morphisms in $E(x)$ with β. Observe that $E(x)$ has a terminal object, namely 1_x, and so the nerve of $E(x)$ is contractible.

(5.3) THEOREM ([33], [11, App. *II, 3.3]). Let* $\mathcal{C}$ *be a small category. Then* $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, \underline{R}) \cong H^*(|\mathcal{C}|; R)$, *the cohomology ring over* R *of the nerve* $|\mathcal{C}|$ *of* $\mathcal{C}$. *Similarly* $\mathrm{Tor}_*^{R\mathcal{C}}(\underline{R}, \underline{R}) \cong H_*(|\mathcal{C}|; R)$ *is the homology of* $|\mathcal{C}|$.

Proof. We follow the proof given in [15, Prop. 2.6]. Other accounts of the proof can be found in [11, App. II, 3.3] and [32]. We first construct a resolution $\mathcal{P} \to \underline{R}$ of the constant functor $\underline{R}$. The resolution we construct is a chain complex of functors $\mathcal{C} \to R$-mod, but this is the same thing as a functor from $\mathcal{C}$ to chain complexes of R-modules, and we specify it as $\mathcal{P} = C_* \circ E$, where C_* is the functor which associates to each small category the chain complex over R of its nerve. Thus $\mathcal{P}(x) = C_*(\mathcal{C}\downarrow_x)$, the chain complex over R of the nerve of the overcategory $\mathcal{C}\downarrow_x$. Since the overcategory is contractible, $\mathcal{P}(x)$ is a complex which is acyclic except in degree zero, where its homology is R. From this it follows that $\mathcal{P}$ is a resolution of the constant functor. We claim that in each degree $\mathcal{P}$ is a projective $R\mathcal{C}$-module. To see this observe that in degree n, $\mathcal{P}_n(x)$ is the free R-module with basis the chains of morphisms $x_0 \to \cdots \to x_n \to x$, and if $x \to y$ is a morphism in $\mathcal{C}$, the $R\mathcal{C}$-module structure is that this chain is sent to $x_0 \to \cdots \to x_n \to y$, obtained by composition with $x \to y$. We see that the chains which start $x_0 \to \cdots \to x_n$ span a submodule of $\mathcal{P}_n$ which is isomorphic to the projective functor F_{x_n}, and $\mathcal{P}_n$ is the direct sum of such projective subfunctors. We have thus shown that $\mathcal{P}$ is a projective resolution of $\underline{R}$.

If M is any representation of $\mathcal{C}$ then $\mathrm{Hom}(\mathcal{P}_n, M)$ is the direct sum, taken over chains $x_0 \to \cdots \to x_n$, of R-modules $M(x_n)$, since $\mathrm{Hom}(F_{x_n}, M) \cong M(x_n)$ by Yoneda's lemma. The chain complex structure is given in a similar way to the boundary map in $C^*(|\mathcal{C}|, R)$. Taking $M = \underline{R}$ we obtain exactly the cochain complex $C^*(|\mathcal{C}|, R)$, and its cohomology is $H^*(|\mathcal{C}|, R)$.

For the statement about homology we proceed in a similar way, but use the isomorphism $\underline{R} \otimes_{R\mathcal{C}} F_x \cong \underline{R} \otimes_{R\mathcal{C}} R\mathcal{C}1_x \cong \underline{R}1_x \cong R$, where now $\underline{R}$ denotes the constant functor regarded as a right $R\mathcal{C}$-module, or contravariant functor. $\square$

If M is any representation of $\mathcal{C}$ we may define $H^*(\mathcal{C}, M)$ to be the cohomology of the complex considered in the proof of the last result, and we obtain the result $\mathrm{Ext}^*_{R\mathcal{C}}(\underline{R}, M) \cong H^*(\mathcal{C}, M)$. When $\mathcal{C}$ is a group the cohomology of $\mathcal{C}$ coincides with the usual group cohomology, defined as Ext groups with the trivial module in the first variable. Theorem 5.3 shows that this is also the cohomology of the nerve of $\mathcal{C}$. Group cohomology may also be defined as the cohomology of an Eilenberg-MacLane space $K(\mathcal{C}, 1)$, and in fact when $\mathcal{C}$ is a group the nerve of $\mathcal{C}$ is indeed a $K(\mathcal{C}, 1)$, as shown in an exercise at the end of this section.

For a general category $\mathcal{C}$ the usual identifications of low-dimensional homology provide corollaries of Theorem 5.3. Thus the identification of H^0 and H_0 yields that

$$\varprojlim \underline{R} \cong \mathrm{End}_{R\mathcal{C}}(\underline{R}) \cong R^{\pi_0(\mathcal{C})}$$

and

$$\varinjlim \underline{R} \cong \underline{R} \otimes_{R\mathcal{C}} \underline{R} \cong R^{\pi_0(\mathcal{C})},$$

where $\pi_0(\mathcal{C})$ is the set of connected components of $\mathcal{C}$. The connected components of $\mathcal{C}$ may be described in a direct fashion as the equivalence classes of objects of $\mathcal{C}$ under the relation $x \sim y$ if and only if there is a morphism $x \to y$ or $y \to x$. We also have the identification of $H_1(\mathcal{C}, \mathbb{Z})$ as the abelianization of the direct product of the fundamental groups of the connected components of $\mathcal{C}$.

The analysis of the projective resolution in the proof Theorem 5.3 leads to a more direct description of it, but a less conceptual way to prove that it is acyclic. The resolution is

$$\cdots \to P_2 \to P_1 \to P_0 \to \underline{R} \to 0$$

where in degree n,

$$P_n = \bigoplus_{x_0 \to x_1 \to \cdots \to x_n} F_{x_n},$$

the direct sum taken over n-tuples of composable morphisms in $\mathcal{C}$. The boundary maps come from the simplicial structure on the nerve of $\mathcal{C}$. Thus the boundary map has component $(-1)^i : F_{x_n} \to F_{x_n}$ for each face $x_0 \to \cdots \to \hat{x}_i \to \cdots \to x_n$ with $i < n$, and a component map $(-1)^n$ times composition with $x_{n-1} \to x_n$ from F_{x_n} to $F_{x_{n-1}}$. This approach is described, for example, in [32] and it will be useful when we come to consider explicit cocycles in the context of extensions of categories.

Computing higher limits is an important and rather difficult problem which has an extensive literature. In the context of groups the theory is that of group cohomology, and a multitude of books has been written on the subject. When the category we are representing is a poset there is also a literature, and we mention [6, 13, 21] for example, and the following theorem.

(5.4) THEOREM (Gerstenhaber and Schack [13]). *Let $\mathcal{C}$ be a poset. Then the cohomology ring $H^*(\mathcal{C}, R)$ is isomorphic to the Hochschild cohomology ring $HH^*(R\mathcal{C}, R\mathcal{C})$.*

This result for posets is not true for categories in general. Thus the cohomology ring of a finite group is usually not isomorphic to its Hochschild cohomology ring ([3, 2.11.2]).

When $\mathcal{C}$ is an EI-category a technique for computing higher limits has been developed by Jackowski, McClure and Oliver [22, 23] in which a functor is filtered with 'atomic' factors, that is, factors which are only non-zero on one isomorphism class of objects. They have a method to compute the higher limits of the atomic factors, and they relate these higher limits to the limits of the original functor by means of a spectral sequence. This is particularly effective if the atomic factors have higher limits which are zero, in which case the spectral sequence immediately gives that the higher limits of the original functor are zero. This technique is exploited by Oliver in [32].

Exercises. 1. Although the topic of this exercise moves outside the strict area of representation theory, it is very tempting to include it at this point, because it provides a nice way to show that the classifying space of a discrete group can be realized as the nerve of the group, regarded as a category. We assume some familiarity with topological arguments. Let $\mathcal{C}$ be a group G, and let EG denote the category $E(*)$ defined before where $*$ is the unique object of G, namely the overcategory over $*$. Show that EG is a groupoid in which the only endomorphisms are the identity morphisms. Show that there is an action of G on EG specified on objects by $g \cdot (* \xrightarrow{h} *) = (* \xrightarrow{gh} *)$ in which G acts freely (i.e. with trivial stabilizers), both on objects and on morphisms. Deduce that G acts freely on the nerve $|EG|$, and that this is a contractible space. Show that the functor $EG \to G$ which on morphisms is

$$\begin{pmatrix} * & \xrightarrow{h} & * \\ \Big\downarrow{\scriptstyle gh} & & \Big\downarrow{\scriptstyle g} \\ * & = & * \end{pmatrix} \quad \mapsto \quad (* \xrightarrow{h} *)$$

defines a mapping $|EG| \to |\mathcal{C}|$ which identifies $|\mathcal{C}|$ with the quotient of $|EG|$ by G. Deduce that $|EG| \to |\mathcal{C}|$ is the universal cover of $|\mathcal{C}|$. Deduce that $\pi_1(|\mathcal{C}|) = G$ and $\pi_i(|\mathcal{C}|) = 0$ if $i > 1$, and hence $|\mathcal{C}|$ is an Eilenberg MacLane space $K(G, 1)$.

2. When R is a field, show that $\underline{R}$ is a simple module if and only if for every pair of objects x and y in $\mathcal{C}$ there is both a morphism $x \to y$ and also a morphism $y \to x$. More generally, place an equivalence relation on $\mathcal{C}$ by putting $x \sim y$ if and only if there is both a morphism $x \to y$ and also a morphism $y \to x$. Show that the composition factors of $\underline{R}$ are in bijection with the equivalence classes of objects of $\mathcal{C}$.

6. Augmentation ideals, derivations and H^1

We write the domain and codomain of a morphism α as $\mathrm{dom}(\alpha)$ and $\mathrm{cod}(\alpha)$ and define the *left augmentation map* $\epsilon : R\mathcal{C} \to \underline{R}$ to be the $R\mathcal{C}$-module homomorphism specified by $\epsilon(\alpha) = 1_{\mathrm{cod}(\alpha)}$ for each morphism α in $\mathcal{C}$, extended to the whole of $R\mathcal{C}$ by R-linearity. The *left augmentation ideal* of $R\mathcal{C}$ is the kernel $I\mathcal{C}$ of this map, so we have a short exact sequence of left $R\mathcal{C}$-modules $0 \to I\mathcal{C} \to R\mathcal{C} \to \underline{R} \to 0$. There is also a *right augmentation map* $R\mathcal{C} \to \underline{R}$ specified by $\alpha \mapsto 1_{\mathrm{dom}(\alpha)}$ and this is a homomorphism of right $R\mathcal{C}$-modules, whose kernel is the *right augmentation ideal.*

(6.1) LEMMA. *$I\mathcal{C}$ is free as an R-module with basis the elements $\alpha - 1_{\mathrm{cod}(\alpha)}$ where α ranges over the non-identity morphisms in $\mathcal{C}$.*

Proof. The elements $\alpha - 1_{\mathrm{cod}(\alpha)}$ where α ranges over the non-identity morphisms lie in $I\mathcal{C}$ and are independent, since the morphisms of $\mathcal{C}$ form an R-basis for $R\mathcal{C}$. We show that they span $I\mathcal{C}$. Suppose that $\sum \lambda_\alpha \alpha$ lies in $I\mathcal{C}$, so that for each object x of $\mathcal{C}$ we have $\sum_{\mathrm{cod}(\alpha)=x} \lambda_\alpha = 0$. Then $\sum \lambda_\alpha \alpha = \sum \lambda_\alpha \alpha - \sum_x 1_x \sum_{\mathrm{cod}(\alpha)=x} \lambda_\alpha = \sum \lambda_\alpha(\alpha - 1_{\mathrm{cod}(\alpha)})$, which is a linear combination of the $\alpha - 1_{\mathrm{cod}(\alpha)}$. □

Let M be an $R\mathcal{C}$-module. We define a *derivation* $d : \mathcal{C} \to M$ to be a mapping from the morphisms of $\mathcal{C}$ to M so that $d(\alpha) \in M(\mathrm{cod}(\alpha))$ and so that $d(\alpha\beta) = M(\alpha)d(\beta) + d(\alpha)$. Given any set of elements $\{u_x \in M(x) \mid x \in \mathrm{Ob}(\mathcal{C})\}$ we obtain a derivation specified by $d(\alpha) = M(\alpha)(u_{\mathrm{dom}(\alpha)}) - u_{\mathrm{cod}(\alpha)}$, and any derivation obtained in this way is called an *inner derivation.* The set of derivations forms an R-module $\mathrm{Der}(\mathcal{C}, M)$ and the inner derivations form an R-submodule $\mathrm{IDer}(\mathcal{C}, M)$.

(6.2) LEMMA.
(1) $\mathrm{Der}(\mathcal{C}, M) \cong \mathrm{Hom}_{R\mathcal{C}}(I\mathcal{C}, M)$ *as R-modules.*
(2) $H^1(\mathcal{C}, M) \cong \mathrm{Der}(\mathcal{C}, M)/\mathrm{IDer}(\mathcal{C}, M)$.

Proof. (1) Given a homomorphism $\delta : I\mathcal{C} \to M$ we define a derivation $d : \mathcal{C} \to M$ by $d(\alpha) = \delta(\alpha - 1_{\mathrm{cod}(\alpha)})$, and given a derivation d we define a module homomorphism $\delta : I\mathcal{C} \to M$ by $\delta(\alpha - 1_{\mathrm{cod}(\alpha)}) = d(\alpha)$. We must verify that we do indeed construct a derivation and a module homomorphism by this means, and it is so since $\delta(\alpha(\beta - 1_{\mathrm{cod}(\beta)})) = M(\alpha)\delta(\beta - 1_{\mathrm{cod}(\beta)})$ if and only if $\delta(\alpha\beta - 1_{\mathrm{cod}(\alpha)}) - \delta(\alpha - 1_{\mathrm{cod}(\alpha)}) = M(\alpha)\delta(\beta - 1_{\mathrm{cod}(\beta)})$, if and only if $d(\alpha\beta) - d(\alpha) = M(\alpha)d(\beta)$.

(2) We use the short exact sequence $0 \to I\mathcal{C} \to R\mathcal{C} \to \underline{R} \to 0$ to compute the first cohomology by means of the exact sequence

$$\mathrm{Hom}_{R\mathcal{C}}(R\mathcal{C}, M) \to \mathrm{Hom}_{R\mathcal{C}}(I\mathcal{C}, M) \to H^1(\mathcal{C}, M) \to 0.$$

Each homomorphism $\eta : R\mathcal{C} \to M$ is specified by the elements $u_x = \eta(1_x) \in M(x)$, and the restriction of this homomorphism to $I\mathcal{C}$ corresponds to the inner derivation determined by the u_x. □

Exercises. 1. If $\mathcal{C}$ is generated by morphisms α_i where $i \in I$ then $I\mathcal{C}$ is generated as a left $R\mathcal{C}$-module by the elements $\alpha_i - 1_{\mathrm{cod}(\alpha_i)}$, $i \in I$.

2. Let $I\mathcal{C}^R$ denote the right augmentation ideal of $R\mathcal{C}$. Show that

$$\underline{R} \otimes_{R\mathcal{C}} M \cong M/(I\mathcal{C}^R \cdot M).$$

3. In this exercise we generalize the familiar result from group homology that $H_1(G, \mathbb{Z}) \cong IG/IG^2 \cong G/G'$ when G is a group, IG denoting the usual augmentation ideal. Let ${}^L I\mathcal{C}$ denote the left augmentation ideal and $I\mathcal{C}^R$ the right augmentation ideal of $R\mathcal{C}$. Show that $\mathrm{Tor}_1^{R\mathcal{C}}(\underline{R}, \underline{R}) \cong (I\mathcal{C}^R \cap {}^L I\mathcal{C})/(I\mathcal{C}^R \cdot {}^L I\mathcal{C})$. Thus from the identification $\mathrm{Tor}_1^{R\mathcal{C}}(\underline{R}, \underline{R}) \cong H_1(|\mathcal{C}|, R)$, deduce when $R = \mathbb{Z}$ that this group is isomorphic to the abelianization of the direct product of the fundamental groups of the connected components of $|\mathcal{C}|$.

7. Extensions of categories and H^2

The usual theory of extensions of groups can be made to work in the more general setting of categories, and we explain here how it may be done. A key point in the theory is a bijection between equivalence classes of extensions and elements of a second cohomology group. If one were constructing a theory of extensions of categories one might take this result as a goal and then set about working out what an extension of categories must be so that the theorem is true, a task which does not have a completely obvious solution.

There is in fact more than one notion of an extension of a category in the literature. Hoff [20] makes two definitions, and we will describe one of them in this section, allowing extensions of a category by groups which may be non-abelian. There is another definition due to Baues and Wirsching [2] which requires the kernel groups to be abelian, but which is otherwise more general than Hoff's definition. We will explore the axioms for extensions in the sense of Hoff and prove the correspondence with elements of second cohomology.

Applications of the theory of extensions of categories can be found in recent work on p-local finite groups and fusion systems, see [4], [28]. There is also a discussion of extensions of categories in [17] in the context of complexes of groups (extending the notion of a graph of groups from Bass-Serre theory). We mention also that there is a different interpretation of the second cohomology of a category in terms of 'twisted category algebras' which is described in [27].

Following [20] we shall say that an extension of categories is a sequence of functors

$$\mathcal{M} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{C}$$

between categories $\mathcal{M}$, $\mathcal{E}$ and $\mathcal{C}$ for which

(i) $\mathcal{M}$, $\mathcal{E}$ and $\mathcal{C}$ all have the same objects, i and p are the identity on objects, i is injective on morphisms, and p is surjective on morphisms;

(ii) whenever f and g are morphisms in $\mathcal{E}$ then $p(f) = p(g)$ if and only if there exists a morphism $m \in \mathcal{M}$ for which $f = i(m)g$. In that case, the morphism m is required to be unique.

As always we are composing mappings on the left, so that in axiom (ii) the domain of $i(m)$ is the codomain of g. There is an asymmetry here in the definition, in that it would instead have been possible to require $f = gi(m)$ in axiom (ii). As things stand, we could regard an extension satisfying this opposite requirement as an extension of the opposite category of $\mathcal{C}$. The approach of Baues and Wirsching [2] (described also in [12]) removes the asymmetry and generalizes both possibilities for axiom (ii) at the same time. We choose here the version using axioms (i) and (ii) as stated because it is intuitively easy to understand and more straightforward to explain. It may also turn out to be adequate for the reader's intended application! We refer the reader in need of the more general version to [2].

While we are discussing the different possibilities for the axioms for extensions of categories we comment that it would be possible to make an apparently more general definition in which $\mathcal{M}$, $\mathcal{E}$ and $\mathcal{C}$ need not have the same objects, but instead we require that p and i be bijective on objects. To do this would add to the notational complexity without enriching the theory.

We now reformulate axiom (ii) in a way which brings to light certain implications and is more in the spirit of Baues and Wirsching [2]. We write the domain and codomain of a morphism α as $\mathrm{dom}(\alpha)$ and $\mathrm{cod}(\alpha)$.

(7.1) PROPOSITION. *In the above definition axiom (ii) can be replaced by (iia) and (iib) as follows:*

(iia) $\mathcal{M}$ is a disjoint union of groups $M(x) = p^{-1}(1_x)$ in bijection with the objects x of $\mathcal{C}$. Thus

$$\mathcal{M} = \coprod_{x \in \mathrm{Ob}(\mathcal{C})} M(x).$$

(iib) For each morphism α in $\mathcal{C}$, the action of $M(\mathrm{cod}(\alpha))$ on $p^{-1}(\alpha)$ given by composition $(m, f) \mapsto i(m)f$ has a single regular orbit.

Proof. Assume conditions (i) and (ii). We show that in $\mathcal{M}$ every endomorphism is an isomorphism and if $m : x \to y$ is a morphism in $\mathcal{M}$ then $x = y$. The latter follows from the observation that if $m : x \to y$ then $i(m) = i(m)1_x$ and so $p(i(m)) = p(1_x) = 1_x$ which forces $y = x$. Furthermore since $p(i(m)) = p(1_x)$ there exists $m' \in \mathcal{M}$ with $1_x = i(m')i(m) = i(m'm)$ so $m'm = 1_x$ in $\mathcal{M}$ by injectivity of i and m has a left inverse. Similarly m' has a left inverse, and also a right inverse, namely m. The left and right inverse are equal, to m, and hence m is invertible.

The regular orbit in condition (iib) is equivalent to the statement in condition (ii) that m exists and is unique.

All this shows that condition (ii) implies (iia) and (iib). We have already shown part of the converse, and condition (iia) allows us to deduce that if $f = i(m)g$ as in (ii) then $p(f) = p(g)$, which is all that remains to be shown. $\square$

Whenever we have an extension of groups $1 \to N \to E \to G \to 1$ (a short exact sequence) there is associated to it an action by conjugation of E on N, and if N happens to be abelian this passes to an action of G on N, which makes N into a representation of G. A similar thing happens with extensions of categories. Given an extension $\mathcal{M} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{C}$ we may define a functor $M : \mathcal{E} \to$ Groups as follows. We have already defined for each object x of $\mathcal{E}$ a group $M(x) = p^{-1}(1_x)$. If $f : x \to y$ is a morphism in $\mathcal{C}$ and m is an element of $M(x)$ (i.e. a morphism in this group regarded as a category) then since $p(f) = p(fi(m))$ there is a unique morphism ${}^f m$ in $M(y)$ for which $fi(m) = i({}^f m)f$. In the case of group extensions this equation is simply the requirement that the kernel is a normal subgroup.

(7.2) PROPOSITION. *(Hoff [20]). Given an extension of categories $\mathcal{M} \to \mathcal{E} \to \mathcal{C}$ the above specification defines a functor $M : \mathcal{E} \to$ Groups, where for each morphism $f : x \to y$ the effect of $M(f) : M(x) \to M(y)$ is $m \mapsto {}^f m$.*

Proof. We check that $M(fg) = M(f)M(g)$ and that $M(1_x) = 1_{M(x)}$. The latter is immediate, and for the former we have

$$fg(i(m)) = i({}^{(fg)}m)fg = fi({}^g m)g = i({}^f({}^g m))fg$$

from which we deduce ${}^{(fg)}m = {}^f({}^g m)$ by uniqueness of these morphisms. $\square$

The functor M just defined carries a lot of information from the extension $\mathcal{M} \to \mathcal{E} \to \mathcal{C}$ and it has the category $\mathcal{M}$ implicit in it. Thus we may instead regard the extension as an extension of $\mathcal{C}$ not by a category $\mathcal{M}$, but by a functor M, a notational approach which is consistent with that of Baues and Wirsching [2].

We define equivalence of extensions by the existence of a commutative diagram of functors:

$$\begin{array}{ccccc} \mathcal{M} & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{C} \\ \| & & \downarrow & & \| \\ \mathcal{M} & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \mathcal{C} \end{array}.$$

We say that the extension is *split* if there is a functor $s : \mathcal{C} \to \mathcal{E}$ so that $ps = 1_{\mathcal{C}}$, and in this situation we obtain a functor $\mathcal{C} \to$ Groups by composition $\mathcal{C} \xrightarrow{s} \mathcal{E} \xrightarrow{M}$ Groups. Conversely, given a functor $M : \mathcal{C} \to$ Groups we may always construct a split extension which realizes M by means of the 'Grothendieck construction' $Gr_{\mathcal{C}}(M) = M \rtimes \mathcal{C}$ (see [5], [9] for the general construction and its properties). We comment that the notation $M \rtimes \mathcal{C}$ is not standard, but it seems appealing by analogy with the notation for a semidirect product in group theory. In our special situation we may take $M \rtimes \mathcal{C}$ to be the category which has the same objects as $\mathcal{C}$, and if x and y are such objects then $\mathrm{Hom}_{M\rtimes\mathcal{C}}(x,y)$ is the set of pairs (m,α) where $\alpha : x \to y$ is a morphism in $\mathcal{C}$ and $m \in M(\mathrm{cod}(\alpha))$. The composition of such morphisms is $(m,\alpha)\circ(n,\beta) = (m\cdot M(\alpha)(n), \alpha\beta)$ when $\mathrm{cod}(\beta) = \mathrm{dom}(\alpha)$. We may verify that this composition is associative, the calculation being similar to that which shows that multiplication in a semidirect product of groups is associative.

There is a functor $p : M \rtimes \mathcal{C} \to \mathcal{C}$ which is the identity on objects and which sends a morphism (m,α) to α, and there is also a functor $s : \mathcal{C} \to M \rtimes \mathcal{C}$ with $s(x) = x$, $s(\alpha) = (1_{\mathrm{cod}(\alpha)}, \alpha)$ satisfying $ps = 1_{\mathcal{C}}$, where here $1_{\mathrm{cod}(\alpha)}$ denotes the identity element of the group $M(\mathrm{cod}(\alpha))$ written in multiplicative notation. Letting $\mathcal{M}$ be the category with the same objects as $\mathcal{C}$, $\mathrm{End}_{\mathcal{M}}(x) = M(x)$ and $\mathrm{Hom}_{\mathcal{M}}(x,y) = \emptyset$ if $x \neq y$ we have a functor $i : \mathcal{M} \to M \rtimes \mathcal{C}$ which is the identity on objects, and which on morphisms $m \in \mathrm{End}_{\mathcal{M}}(x)$ is $i(m) = (m, 1_x)$. We leave it as an exercise to show that $\mathcal{M} \xrightarrow{i} M \rtimes \mathcal{C} \xrightarrow{p} \mathcal{C}$ is an extension of categories.

Consider the case of an extension in which the groups $M(x)$ are all abelian. In this case the action of $i(\mathcal{M})$ on $\mathcal{M}$ given by the functor M is trivial, since if $m, m' \in M(x)$ then (in multiplicative notation) $m'm = {}^{m'}m m' = mm'$, so that ${}^{m'}m = m = M(m')(m)$. It follows that the functor $M : \mathcal{E} \to \mathrm{Groups}$ induces a well-defined functor $M : \mathcal{C} \to \mathrm{AbelianGroups}$, or in other words a representation of $\mathcal{C}$. Such extensions are called *linear extensions* in [2], and we say that the extension is an extension of $\mathcal{C}$ by the representation M. Thus an extension of $\mathcal{C}$ by M, where $M : \mathcal{C} \to R$-mod is a representation of $\mathcal{C}$, is an extension $\mathcal{M} \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{C}$ satisfying conditions (i) and (ii) (or (iia) and (iib)) and where

$$\mathrm{End}_{\mathcal{M}}(x) = M(x) = p^{-1}(1_x)$$

and if $f : x \to y$ in $\mathcal{E}$, $m \in \mathrm{End}_{\mathcal{M}}(x)$, then

$$fi(m) = i(M(f)(m))f.$$

We come now to the parametrization of equivalence classes of extensions by elements of $H^2(\mathcal{C}, M) = \mathrm{Ext}^2_{\mathbb{Z}\mathcal{C}}(\underline{\mathbb{Z}}, M)$.

(7.3) PROPOSITION. *Let M be a representation of $\mathcal{C}$ and let $\mathcal{M} = \coprod_{x\in \mathrm{Ob}(\mathcal{C})} M(x)$ be the category constructed from M as above. Then equivalence classes of extensions of $\mathcal{C}$ by $\mathcal{M}$ are in bijection with $H^2(\mathcal{C}, M)$ in such a way that the zero element corresponds to the Grothendieck construction $M \rtimes \mathcal{C}$.*

Proof. There is more than one way of exhibiting this correspondence. Because it is appealing conceptually, and partly because it seems less well known, we will copy the approach which is given for group extensions in [16, 10.5]. After this proof we will describe a different approach using explicit cocycles. In what follows, all our augmentation ideals will be left augmentation ideals.

We will show that category extensions of $\mathcal{C}$ biject with $R\mathcal{C}$-module extensions of the augmentation ideal $I\mathcal{C}$. Shifting dimensions by means of the short exact sequence $0 \to I\mathcal{C} \to R\mathcal{C} \to \underline{R} \to 0$ we have $\mathrm{Ext}^2_{R\mathcal{C}}(\underline{R}, M) \cong \mathrm{Ext}^1_{R\mathcal{C}}(I\mathcal{C}, M)$, and we know from homological algebra that the latter group bijects with equivalence classes of $R\mathcal{C}$-module extensions of $I\mathcal{C}$ by M. Putting these bijections together gives the result.

Given an $R\mathcal{C}$-module extension $0 \to M \to E \to I\mathcal{C} \to 0$ we show how to obtain an extension of categories $\mathcal{M} \to E \rtimes \mathcal{C} \to I\mathcal{C} \rtimes \mathcal{C}$. There is a functor $\mathcal{C} \to I\mathcal{C} \rtimes \mathcal{C}$ which splits the right hand term, specified as the identity on objects, and sending a morphism α to $(\alpha - 1_{\mathrm{cod}(\alpha)}, \alpha)$. We now form the diagram

$$\begin{array}{ccccc} \mathcal{M} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{C} \\ \| & & \downarrow & \text{pullback} & \downarrow \\ \mathcal{M} & \longrightarrow & E \rtimes \mathcal{C} & \longrightarrow & I\mathcal{C} \rtimes \mathcal{C} \end{array}$$

in which the right hand square is a pullback in the category of small categories, and this constructs an extension $\mathcal{M} \to \mathcal{E} \to \mathcal{C}$.

Conversely, given an extension of categories $\mathcal{M} \to \mathcal{E} \to \mathcal{C}$ we may construct an extension of $R\mathcal{C}$-modules as follows. The functor $\mathcal{E} \to \mathcal{C}$ yields an algebra homomorphism. $R\mathcal{E} \to R\mathcal{C}$, and we claim that the kernel is the ideal of $R\mathcal{E}$ generated by the augmentation ideal $I\mathcal{M}$. We will denote this ideal $\overline{I\mathcal{M}}$. Since it is contained in $I\mathcal{E}$ we obtain a short exact sequence of $R\mathcal{E}$-modules: $0 \to \overline{I\mathcal{M}} \to I\mathcal{E} \to I\mathcal{C} \to 0$, and hence a sequence of $R\mathcal{C}$-modules $0 \to \overline{I\mathcal{M}}/(\overline{I\mathcal{M}} \cdot I\mathcal{E}) \to I\mathcal{E}/(\overline{I\mathcal{M}} \cdot I\mathcal{E}) \to I\mathcal{C} \to 0$.

We claim that $\overline{I\mathcal{M}}/(\overline{I\mathcal{M}} \cdot I\mathcal{E}) \cong M$ as $R\mathcal{C}$-modules, the isomorphism being

$$m_x - 1_x + (\overline{I\mathcal{M}} \cdot I\mathcal{E}) \leftrightarrow m_x$$

where $m_x \in M(x)$. We may see this from the structure of $\overline{I\mathcal{M}}$, which is spanned by elements of the form $\alpha(m-1)\beta$ where m is a morphism in $\mathcal{M}$ and α, β are morphisms in $\mathcal{E}$. Since $\alpha(m-1) = ({}^{\alpha}m - 1)\alpha$, this ideal is in fact spanned by elements of the form $(m-1)\beta = (m-1)(\beta-1) + (m-1)$, from which we see that the images of the elements $(m-1)$ with m in M span $\overline{I\mathcal{M}}/(\overline{I\mathcal{M}} \cdot I\mathcal{E})$. The mapping $M \to \overline{I\mathcal{M}}/(\overline{I\mathcal{M}} \cdot I\mathcal{E})$ specified by $m \mapsto (m-1) + (\overline{I\mathcal{M}} \cdot I\mathcal{E})$ is a surjective group homomorphism. It has an inverse, induced by $(m-1)\beta \mapsto m$. Note that under this assignment a product $(m-1)(\beta-1) = (m-1)\beta - (m-1)$ is sent to zero, so the inverse mapping is well defined. Note also that since

$${}^{\alpha}m - 1 + (\overline{I\mathcal{M}} \cdot I\mathcal{E}) = ({}^{\alpha}m - 1)\alpha + (\overline{I\mathcal{M}} \cdot I\mathcal{E}) = \alpha(m-1) + (\overline{I\mathcal{M}} \cdot I\mathcal{E})$$

the isomorphism is one of $R\mathcal{C}$-modules.

We conclude that we have constructed an extension of $R\mathcal{C}$-modules

$$0 \to M \to I\mathcal{E}/(\overline{I\mathcal{M}} \cdot I\mathcal{E}) \to I\mathcal{C} \to 0.$$

We may check that the two operations we have described, sending category extensions of $\mathcal{C}$ to $R\mathcal{C}$-module extensions of $I\mathcal{C}$ and vice-versa, are mutually inverse bijections on equivalence classes. ◻

Another way to demonstrate the correspondence between equivalence classes of extensions of $\mathcal{C}$ and elements of second cohomology is to use explicitly given 2-cocycles. Given an extension $\mathcal{M}\xrightarrow{i}\mathcal{E}\xrightarrow{p}\mathcal{C}$ we may choose a *section* for p, that is, for each morphism $\alpha : x \to y$ in $\mathcal{C}$ a morphism $s(\alpha) : x \to y$ in $\mathcal{E}$ with $ps(\alpha) = \alpha$. Now if $\alpha : x \to y$ and $\beta : y \to z$ in $\mathcal{C}$ then $s(\beta\alpha) = i(\tau(\beta, \alpha))s(\beta)s(\alpha)$ for some unique $\tau(\beta, \alpha) \in M(z)$ and associativity of composition in $\mathcal{C}$ implies the *2-cocycle condition*

$$\tau(\gamma\beta, \alpha) + \tau(\gamma, \beta) = \tau(\gamma, \beta\alpha) + M(\gamma)\tau(\beta, \alpha)$$

as we may readily check by a standard calculation. Such mappings τ defined on pairs of composable morphisms are called *explicit 2-cocycles* or *factor sets*. Mappings of the form

$$\tau(\beta, \alpha) = f(\beta\alpha) - f(\beta) - M(\beta)f(\alpha),$$

where for each morphism $\alpha : x \to y$ in $\mathcal{C}$ we have an element $f(\alpha) \in M(y)$, automatically satisfy this condition, and may be called *explicit 2-coboundaries.* If $\mathcal{P} \to \underline{R}$ is the resolution constructed in the proof of Theorem 5.3 and described explicitly immediately afterwards we see that these are in bijection with the 2-cocycles and 2-coboundaries in the complex $\mathrm{Hom}(\mathcal{P}, M)$. To see this, observe that P_2 is the direct sum of representable functors F_z for each pair of composable morphisms $x\xrightarrow{\alpha}y\xrightarrow{\beta}z$ in $\mathcal{C}$ and homomorphisms $F_z \to M$ are in bijection with the elements of $M(z)$, by Yoneda's lemma. Thus homomorphisms $P_2 \to M$ are in bijection with elements $\tau(\beta, \alpha) \in M(z)$, and from the description of the boundary map on $\mathrm{Hom}(\mathcal{P}, M)$ we see that it is a 2-cocycle if and only if it satisfies the above 2-cocycle condition, and a 2-coboundary if and only if it satisfies the above 2-coboundary condition. This gives an identification of $H^2(\mathcal{C}, M)$ as the quotient of 2-cocycles by 2-coboundaries in the above sense. We may verify that 2-cocycles are homologous if and only if the extensions which produced them are equivalent, and also that a change in the choice of a section produces a cohomologous 2-cocycle. This gives a different approach to the correspondence between equivalence classes of extensions of $\mathcal{C}$ by M, and $H^2(\mathcal{C}, M)$.

Exercises. Most of these exercises are analogues of standard results in group cohomology, and they may be done by using the usual ideas and transfering them to the context of categories.

1. Prove that if two extensions of categories are equivalent then the categories which are the middle terms of the sequences of functors are isomorphic.

2. Show that the composition of morphisms in $M \rtimes \mathcal{C}$ is associative. Show also that the functors $\mathcal{M}\xrightarrow{i}M \rtimes \mathcal{C}\xrightarrow{p}\mathcal{C}$ described in the text give an extension of categories.

3. Show that an extension $\mathcal{M} \to \mathcal{E} \to \mathcal{C}$ is split if and only if it is equivalent to the extension $\mathcal{M} \to M \rtimes \mathcal{C} \to \mathcal{C}$. In other words: prove that any two split extensions of $\mathcal{C}$ by M are equivalent.

4. Given an extension $\mathcal{M} \to \mathcal{E} \to \mathcal{C}$ show that $R\mathcal{E}$ is free as an $R\mathcal{M}$-module.

5. Given an extension $\mathcal{M} \to \mathcal{E} \to \mathcal{C}$ and an object x of $\mathcal{C}$ show that if $\mathrm{End}_{\mathcal{C}}(x)$ is a group then $\mathrm{End}_{\mathcal{E}}(x)$ is a group.

6. Show that with the correspondence between $H^2(\mathcal{C}, M)$ and equivalence classes of extensions given above, the zero element of $H^2(\mathcal{C}, M)$ corresponds to the extension $\mathcal{M} \to M \rtimes \mathcal{C} \to \mathcal{C}$ given by the Grothendieck construction.

7. Let M be a representation of $\mathcal{C}$, and consider a section s for a split extension $\mathcal{M} \xrightarrow{i} M \rtimes \mathcal{C} \xrightarrow{p} \mathcal{C}$, so that for each morphism α in $\mathcal{C}$ we have a morphism $s(\alpha)$ in $M \rtimes \mathcal{C}$ with the property that $ps(\alpha) = \alpha$. From the definition of $M \rtimes \mathcal{C}$ we have $s(\alpha) = (d_s(\alpha), \alpha)$ for some uniquely defined $d_s(\alpha) \in \mathrm{End}_{\mathcal{M}}(\mathrm{cod}(\alpha))$.

(i) Show that s is a functor $\mathcal{C} \to M \rtimes \mathcal{C}$ if and only if d_s is a derivation of $\mathcal{C}$ in M, so that in such a case s splits the extension.

(ii) Show that there is an action of the abelian group $\bigoplus_{x \in \mathrm{Ob}\, \mathcal{C}} M(x)$ on the set of splittings s of the extension in which an element $u = \sum_{x \in \mathrm{Ob}\, \mathcal{C}} m_x$ where $m_x \in M(x)$ sends s to ${}^u s$, where ${}^u s(\alpha) = i(m_{\mathrm{cod}(\alpha)}) s(\alpha) i(-m_{\mathrm{dom}(\alpha)})$.

(iii) Consider now two splittings s and t corresponding to derivations d_s and d_t. Show that $d_t - d_s$ is an inner derivation if and only if s and t lie in the same orbit under the 'conjugation' action of M described in (ii). Deduce that $H^1(\mathcal{C}, M)$ is in bijection with the orbits of M on the complements to $i(\mathcal{M})$ in $M \rtimes \mathcal{C}$ under conjugation.

8. Show that the group structure on equivalence classes of extensions is given by the following construction which in the case of group extensions is the Baer sum. We suppose we have two extensions $\mathcal{M} \xrightarrow{i_j} \mathcal{E}_j \xrightarrow{p_j} \mathcal{C}$, $j = 1, 2$ and construct a new category $\mathcal{E}_3$ whose objects are the same as those of $\mathcal{C}$. We define $\mathrm{Hom}_{\mathcal{E}_3}(x, y) = A \backslash B$, the set of orbits of A on B in a left action, where

$$B = \{(\alpha_1, \alpha_2) \mid \alpha_1 \in \mathrm{Hom}_{\mathcal{E}_1}(x, y),\ \alpha_2 \in \mathrm{Hom}_{\mathcal{E}_2}(x, y),\ p_1(\alpha_1) = p_2(\alpha_2)\}$$

and

$$A = \{(i_1(m), i_2(m)^{-1}) \mid m \in \mathrm{End}_{\mathcal{M}}(y)\}.$$

The functors $\mathcal{M} \xrightarrow{i_3} \mathcal{E}_3 \xrightarrow{p_3} \mathcal{C}$ are defined by $i_3(m) = A(m, 1)$ and $p_3(A(\alpha_1, \alpha_2)) = p_1(\alpha_1)$.

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